The Spatially Closed Universe

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Abstract: The general world model for homogeneous and isotropic universe has been proposed. For this purpose, we introduce a global and fiducial system of reference (world reference frame) constructed on a (4+1)-dimensional space-time, and assume that the universe is spatially a 3-dimensional hypersurface embedded in the 4-dimensional space. The simultaneity for the entire universe has been specified by the global time coordinate. We define the line element as the separation between two neighboring events on the expanding universe that are distinct in space and time, as viewed in the world reference frame. The information that determines the kinematics of the geometry of the universe such as size and expansion rate has been included in the new metric. The Einstein's field equations with the new metric imply that closed, flat, and open universes are filled with positive, zero, and negative energy, respectively. The curvature of the universe is determined by the sign of mean energy density. We have demonstrated that the flat universe is empty and stationary, equivalent to the Minkowski space-time, and that the universe with positive energy density is always spatially closed and finite. In the closed universe, the proper time of a comoving observer does not elapse uniformly as judged in the world reference frame, in which both cosmic expansion and time-varying light speeds cannot exceed the limiting speed of the special relativity. We have also reconstructed cosmic evolution histories of the closed world models that are consistent with recent astronomical observations, and derived useful formulas such as energy-momentum relation of particles, redshift, total energy in the universe, cosmic distance and time scales, and so forth. The notable feature of the spatially closed universe is that the universe started from a non-singular point in the sense that physical quantities have finite values at the initial time as judged in the world reference frame. It has also been shown that the inflation with positive acceleration at the earliest epoch is improbable.

Keywords: line element, world reference frame, curvature, spatially closed universe, cosmic evolution

1. Introduction

The main goal of modern cosmology is to build a cosmological model that is consistent with astronomical observations. To achieve this goal, tremendous efforts have been made both on theories and on observations since the general theory of relativity was developed. So far the most successful model of the universe is the Friedmann-Robertson-Walker (FRW) world model (Friedmann 1922, 1924; Robertson 1929; Walker 1935). The FRW world model predicts reasonably well the current observations of the cosmic microwave background

(CMB) radiation and the large-scale structures in the universe. The precisely determined cosmological parameters of the FRW world model imply that our universe is consistent with the spatially flat world model dominated by dark energy and cold dark matter (ΛCDM) with adiabatic initial condition driven by inflation (Spergel et al. 2007; Tegmark et al. 2006; see Planck Collaboration 2018a for recent results).

Although the flat FRW world model is currently the most reliable physical world model, one may have the following fundamental questions on the nature of the FRW world model. First, mathematically, if a manifold is flat, then the Riemann curvature tensor should vanish, and vice versa. However, the spatial part of the Riemann curvature (or Ricci) tensor of the flat FRW world model does not vanish unless the cosmic expansion speed and acceleration are zeros, which implies that the physical space of the flat FRW world is not geometrically flat but curved. Only its spatial

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section at a constant time is flat. In the FRW metric, the kinematics of the geometry of the universe is described by the expansion scale factor $a(t)$ alone. Generally, the kinematic state of an expanding space is completely determined by its size and expansion rate at a given time. Thus, it is natural to expect that the space-time metric for the non-stationary universe should include the two parameters [$a(t)$ and $\dot{a}(t)$; Sec. 2].

Secondly, the cosmic evolution equations of the FRW world model can be derived from an application of the Newton's gravitation and the local energy conservation laws to the dynamical motion of an expanding sphere with finite mass density (Milne 1934; McCrea and Milne 1934). Besides, the Newton's gravitation theory has been widely used to mimic the non-linear clustering of large-scale structures in the universe even on the horizon-sized N-body simulations (Colberg et al. 2000; Park et al. 2005). On large scales, the close connection between the FRW world model and the Newton's gravitation law is usually attributed to the fact that the linear evolution of large-scale density perturbations satisfies the weak gravitational field condition.[1] However, one may have a different point of view that the Newton's gravitational action at a distance appears to be valid even on the super-horizon scales in the FRW world just because the world model does not reflect the full nature of the relativistic theory of gravitation.

Thirdly, according to the FRW world model, the universe at sufficiently early epoch ($z \gtrsim 1000$) is usually regarded as flat since the curvature parameter contributes negligibly to the total density. The present non-flat universe should have had the density parameter approaching to $\Omega = 1$ with infinitely high precision just after the big-bang (flatness problem). On the other hand, if we imagine the surface of an expanding balloon with positive curvature, then the

curvature of the surface is always positive and becomes even higher as the balloon is traced back to the earlier epoch when it was smaller. This prediction from the common sense contradicts the FRW world model.

Observationally, the flat ΛCDM universe is favored by the joint cosmological parameter estimation using the CMB (Hinshaw et al. 2007; Page et al. 2007), large-scale structures (Cole et al. 2005; Tegmark et al. 2004), type Ia supernovae (SNIa; Riess et al. 2007; Wood-Vasey et al. 2007), Hubble constant (Freedman et al. 2001; Macri et al. 2006; Sandage et al. 2006), baryonic acoustic oscillation data (Eisenstein et al. 2005), and so on. However, the Wilkinson Microwave Anisotropy (WMAP) CMB data alone is more compatible with the non-flat FRW world model (Spergel et al. 2007, §7.3; Tegmark et al. 2006, Table III). Besides, some parameter estimations using SNIa data or angular size-redshift data of distant radio sources alone suggest a possibility of the closed universe (Clocchiatti et al. 2006; Jackson and Jannetta 2006). The WMAP 5-year data analysis (Hinshaw et al. 2009) shows that the WMAP data alone favors the spatially closed universe with the current curvature density parameter $\Omega_k = -0.099^{+0.085}_{-0.100}$ in the tilted non-flat ΛCDM FRW universe (Dunkley et al. 2009). The analysis of Planck 2018 CMB data alone gives $\Omega_k = -0.044^{+0.033}_{-0.034}$ with 95% confidence level (Planck Collaboration 2018a, 2018b). Although the CMB data alone poorly constrains the magnitude of the curvature, it precisely determines the sign of the curvature, excluding flat and open universes ($\Omega_k \geq 0$) with over 95% significance level.[2]

The questions above and the observational constraints on the cosmological model may bring about possibilities of non-flat or non-FRW world models. Interestingly, Einstein (1922) claimed that our universe is spatially bounded or closed. The

[1] Hwang and Noh (2006) show that the relativistic fluid equations perturbed to second order in a flat FRW background world coincide exactly with the Newtonian results, and prove that the Newtonian numerical simulation is valid in all cosmological scales up to the second order.

primary reason for his preference to the closed universe is because Mach's idea (Mach 1893; Misner et al. 1973, §21.12) that the inertia depends upon the mutual action of bodies is compatible only with the finite universe, not with a quasi-Euclidean, infinite universe. According to Einstein's argument, an infinite universe is possible only if the mean density of matter in the universe vanishes, which is unlikely due to the fact that there is a positive mean density of matter in the universe.[3]

In this paper, we propose the general world model for homogeneous and isotropic universe which supports Einstein's perspective on the physical universe. The outline of this paper is as follows. In Sec. 2, we review and criticize the FRW metric and present a new method to define the metric where the kinematics of the geometry of the universe is fully considered. The metric and the cosmic evolution equations for flat, closed, and open universes are derived in Sec. 3. It will be shown that our universe is spatially closed. In Sec. 4, we reconstruct cosmic evolution histories of the closed world models, and derive interesting properties of the closed universe. In Sec. 5, we discuss whether the inflation theory is compatible with the closed world model or not. Conclusion follows in Sec. 6.

Throughout this paper, we adopt a sign convention $(+,-,-,-)$ for the metric tensor g_{ik}, and denote a 4-vector in space-time as p^i $(i=0,1,2,3)$ and a 3-vector in space as p^α $(\alpha=1,2,3)$ or $\mathbf{p}$. The Einstein's field equations are

$$R_{ik} - \frac{1}{2}g_{ik}R = 8\pi G T_{ik} + \Lambda g_{ik}, \qquad (1)$$

where $R_{ik} = R^a{}_{iak}$ is the Ricci tensor, $R = R^i{}_i$ the Ricci scalar, T_{ik} the energy-momentum tensor, G the Newton's gravitational constant, and Λ the cosmological constant. The Riemann curvature tensor is given by $R^a{}_{ibk} = \partial_b \Gamma^a_{ki} - \partial_k \Gamma^a_{bi} + \Gamma^a_{bn}\Gamma^n_{ki} - \Gamma^a_{kn}\Gamma^n_{bi}$, with the Christoffel symbol $\Gamma^a_{ik} = \frac{1}{2}g^{ab}(\partial_i g_{kb} + \partial_k g_{ib} - \partial_b g_{ik})$. The energy-momentum tensor for perfect fluid is

$$T_{ik} = (\varepsilon_{\rm b} + P_{\rm b})u_i u_k - P_{\rm b}g_{ik}, \qquad (2)$$

where $\varepsilon_{\rm b}$ and $P_{\rm b}$ are background energy density and pressure of ordinary matter and radiation, and u_i is the 4-velocity of a fundamental observer. We assume that the cosmological constant acts like a fluid with effective energy density $\varepsilon_\Lambda = \Lambda/8\pi G$ and pressure $P_\Lambda = -\varepsilon_\Lambda$. The limiting speed in the special theory of relativity is set to unity $(c \equiv 1)$.

2. How to define metric for homogeneous and isotropic universes?

(a) Friedmann-Robertson-Walker metric

The starting point for constructing a physical world model is to define the space-time separation between two neighboring events, i.e., the line element

$$ds^2 = g_{ik}(x)dx^i dx^k, \qquad (3)$$

where $g_{ik}(x)$ is the metric tensor which determines all the geometric properties of space-time in a system of coordinates. In Eq. (3), the two events are generally distinct in space and time, separated by $dx^i = (dx^0, dx^1, dx^2, dx^3)$. In the special theory of

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[2] Park and Ratra (2018a, 2019a, 2019b) constrained untilted non-flat dark energy models using the Planck 2015 CMB data, type Ia supernovae data, baryonic acoustic oscillations data, Hubble parameter measurements, and growth rate observations. The estimated curvature parameter (Ω_k) implies the spatially closed universe, deviating from the spatial flatness by over 5.1σ, 3.4σ, 3.1σ in ΛCDM, XCDM, and ϕCDM models, respectively. The constraint on the spatial curvature from the non-CMB data alone are also consistent with, but weaker than, that from the joint analysis of CMB and non-CMB data sets (Park and Ratra 2019c).

[3] However, in the appendix to the second edition of his book (Einstein 1922), Einstein summarized Friedmann's world models and discussed a universe with vanishing spatial curvature and non-vanishing mean matter density, which differs from his original argument.

relativity, a separation between two distinct events is given by

$$ds^2 = dt^2 - d\mathbf{r}^2, \qquad (4)$$

which is invariant in all inertial reference frames. The metric $g_{ik} = \mathrm{diag}(1,-1,-1,-1)$ is called the Minkowski metric.

The early development in modern cosmology was focused on finding the metric appropriate for the real universe whose space-time structure is inconsistent with the Minkowski metric due to the expansion of the universe and the presence of matter in it. As a pioneer, Einstein (1917) developed a model of the static closed universe that is spatially homogeneous and isotropic, by adopting a metric with $g_{00} = 1$ and $g_{0\alpha} = 0$ from the static condition. Friedmann (1922) developed a more general world model that includes both stationary and non-stationary universes of positive spatial curvature, by assuming that one can make $g_{00} = 1$ and $g_{0\alpha} = 0$ by an appropriate choice of time coordinate. Weyl (1923) postulated that in a cosmological model the world lines of particles (e.g., galaxies) form a 3-bundle of nonintersecting geodesics orthogonal to a series of space-like hypersurfaces, which implies that g_{00} depends on $x^0 = t$ only and $g_{0\alpha} = 0$. Robertson (1929) argued that the line element may be expressed as $ds^2 = dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta$ for a universe where the matter has on the whole the time-like geodesics $x^\alpha = \mathrm{const.}$ as world lines, and the coordinate t can be interpreted as a mean time which serves to define proper time and simultaneity for the entire universe. Walker (1935) also argued that the coordinates can be chosen so that the line element for the non-stationary space-time manifold has a form $ds^2 = dt^2 + U(t)d\sigma^2$, where $d\sigma^2 = h_{\alpha\beta}dx^\alpha dx^\beta$ is the metric of the 3-dimensional Riemannian space of constant curvature, if the space has spherical symmetry about each of time-like geodesics.

The resulting line element for spatially homogeneous and isotropic universe (FRW metric) is concisely written as

$$\begin{aligned}
ds^2 &= dt^2 - a^2(t)d\sigma^2 \\
&= dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]
\end{aligned}$$

$$(5)$$

where $a(t)$ is a cosmic expansion scale factor, $d\sigma$ is the comoving-space separation between events, and $S_k(\chi) = \sinh\chi$, χ, and $\sin\chi$ for open ($k=-1$), flat ($k=0$), and closed ($k=+1$) spaces, respectively (see Weinberg 1972 for a detail derivation of the FRW metric). The corresponding cosmic evolution equations known as Friedmann equations are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon_{\mathrm{b}} + 3P_{\mathrm{b}}) + \frac{\Lambda}{3} \qquad (6)$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon_{\mathrm{b}} - \frac{k}{a^2} + \frac{\Lambda}{3}. \qquad (7)$$

The dot denotes a differentiation with respect to time.

Let us examine the property of the FRW line element that has been considered as the general metric for homogeneous and isotropic universe. To determine the metric, we need to locate space-time coordinates of neighboring events. So it is essential to specify 'a given moment of time', the simultaneity for the entire universe. In the Minkowski space-time, once an inertial reference frame is set, the simultaneity for the entire space is guaranteed. However, in the curved space-time, the concept of the simultaneity is ambiguous because generally there is no global inertial reference frame. The traditional method to avoid this problem was to replace the concept of simultaneity with that of a 3-dimensional space-like hypersurface, a slice of simultaneity (Robertson 1929; Misner et al. 1973, §27.3). At each event on a space-like hypersurface, there is a locally inertial frame whose plane of simultaneity coincides locally with the hypersurface and whose 4-velocity is orthogonal to the hypersurface. Planes of simultaneity defined by locally inertial frames at various events on the hypersurface overlap to form the space-like

hypersurface itself. Therefore, in the FRW world 'at a given moment of time' just translates into 'on a given space-like hypersurface'. The space-time spanned by the universe is represented as foliation by space-like hypersurfaces at a series of moments of time.

Figure 1 is a simplified picture showing how spatial and temporal separations of the FRW line element is related to space-time events on the expanding hypersurfaces. Here, the local time coordinate t of a given event is the proper time as measured along the world line of the cosmological fluid element that passes through the event. Let us imagine two locally inertial frames specified at events 1 and 2, infinitesimally separated by $a(t_1)d\sigma$ on the hypersurface at t_1. After an infinitesimal time interval dt, the two frames will be located at events $1'$ and $2'$, separated by $a(t_2)d\sigma$ on the hypersurface at t_2. The FRW line element (5) is given as a combination of the temporal separation dt and the spatial separation $a(t)d\sigma$.

The simultaneity for the entire universe as the space-like hypersurface or the overlapping planes of simultaneity of the locally inertial frames appears to be accurate. However, due to the following reasons, we see that even the planes of simultaneity of two locally inertial frames separated by an infinitesimal comoving separation $d\sigma$ do not coincide with an accuracy better than $d\sigma$.

First, on a space-like hypersurface at a constant time, two planes of simultaneity of the locally inertial frames do not coincide but deviate with an angle $d\sigma$ for the non-flat hypersurface (Of course, the angle is zero for the flat hypersurface). Such an effect of the spatial curvature has been reflected in the space-space part of the FRW metric ($g_{\alpha\beta}$), which describes the geometry of the hypersurface at a constant time in terms of the global coordinate system $x^\alpha = (\chi, \theta, \phi)$. As is known, the metric for 3-dimensional space-like hypersurfaces of constant curvature is unique (Misner et al. 1973, §27.6).

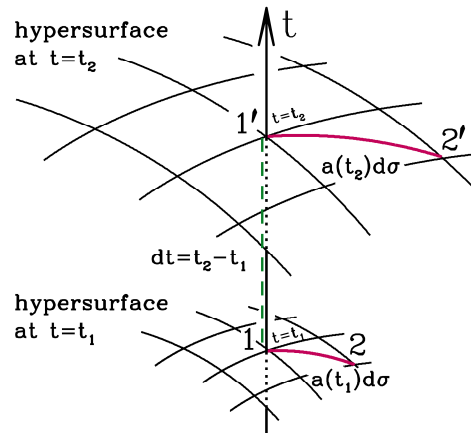Secondly, since the universe is expanding, as observed in the inertial frame at event 1, the frame

Fig. 1. A simplified picture showing the relationship between the FRW line element and space-time events on the expanding universe. Shown are two hypersurfaces at different time epochs (t_1 and t_2) infinitesimally separated by $dt = t_2 - t_1$. The local time coordinate t is defined as the proper time as measured along the world line of the cosmological fluid. Denoted as 1 and 2 are two neighboring events with a spatial separation $a(t_1)d\sigma$ on the same hypersurface at a constant time t_1, and likewise for events $1'$ and $2'$ on the hypersurface at t_2. Reproduced from Park (2012).

at event 2 recedes with relative velocity $\dot{a}(t_1)d\sigma$, and vice versa. For the expanding flat hypersurface, the deviation angle between two planes of simultaneity is given by $\arctan(\dot{a}d\sigma) \simeq \dot{a}d\sigma$ as in the special relativity, which is also comparable to the order of the infinitesimal comoving separation $d\sigma$. Although the readings of clocks at events 1 and 2 ($1'$ and $2'$) are always synchronous in the FRW viewpoint where the kinematic state (expansion speed) of a hypersurface at an instant of time is ignored, they are not as viewed in the locally inertial frame at event 1 or 2 ($1'$ or $2'$). Even if the clocks are forced to be synchronized at t_1, the synchronization will not be maintained thereafter.

We note that the effects of curvature and expansion become negligible only for a pair of locally inertial frames whose separation is far smaller than $d\sigma$. Thus, the conditions for the time-time and time-space components of the FRW metric, $g_{00} = 1$ and $g_{0\alpha} = 0$, which have been

justified by the local proper time of a locally inertial frame and the orthogonality of the world line of the locally inertial frame to the space-like hypersurface, respectively (Misner et al. 1973, §27.4), are valid extremely locally, and are not relevant to the geometrical and kinematic properties of the expanding universe, but relevant to the locally inertial frame itself.

Furthermore, we see a serious inconsistency of the FRW line element with the definition of line element. As shown in Fig. 1, in the FRW metric (5) the real-space separation $a(t)d\sigma$ is measured for events on the hypersurface at a constant time (e.g., events 1 and 2 at t_1),[4] while the temporal separation dt is measured for events at different time epochs (e.g., events 1 and 1$'$). In other words, spatial and temporal distances are related to two different pairs of events, in contradiction to the general definition of the line element [Eq. (3)] in which only a pair of events should be involved.

Generally, the kinematics of a free particle is determined by its position and velocity at a given time. Similarly, the kinematics of the expanding universe is determined by its size $[a(t)]$ and expansion speed $[\dot{a}(t)]$, aside from the gravitational field over the entire space, which is given by the Einstein's field equations and the energy content of the universe. In the FRW metric, however, the kinematics of the geometry of the universe is described by a single function of time, the expansion scale factor $[a(t)]$ alone. In this sense, the FRW metric does not include the full information about the kinematics of the geometry of the universe. In conclusion, the FRW metric describes the expansion of the homogeneous and isotropic universe incompletely.

(b) Global reference frame and new metric

In the previous subsection, it has been shown that the effect of cosmic expansion on the space-time geometry has not been correctly reflected in the FRW metric. The primary reason comes from the fact that the local time coordinates of locally inertial frames have been used as global time to specify the simultaneity for the universe. As a result, the expansion speed of the space-like hypersurface has been ignored in the metric. Besides, the way the line element is defined does not follows the general rule that a pair of neighboring events that are distinct in space and time should be used. These problems can be overcome by defining a line element based on a global inertial reference frame, by which the kinematics of the geometry of the universe is completely described. The method proposed here can be applicable to the kinematics of an expanding 2-dimensional surface such as plane or sphere in the 3+1 Minkowski space-time.

To derive the general form of line element for homogeneous and isotropic universe, let us introduce a 4+1 Minkowski space-time composed of 4-dimensional Euclidean space and 1-dimensional time, and assume that our universe is spatially a 3-dimensional hypersurface with uniform curvature embedded in the 4-dimensional space. In fact, the hypersurface with negative curvature (open space) cannot be embedded in the Euclidean space. For the present we restrict our attention to flat and closed spaces, deferring the discussion about the open space to Sec. 3(c).

Throughout this paper, we call a reference frame constructed on the high dimensional Minkowski space-time as world reference frame. It is a global system of reference provided with a rigid measuring rod and a number of clocks to indicate position and time (world time)[5] of an event. The space-time distance between events on the universe will be measured based on this fiducial system. The simultaneity for the entire universe is specified by

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[4] The spatial separation of the FRW line element is defined in the same way as the length of a rigid rod is measured at fixed time in an inertial reference frame of the special relativity.

[5] The world time coordinate is equivalent to the time coordinate of any locally inertial frame on the stationary universe.

the global time coordinate of the world reference frame.

Although the world reference frame has been introduced for mathematical convenience, it is useful in that the space-time coordinates of the frame are independent of the dynamics of the universe. As will be shown later, the proper time as measured by an observer in the universe does not elapse uniformly, being affected by the cosmic expansion. Thus, it is natural to use the world time coordinate with uniform lapse for a fair comparison of physical phenomena in the expanding universe. The space coordinates of the world reference frame can be conveniently used to describe the geometry of a 3-dimensional hypersurface embedded in the 4-dimensional Euclidean space [see Sec. 3(b)].

An example of the expanding 1-dimensional flat hypersurface (with uniform and zero curvature) is shown in Fig. 2a. At initial time t, a hypersurface is given as a straight line, on which there are equally spaced events (open circles) with mutual comoving separation δx, all at rest with respect to the comoving coordinate system whose spatial coordinate x is related to the real-space coordinate by $r = a(t)x$. After an infinitesimal time δt, the straight line has been expanded, and the proper separation between neighboring events on the hypersurface has increased from $a(t)\delta x$ to $a(t+\delta t)\delta x$.

We define the line element δs^2 as the space-time separation between two distinct events located at (t, r) and $(t+\delta t, r+\delta r)$, which correspond to events 1 and 2$'$ in Fig. 2a without loss of generality. The spatial separation between events 1 and 2$'$ as measured in the world reference frame is $\delta r_{12'} = a(t+\delta t)[x+\delta x] - a(t)x \simeq \dot{a}(t)\,x\,\delta t + a(t)\,\delta x$ up to the first order of δt and δx. Therefore, we get

$$\begin{aligned}
\delta s_{12'}^2 &= \delta t^2 - \delta r_{12'}^2 \\
&= (1 - \dot{a}^2 x^2)\delta t^2 - 2\dot{a}ax\,\delta t\,\delta x - a^2\delta x^2,
\end{aligned} \quad (8)$$

where the effect of cosmic expansion on the physical separation between events has been

included explicitly. Note that the line element (8) implies that generally g_{00} is a function of both time- and space-coordinates and $g_{0\alpha} \neq 0$, which violates the Weyl's postulate unless $\dot{a} = 0$. As shown in Sec. 3(a), the homogeneity of the universe also implies $\dot{a} = 0$.

The line element for non-flat spaces can be defined analogously. As an example of the closed space, Fig. 2b shows an expanding 1-dimensional circle (1-sphere) with radius $a(t)$ at two distinct (infinitesimally separated) world times. The expanding circle is a 1-dimensional hypersurface with uniform and positive curvature embedded in the 2-dimensional Euclidean space, rw-plane. Events on the circle are equally spaced out with $\delta\chi$, where χ is a comoving coordinate related to r-coordinate by $r = a\sin\chi$. The line element is defined as the space-time separation between distinct events 1 and 2$'$, written concisely as

$$\delta s_{12'}^2 = \delta t^2 - \delta l_{12'}^2 = (1 - \dot{a}^2)\delta t^2 - a^2\delta\chi^2 \ , \quad (9)$$

where $\delta l_{12'}^2 = \delta r_{12'}^2 + \delta w_{12'}^2$ is the spatial separation between events 1 and 2$'$ as measured in the world reference frame. The spatial distances projected on r- and w-axes are given by $\delta r_{12'} = a(t+\delta t)\sin(\chi+\delta\chi) - a(t)\sin\chi$ and $\delta w_{12'} = a(t+\delta t)\cos(\chi+\delta\chi) - a(t)\cos\chi$, respectively. In the second equality of Eq. (9), $\delta r_{12'}$ and $\delta w_{12'}$ have been expanded up to the first order of δt and $\delta\chi$. From the two cases, it is clear that the expansion of space affects both space and time intervals in the line element, which is the main difference from the FRW metric. Generally, the line element (metric) should reflect the fact that the cosmic expansion is a kinematic phenomenon.

Eq. (8) implies that the FRW line element for the 1-dimensional flat space is valid only at a local region around an observer at $x = 0$. The FRW line element for the closed space also has a similar form to the flat case, i.e., $\delta s^2 = \delta\tau^2 - a^2\delta\chi^2$, where, to be consistent with Eq. (9), $\delta\tau$ should be interpreted as a proper time interval measured by a local observer

$(\delta\tau = \delta t \sqrt{1 - \dot{a}^2}\,)$. To such an observer who may be located between events 1 and 2 (or between events $1'$ and $2'$ after δt), the spatial separation between neighboring events appears to be $a\delta\chi$ (e.g., the arc length between events 1 and 2 in Fig. 2b). Therefore, both FRW line elements describe the space-time separation between events near an observer, which is the local nature of the FRW metric.

3. Metric and evolution equations of expanding universes

In this section, we define the general forms of metric for homogeneous and isotropic universes of various spatial curvature types, and derive the corresponding cosmic evolution equations from the Einstein's field equations.

(a) Flat universe

Suppose that the universe is spatially an expanding 3-dimensional flat hypersurface (with uniform and zero curvature) embedded in a 4-dimensional Euclidean space with the Cartesian coordinates (r_1, r_2, r_3, r_4). The embedded flat space is infinite, homogeneous, and isotropic, and is Euclidean at an instant of time. To simplify the problem, let us assume that the hypersurface is orthogonal to the r_4-axis so that the fourth Cartesian coordinate can be ignored. Then, each event on the flat hypersurface is labelled by the world time t and the real-space position $\mathbf{r}$ defined as

$$\mathbf{r} = a(t)\,\mathbf{x}, \tag{10}$$

where $\mathbf{x}$ is the comoving-space position vector with the Cartesian coordinates $(x_1, x_2, x_3) = (x\sin\theta\cos\phi, x\sin\theta\sin\phi, x\cos\theta)$, expressed in terms of the comoving spherical coordinate $x^\alpha = (x, \theta, \phi)$ with $x = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

With the help of the differential of Eq. (10)

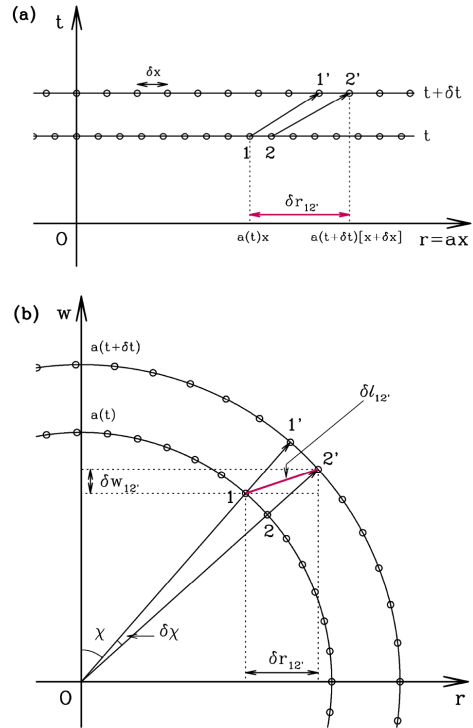$$d\mathbf{r} = \dot{a}(t)\,dt\,\mathbf{x} + a(t)d\mathbf{x}, \tag{11}$$

Fig. 2. Schematic diagrams showing the expansion of (a) flat and (b) closed 1-dimensional homogeneous and isotropic hypersurfaces. Events denoted as open circles are equally spaced out on two hypersurfaces that are infinitesimally separated by δt. Note that the time axis, which is orthogonal to r- and w-axes, is omitted in (b). The line element is defined as the space-time separation between two distinct events 1 and $2'$ (see text).

the line element is defined as the space-time separation between distinct events located at $(t, \mathbf{r})$ and $(t+dt, \mathbf{r}+d\mathbf{r})$:

$$\begin{aligned}
ds^2 &= dt^2 - d\mathbf{r}^2 \\
&= (1 - \dot{a}^2\mathbf{x}^2)\,dt^2 - 2\dot{a}a\,dt\,\mathbf{x}\cdot d\mathbf{x} - a^2 d\mathbf{x}^2 \\
&= (1 - \dot{a}^2 x^2)dt^2 - 2\dot{a}a\,x\,dt\,dx \\
&\quad - a^2[dx^2 + x^2(d\theta^2 + \sin^2\theta\,d\phi^2)].
\end{aligned} \tag{12}$$

The metric tensor in the coordinate system $x^i = (t, x, \theta, \phi)$ is

$$g_{ik} = \begin{pmatrix} 1 - \dot{a}^2 x^2 & -\dot{a}ax & 0 & 0 \\ -\dot{a}ax & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 x^2 & 0 \\ 0 & 0 & 0 & -a^2 x^2 \sin^2\theta \end{pmatrix}. \tag{13}$$

We calculate the Christoffel symbols and the Riemann curvature tensor from the metric tensor. The non-zero Christoffel symbols Γ_{ik}^a $(=\Gamma_{ki}^a)$ are

$$\Gamma_{00}^1 = \frac{\ddot{a}}{a}x, \quad \Gamma_{01}^1 = \Gamma_{02}^2 = \Gamma_{03}^3 = \frac{\dot{a}}{a},$$

$$\Gamma_{22}^1 = -x, \quad \Gamma_{33}^1 = -x\sin^2\theta, \quad \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{1}{x}, \quad (14)$$

$$\Gamma_{33}^2 = -\sin\theta\cos\theta, \quad \Gamma_{23}^3 = \cot\theta.$$

One easily verifies that the Riemann curvature tensor vanishes:

$$R^a{}_{ibk} = 0. \quad (15)$$

Thus the Ricci tensor R_{ik} and the Ricci scalar R also vanish. This demonstrates explicitly that the space-time curvature of the expanding flat universe is zero.

The proper time interval as measured by an observer in arbitrary motion is obtained from Eq. (12) as

$$d\tau = dt\left[1 - \dot{a}^2 x^2 - 2\dot{a}ax\left(\frac{dx}{dt}\right) - a^2 v^2\right]^{1/2}, \quad (16)$$

where $av = (-v_\alpha v^\alpha)^{1/2}$ is the magnitude of proper 3-velocity, $v^\alpha = dx^\alpha/dt$ is the 3-velocity in the comoving coordinate system, and $v_\alpha = g_{\alpha\beta}v^\beta$. For an observer who is at rest $(v^\alpha = 0)$ in the comoving coordinate system (hereafter a comoving observer), we get the proper time interval

$$d\tau = dt(1 - \dot{a}^2 x^2)^{1/2} \quad (17)$$

and the 4-velocity

$$u^i = \frac{dx^i}{d\tau} = \left(\frac{1}{\sqrt{1 - \dot{a}^2 x^2}}, 0, 0, 0\right) \quad (18)$$

of the observer. Inserting Eqs. (13) and (18) into Eq. (2) gives the energy-momentum tensor, whose non-zero components are

$$T_{00} = \varepsilon_b(1 - \dot{a}^2 x^2), \quad T_{01} = T_{10} = -\varepsilon_b \dot{a}ax,$$

$$T_{11} = \frac{\varepsilon_b \dot{a}^2 a^2 x^2 + P_b a^2}{1 - \dot{a}^2 x^2}, \quad T_{22} = P_b a^2 x^2, \quad (19)$$

$$T_{33} = P_b a^2 x^2 \sin^2\theta.$$

From Eqs. (13), (15) and (19), the Einstein's field

equations (1) reduce to

$$P_b = -\varepsilon_b = \frac{\Lambda}{8\pi G}. \quad (20)$$

The ε_b and P_b should not be negative because they are energy density and pressure of ordinary matter and radiation, suggesting that

$$\varepsilon_b = P_b = 0 \text{ and } \Lambda = 0. \quad (21)$$

Even if non-zero energies of matter and radiation with an equation of state (20) can exist, the total energy density and pressure should vanish:

$$\varepsilon_b + \varepsilon_\Lambda = 0 \text{ and } P_b + P_\Lambda = 0. \quad (22)$$

Therefore, in the domain of classical physics the flat universe is empty, which is consistent with the Einstein's claim that the infinite universe has vanishing mean density (Einstein 1922).

Eq. (20) does not give any information about the cosmic expansion history. Actually, the homogeneity condition constrains the flat universe to be stationary. The homogeneity of the universe means that the physical conditions such as energy density, pressure, and the curvature of space are identical at every event on the hypersurface at a constant time (Misner et al. 1973, §27.3). In Eq. (16), the proper time interval of an arbitrary observer depends on the choice of the comoving coordinate system. The proper time of an observer moving faster at farther distance from the origin goes slower. Only at $x = 0$ or if $\dot{a} = 0$, it becomes $d\tau = dt\sqrt{1 - a^2 v^2}$, the same form of the proper time as in the special relativity, irrespective of the choice of the comoving coordinate system. Since the dependence of the proper time interval on the choice of reference frame is contradictory to the homogeneity condition, the flat universe should be stationary $(\dot{a} = 0)$. In conclusion, the flat universe is empty and stationary, and therefore is equivalent to the Minkowski space-time.

The formalism given above for $\dot{a} \neq 0$ is still applicable to a finite region of the expanding flat space.

(b) Closed universe

The homogeneous and isotropic closed space is usually described by the spatially finite hypersphere with uniform and positive curvature. By extending the example in Fig. 2b, let us consider our universe as an expanding 3-sphere of curvature radius $a(t)$, embedded in a 4-dimensional Euclidean space where each point is labelled by the Cartesian coordinates (x, y, z, w) and the world time t. The equation of the 3-sphere in x-y-z-w coordinate system is

$$x^2 + y^2 + z^2 + w^2 = r^2 + w^2 = a^2, \quad (23)$$

where r is the radial distance in x-y-z coordinate system. The coordinates x, y, z have transformation relations with the spherical coordinates r, θ, ϕ as $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.

The line element is defined as the space-time distance between two infinitesimally separated events located at (t, x, y, z, w) and $(t+dt, x+dx, y+dy, z+dz, w+dw)$:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2 = dt^2 - dl^2, \quad (24)$$

where dl^2 is a spatial separation as measured in the world reference frame. For two distinct events on the expanding 3-sphere, the spatial separation is written as[6]

$$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{(ada - rdr)^2}{a^2 - r^2} \quad (25) \\ &= da^2 + a^2[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)], \end{aligned}$$

where w has been removed by Eq. (23) and its differential

$$rdr + wdw = ada = a\dot{a}dt, \quad (26)$$

and r has been replaced with the comoving coordinate χ by a parametrization

$$r = a\sin\chi \quad (0 \le \chi \le \pi), \quad (27)$$

and its differential

$$dr = \dot{a}\sin\chi\, dt + a\cos\chi\, d\chi. \quad (28)$$

Therefore, the general form of line element for the non-stationary closed universe is

$$\begin{aligned} ds^2 &= [1 - \dot{a}^2(t)]\, dt^2 \\ &\quad - a^2(t)\,[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)]. \end{aligned} \quad (29)$$

The metric tensor in the coordinate system $x^i = (t, \chi, \theta, \phi)$ is

$$g_{ik} = \text{diag}[1 - \dot{a}^2, -a^2, -a^2\sin^2\chi, -a^2\sin^2\chi\sin^2\theta]. \quad (30)$$

We calculate the Christoffel symbols and the Ricci tensor from the metric tensor. The non-zero Christoffel symbols are

$$\begin{aligned} &\Gamma^0_{00} = \frac{-\dot{a}\ddot{a}}{1-\dot{a}^2}, \quad \Gamma^0_{11} = \frac{a\dot{a}}{1-\dot{a}^2}, \quad \Gamma^0_{22} = \frac{a\dot{a}}{1-\dot{a}^2}\sin^2\chi, \\ &\Gamma^0_{33} = \frac{a\dot{a}}{1-\dot{a}^2}\sin^2\chi\sin^2\theta, \\ &\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, \quad \Gamma^1_{22} = -\sin\chi\cos\chi, \\ &\Gamma^1_{33} = -\sin\chi\cos\chi\sin^2\theta, \\ &\Gamma^2_{12} = \Gamma^3_{13} = \cot\chi, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta. \end{aligned} \quad (31)$$

The non-zero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -3\left(\frac{\ddot{a}}{a}\right)\frac{1}{1-\dot{a}^2}, \\ R_{11} &= \frac{a\ddot{a}}{(1-\dot{a}^2)^2} + \frac{2}{1-\dot{a}^2}, \quad (32) \\ R_{22} &= R_{11}\sin^2\chi, \quad R_{33} = R_{11}\sin^2\chi\sin^2\theta, \end{aligned}$$

and the Ricci scalar is

$$R = -6\left(\frac{\ddot{a}}{a}\right)\frac{1}{(1-\dot{a}^2)^2} - 6\frac{1}{a^2(1-\dot{a}^2)}. \quad (33)$$

From Eq. (29), we obtain a proper time interval as measured by an observer in arbitrary motion as

$$d\tau = dt\,(1 - \dot{a}^2 - a^2v^2)^{1/2}, \quad (34)$$

and thus express the 4-velocity of the observer as

---

[6] The 3-dimensional version of Eq. (25), $dl^2 = da^2 + a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, is mathematically equivalent to the one presented in Dubrovin et al. [1984, §9(2)], which is a separation between two neighboring points in the 3-dimensional Euclidean space, expressed in terms of spherical coordinates (a, θ, ϕ).

$$u^i = \frac{dx^i}{d\tau} = (\gamma, \gamma \mathbf{v}), \tag{35}$$

where

$$\gamma = (1 - \dot{a}^2 - a^2 v^2)^{-1/2} \tag{36}$$

is a contraction factor. Note that the contraction factor depends on the expansion speed of the universe as well as the peculiar motion of the observer. For a comoving observer, the energy-momentum tensor for perfect fluid is obtained by inserting the 4-velocity $u^i = (1/\sqrt{1-\dot{a}^2}, 0, 0, 0)$ into Eq. (2):

$$T_{ik} = \mathrm{diag}[\varepsilon_\mathrm{b}(1-\dot{a}^2), P_\mathrm{b} a^2, \\ P_\mathrm{b} a^2 \sin^2\chi, P_\mathrm{b} a^2 \sin^2\chi \sin^2\theta], \tag{37}$$

where ε_b and P_b are energy density and pressure as defined in the world reference frame. Inserting Eqs. (30), (32), (33) and (37) into (1), we get evolution equations for homogeneous and isotropic closed universe. They are concisely written as

$$\frac{1}{(1-\dot{a}^2)^2} \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon_\mathrm{b} + 3P_\mathrm{b}) + \frac{\Lambda}{3} \tag{38}$$

and

$$\frac{1}{a^2(1-\dot{a}^2)} = \frac{8\pi G}{3}\varepsilon_\mathrm{b} + \frac{\Lambda}{3}. \tag{39}$$

Combining Eqs. (38) and (39) gives a continuity equation for energy density and pressure

$$-3\left(\frac{\dot{a}}{a}\right)(\varepsilon + P) = \dot{\varepsilon}, \tag{40}$$

where ε and P are total energy density and pressure of radiation (R), matter (M), and the cosmological constant (Λ): $\varepsilon = \sum_I \varepsilon_I$ and $P = \sum_I P_I$ ($I = \mathrm{R}, \mathrm{M}, \Lambda$). Note that $\varepsilon_\mathrm{b} = \varepsilon_\mathrm{R} + \varepsilon_\mathrm{M}$. The continuity equation is equivalent to $T^i_{0;i} = 0$, with a semicolon denoting a covariant derivative. It is worth noting that the time-time component of the metric tensor ($g_{00} = 1 - \dot{a}^2$) is positive according to the sign convention adopted. The positiveness of the

left-hand side of Eq. (39) suggests that the total energy density should be positive in the closed universe ($\varepsilon > 0$).

Introducing an equation of state $P_I = w_I \varepsilon_I$ for each species I, we obtain a solution to Eq. (40) as $\varepsilon_I \propto a^{-3(1+w_I)}$. Thus the total energy density is written as $\varepsilon = \sum_I \varepsilon_{I0}(a/a_0)^{-3(1+w_I)}$. The subscript 0 denotes the present epoch t_0. Hereafter we call universes dominated by radiation, matter, and the cosmological constant as radiation-universe (R-u), matter-universe (M-u), and Λ-universe (Λ-u), respectively. The energy density evolves as $\varepsilon_\mathrm{R} \propto a^{-4}$ in the radiation-universe ($w_\mathrm{R} = 1/3$), $\varepsilon_\mathrm{M} \propto a^{-3}$ in the matter-universe ($w_\mathrm{M} = 0$), and $\varepsilon_\Lambda = \mathrm{const.}$ in the Λ-universe ($w_\Lambda = -1$). For the cosmological constant, $\varepsilon_\Lambda = \varepsilon_{\Lambda 0}$.

Let us define a dimensionless function of redshift $z \equiv a_0/a(t) - 1$,

$$\begin{aligned} A^2(z) &\equiv \frac{a_0^2}{a^2(1-\dot{a}^2)} \\ &= \frac{8\pi G a_0^2}{3}[\varepsilon_{\mathrm{R}0}(1+z)^4 + \varepsilon_{\mathrm{M}0}(1+z)^3 + \varepsilon_\Lambda] \\ &= a_0^2\left[\frac{(1+z)^4}{a_{\mathrm{R}0}^2} + \frac{(1+z)^3}{a_{\mathrm{M}0}^2} + \frac{1}{a_\Lambda^2}\right], \end{aligned} \tag{41}$$

where $a_{I0} = (3/8\pi G\varepsilon_{I0})^{1/2}$. Throughout this paper, a function of time t will be expressed in terms of redshift z interchangeably. The radius parameter $a_{\mathrm{R}0}$ ($a_{\mathrm{M}0}$) can be interpreted as a free-fall time or radius for the gravitational collapse of a stationary radiation- (matter-) universe with the present radiation (matter) energy density, and $a_\Lambda = a_{\Lambda 0}$ as the minimum radius of Λ-universe. By introducing another dimensionless quantity

$$D_I \equiv \frac{8\pi G\varepsilon_I}{3}a^2(1-\dot{a}^2) = \left(\frac{a_0}{a_{I0}}\right)^2 \frac{(1+z)^{3(1+w_I)}}{A^2(z)}, \tag{42}$$

we can rewrite Eq. (39) as

$$D_\mathrm{R} + D_\mathrm{M} + D_\Lambda = 1, \tag{43}$$

which holds during the whole history of the universe. The D_I can be interpreted as the fraction

of energy of species I [see Sec. 4(e)].

(c) Open universe

We now consider the geometry of homogeneous and isotropic expanding 3-dimensional space with uniform and negative curvature. Such a negatively curved space cannot be embedded in a 4-dimensional Euclidean space. At an instant of time, it is a pseudosphere with imaginary radius ia (Landau and Lifshitz 1975, §111). By replacing a^2 with $-a^2$ in Eq. (23), we obtain an expression analogous to Eq. (25) for a spatial separation between two distinct events on the expanding 3-pseudosphere,[7]

$$
\begin{aligned}
dl^2 &= dx^2 + dy^2 + dz^2 + du^2 \\
&= dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - \frac{(ada + rdr)^2}{a^2 + r^2} \\
&= -da^2 + a^2[d\chi^2 + \sinh^2\chi \, (d\theta^2 + \sin^2\theta \, d\phi^2)],
\end{aligned}
\tag{44}
$$

where the radial distance $r = (x^2 + y^2 + z^2)^{1/2}$ in x-y-z coordinate system has been parametrized with the comoving coordinate χ by

$$
r = a \sinh\chi \quad (\chi \geq 0).
\tag{45}
$$

Therefore, the line element for the non-stationary open universe is

$$
ds^2 = [1 + \dot{a}^2(t)]dt^2 \\
- a^2(t)[d\chi^2 + \sinh^2\chi \, (d\theta^2 + \sin^2\theta \, d\phi^2)].
\tag{46}
$$

The non-zero components of the Ricci tensor calculated from the metric (46) are

$$
\begin{aligned}
R_{00} &= -3\left(\frac{\ddot{a}}{a}\right)\frac{1}{1 + \dot{a}^2}, \\
R_{11} &= \frac{a\ddot{a}}{(1 + \dot{a}^2)^2} - \frac{2}{1 + \dot{a}^2}, \\
R_{22} &= R_{11}\sinh^2\chi, \quad R_{33} = R_{11}\sinh^2\chi \sin^2\theta,
\end{aligned}
\tag{47}
$$

and the Ricci scalar is

$$
R = -6\left(\frac{\ddot{a}}{a}\right)\frac{1}{(1 + \dot{a}^2)^2} + 6\frac{1}{a^2(1 + \dot{a}^2)}.
\tag{48}
$$

For a comoving observer ($v^\alpha = 0$) with 4-velocity $u^i = (1/\sqrt{1 + \dot{a}^2}, 0, 0, 0)$, the energy-momentum tensor for perfect fluid becomes

$$
\begin{aligned}
T_{ik} = \mathrm{diag}[\varepsilon_b(1 + \dot{a}^2), P_b a^2, \\
P_b a^2 \sinh^2\chi, P_b a^2 \sinh^2\chi \sin^2\theta].
\end{aligned}
\tag{49}
$$

The resulting evolution equations for the open universe are obtained in the same way as those for the closed universe are obtained. They are

$$
\frac{1}{(1 + \dot{a}^2)^2}\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon_b + 3P_b) + \frac{\Lambda}{3}
\tag{50}
$$

and

$$
\frac{1}{a^2(1 + \dot{a}^2)} = -\frac{8\pi G}{3}\varepsilon_b - \frac{\Lambda}{3}.
\tag{51}
$$

Combining Eqs. (50) and (51) also gives the same continuity equation as Eq. (40). Since the time-time component of the metric tensor ($g_{00} = 1 + \dot{a}^2$) is always positive, Eq. (51) suggests that the total energy density should be negative in the open universe ($\varepsilon < 0$).

(d) Our Universe is spatially closed

In Secs. 3(a)-(c), we have demonstrated that flat universe is equivalent to the Minkowski space-time, which is empty and stationary, and that closed and open universes have positive and negative energy densities, respectively. In other words, the curvature of the universe is determined by the sign of mean energy density, not by the ratio of the energy density to the critical density as in the FRW world.

---

[7] The 3-dimensional hypersurface with uniform and negative curvature can also be represented as a pseudosphere of radius a, satisfying the equation $u^2 - x^2 - y^2 - z^2 = a^2$, embedded in a 4-dimensional pseudo-Euclidean space $\mathbb{R}^4_1$. Here, the spatial separation between two neighboring points on the non-stationary pseudosphere has the same form as Eq. (44) aside from the sign, $dl^2 = du^2 - dx^2 - dy^2 - dz^2 = da^2 - a^2[d\chi^2 + \sinh^2\chi \, (d\theta^2 + \sin^2\theta \, d\phi^2)]$. See Dubrovin et al. (1984), §3.2(12) for the 3-dimensional case in $\mathbb{R}^3_1$.

Table 1. Possible Ranges of Total Energy Density in the FRW and New World Models

Curvature Type	FRW World[a]	New World
closed	$\varepsilon \geq 3/8\pi Ga^2$	$\varepsilon > 0$
flat	$\varepsilon \geq 0$	$\varepsilon = 0$
open	$-3/8\pi Ga^2 \leq \varepsilon < 0$	$\varepsilon < 0$

[a]Equality signs for the FRW world models correspond to cases of stationary universes ($\dot{a} = 0$, $\ddot{a} \neq 0$).

The open universe is unrealistic because the mean density of the universe is known to be positive from astronomical observations. Therefore, we conclude that our universe is spatially closed and finite. The spatial closure of the universe has been deduced from the purely theoretical point of view. Our conclusion verifies the Einstein's claim for the finiteness of the universe.

The new world model does not contradict the homogeneity and isotropy conditions, since the physical condition is identical everywhere on the space-like hypersurface (homogeneity) and a comoving observer cannot distinguish one of his/her spatial directions from the others by any local physical measurement (isotropy). For a given curvature type, the geometrical structure of the space-like hypersurface of the new world is equivalent to that of the FRW world (compare $g_{\alpha\beta}^{\text{new}}$ with $g_{\alpha\beta}^{\text{FRW}}$ at fixed time).

(e) The Friedmann equations

In the non-flat universes, the proper time interval of a comoving observer is related to the world time by

$$d\tau = [1 - k\dot{a}^2(t)]^{1/2}dt, \tag{52}$$

where $k = +1$ for closed and -1 for open universes. The proper time goes slower (faster) than the world time in the closed (open) universe, with non-uniform lapse. Only in the flat universe, the proper time elapses uniformly ($d\tau = dt$).

From Eq. (52), we obtain relations between world- and proper-time derivatives of the curvature radius:

$$\frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2 + \frac{k}{a^2} = \frac{k}{a^2(1 - k\dot{a}^2)} \tag{53}$$

and

$$\frac{1}{a}\left(\frac{d^2a}{d\tau^2}\right) = \frac{1}{(1 - k\dot{a}^2)^2}\frac{\ddot{a}}{a}, \tag{54}$$

where the proper time τ acts as the time of non-flat FRW world models. As will be shown in Sec. 4(e) [Eq. (92)], energy density and pressure defined in the world reference frame are equivalent to those measured by the comoving observer. Therefore, Friedmann equations (6) and (7) for non-flat universes can be derived from the new cosmic evolution equations [Eqs. (38) and (39) for closed and Eqs. (50) and (51) for open universes] by the time-parametrization (52). This, along with the local nature of the FRW metric as discussed in Sec. 2, implies that the Friedmann equations describe the evolution of the local universe around a comoving observer.

Table 1 lists possible ranges of total energy density in the FRW and new world models. In the new world models, the energy density is strictly positive, zero, and negative for closed, flat, and open universes, respectively. On the other hand, the FRW world models have rather complicated ranges of energy density. From Eq. (53), one finds that $(da/d\tau)^2 \geq 0$ for closed, and $0 \leq (da/d\tau)^2 < 1$ for open universes. Thus, Eq. (7) plus constraints on $(da/d\tau)^2$ suggests that the closed FRW world model have positive energy density larger than or equal to $3/8\pi Ga^2$, and that the open model accommodate only negative energy density not smaller than $-3/8\pi Ga^2$. Note that the positive energy density appears to be allowed in the open FRW world model if the constraint on $(da/d\tau)^2$ is not imposed.

The Friedmann equations for the flat universe ($k = 0$) cannot be derived with any world-proper time relation, but can be obtained by neglecting the curvature term $-k/a^2$ in Eq. (7) for the closed model, resulting in $\varepsilon \geq 0$. The flat FRW world model is valid only within regions where the effect

of curvature is negligible or the distance scale of interest is far smaller than the curvature radius. In other words, the flat FRW world is an approximation of the closed universe with large curvature radius. Its hypersurface is a tangent space of a comoving observer on the 3-sphere.

All the differences between FRW and new world models come from a difference between reference frames adopted. In later sections of this paper, to describe physical phenomena in the expanding closed universe, we use both the world reference frame and the comoving observer's frame. The former has a global time t with uniform lapse, while the latter has a local time τ whose lapse depends on the cosmic expansion speed. An event on the expanding 3-sphere may be labelled by coordinates (t, x, y, x, w) or $(t, a, \chi, \theta, \phi)$. By omitting the curvature radius a that is a function of t, we regard the coordinate system $x^i = (t, \chi, \theta, \phi)$ as equivalent to the world reference frame. The comoving observer's frame is a system of physical space and proper time coordinates adopted by an observer like us whose comoving coordinate $x^\alpha = (\chi, \theta, \phi)$ is fixed during the expansion of the universe. Actually, the comoving observer's frame is equivalent to the locally inertial frame.

4. Physical and astronomical aspects of the expanding closed universe

In this section, we investigate interesting properties of the expanding closed universe, such as time-varying light speed, cosmic expansion history, energy-momentum relation of particles, redshift, and cosmic distance and time scales. All the quantities, not otherwise specified, are defined and expressed in the world reference frame, and secondarily in the comoving observer's frame. In the latter frame, all the physical quantities and their evolution are the same as those in the closed FRW world.

(a) Time-varying light speed and cosmic expansion speed

In the special theory of relativity, the speed of light is constant and equal to the limiting speed ($c = 1$), which applies to the Minkowski space-time, or equivalently to the flat universe. In the expanding closed universe, however, the light speed is less than or equal to the limiting speed. From the photon's geodesic equation ($ds = 0$), one can express the speed of light as

$$\eta(t) \equiv (1 - \dot{a}^2)^{1/2} = \frac{1+z}{A(z)}. \tag{55}$$

It should be noted that the light speed varies with time, depending on the cosmic expansion speed $\dot{a}$ and satisfying $\eta^2 + \dot{a}^2 = 1$. Both speeds cannot exceed the limiting speed ($0 \le \eta \le 1$ and $-1 \le \dot{a} \le 1$). The cosmic expansion speed goes to unity as the redshift goes to infinity because $A(z)$ evolves as $(1+z)^2$ in the radiation-dominated era [Eq. (41)]. Therefore, we expect $\dot{a} = 1$ and $\eta = 0$ at the beginning of the universe.

On the other hand, the comoving observer always measures the speed of light as unity because the observer's proper time interval varies in the same way as the world-frame light speed does [Eqs. (52) and (55)]. Besides, the cosmic expansion speed in the comoving observer's frame has no limit ($da/d\tau = \dot{a}/\sqrt{1 - \dot{a}^2} < \infty$).

In the open universe, the speed of light is $\eta = (1 + \dot{a}^2)^{1/2} \ge 1$: the light propagates faster than the limiting speed. There is no upper limit on the cosmic expansion and the light speeds in the world reference frame. However, the comoving observer perceives that the speed of light is always unity and that the cosmic expansion speed is bounded to unity ($da/d\tau = \dot{a}/\sqrt{1 + \dot{a}^2} < 1$), as discussed in Sec. 3(e).

Table 2. Cosmological Parameters of the Two Closed World Models.

Parameters	Symbols	Model I[a]	Model II[b]
		Input	
CMB temperature	T_{cmb}	2.725 K	2.725 K
Matter density	Ω_{M}	0.415	0.315
Dark energy density	Ω_{Λ}	0.630	0.696
Hubble constant	h	0.55	0.71
		Derived	
Curvature radius	a_0	25670 Mpc	40170 Mpc
Speed of light	η_0/c	0.208	0.105
Expansion speed	$\dot{a}_0/c$	0.978	0.995
Expansion acceleration	$\ddot{a}_0$	$6.79\times10^{-7}\,\mathrm{Mpc}^{-1}$	$1.45\times10^{-7}\,\mathrm{Mpc}^{-1}$
Cosmic age (in world time)	t_0	85.3 Gyr	132 Gyr
Cosmic age (in proper time)	τ_0	15.8 Gyr	13.1 Gyr
Radiation free-fall radius	a_{R0}	603.1 Gpc	603.1 Gpc
Matter free-fall radius	a_{M0}	8461 Mpc	7523 Mpc
Minimum radius of Λ-u	a_{Λ}	6867 Mpc	5061 Mpc
Maximum radius of R-u	b_{R}	1093 Mpc	2675 Mpc
Maximum radius of M-u	b_{M}	236.3 Gpc	1145 Gpc
3-volume of the universe	V_0	$6.94\times10^{13}\,\mathrm{Mpc}^3$	$1.34\times10^{14}\,\mathrm{Mpc}^3$
Total radiation energy	E_{R0}	8.50×10^{74} erg	1.64×10^{75} erg
Total matter energy	E_{M0}	4.32×10^{78} erg	1.05×10^{79} erg
Total dark energy	$E_{\Lambda0}$	6.56×10^{78} erg	2.33×10^{79} erg
Fraction of radiation energy	D_{R0}	7.82×10^{-5}	4.85×10^{-5}
Fraction of matter energy	D_{M0}	0.397	0.312
Fraction of dark energy	$D_{\Lambda0}$	0.603	0.688

[a]Based on parameters of a non-flat FRW world model that best fits with the WMAP 3-year data only (Spergel et al. 2007).

[b]Based on parameters of a non-flat FRW world model that jointly fits with the CMB, SNIa, γ-ray bursts, the shape parameter $\Gamma=\Omega_{\mathrm{M}}h$, Hubble constant, matter density ($\Omega_{\mathrm{M}}h^2$), and big-bang nucleosynthesis data (Wright 2007). The Hubble constant is an average of recent measurement values.

(b) Cosmic evolution history

The evolution of homogeneous and isotropic universe is described by the evolution of physical quantities during the history of the universe. The most important quantity is the curvature radius a. Although it is not easy to get the general solution to Eq. (39), there exist analytic solutions for special cases of the universe dominated by energy of the single species. For radiation-universe ($\varepsilon_{\mathrm{M}}=0$, $\varepsilon_{\Lambda}=0$), the curvature radius is given by

$$a(t) = b_{\mathrm{R}}\sin(t/b_{\mathrm{R}}) \quad (0 \le t \le \pi b_{\mathrm{R}}), \qquad (56)$$

where $b_{\mathrm{R}} = (8\pi G\varepsilon_{\mathrm{R0}}a_0^4/3)^{1/2} = (a_0/a_{\mathrm{R0}})a_0$ is the maximum curvature radius of the radiation-universe. At $t=0$, the universe expands with the maximum speed and zero acceleration ($\dot{a}=1$, $\ddot{a}=0$). The positive acceleration is not allowed in the radiation-universe.

For matter-universe ($\varepsilon_{\mathrm{R}}=0$, $\varepsilon_{\Lambda}=0$), the solution for the curvature radius is

$$a(t) = b_{\mathrm{M}} - \frac{1}{4b_{\mathrm{M}}}(t-2b_{\mathrm{M}})^2 \quad (0 \le t \le 4b_{\mathrm{M}}), \quad (57)$$

where $b_{\mathrm{M}} = 8\pi G \varepsilon_{\mathrm{M0}} a_0^3 / 3 = (a_0/a_{\mathrm{M0}})^2 a_0$ is the maximum curvature radius of the matter-universe. The initial condition $a(0) = 0$ has been assumed. The cosmic expansion acceleration is negatively constant in the matter-universe ($\ddot{a} = -1/2b_{\mathrm{M}}$). Due to the negative acceleration, the expanding universe containing only radiation and matter is bound to contract into the single point.

If the universe does not contain the ordinary matter and radiation but is dominated by the cosmological constant or dark energy (Λ-universe), Eq. (39) becomes

$$\frac{da}{dt} = \pm \sqrt{1 - \frac{3}{\Lambda a^2}} = \pm \sqrt{1 - \left(\frac{a_\Lambda}{a}\right)^2}, \qquad (58)$$

where $a_\Lambda \equiv (3/\Lambda)^{1/2} = (3/8\pi G\varepsilon_\Lambda)^{1/2}$ is the (minimum) radius of the Λ-universe at initial time t_i. For the expanding Λ-universe, we get

$$a(t) = [a_\Lambda^2 + (t - t_i)^2]^{1/2} \quad (t \ge t_i). \qquad (59)$$

The expansion speed and acceleration of the Λ-universe are $\dot{a}(t) = (t - t_i)/[a_\Lambda^2 + (t - t_i)^2]^{1/2}$ and $\ddot{a}(t) = a_\Lambda^2/[a_\Lambda^2 + (t - t_i)^2]^{3/2}$, which go over asymptotically into unity and zero, respectively, as t goes to infinity. Starting with $a(t_i) = a_\Lambda$, $\dot{a}(t_i) = 0$, and $\ddot{a}(t_i) = a_\Lambda^{-1}$, the Λ-universe expands eternally.

It is interesting to consider a universe dominated by energy of a hypothetical species with an equation of state $w_{\mathrm{H}} = -1/3$ (Kolb 1989). This universe (H-universe) has a simple expansion history

$$a(t) = t[1 - (a_{\mathrm{H0}}/a_0)^2]^{1/2} \quad (t \ge 0), \qquad (60)$$

where $a_{\mathrm{H0}} = (3/8\pi G\varepsilon_{\mathrm{H0}})^{1/2}$. The important property of the H-universe is that $a \propto t$ and $\ddot{a} = 0$. The universe expands with the constant speed. The energy density of the hypothetical species evolves as $\varepsilon_{\mathrm{H}} \propto a^{-2}$. For an extreme case of $a_{\mathrm{H0}} = 0$

($\varepsilon_{\mathrm{H0}} = \infty$), the H-universe expands with the limiting speed ($a = t$).

To reconstruct the evolution history of the closed universe, we have adopted two world models that are consistent with the recent astronomical observations. The model parameters, which are listed in Table 2, are based on a non-flat ΛCDM FRW world model that best fits with the WMAP CMB data only (Model I; $H_0 = 55$ km s^{-1}Mpc^{-1}, $\Omega_{\mathrm{M}} = 0.415$, $\Omega_\Lambda = 0.630$; Spergel et al. 2007, §3.3) and on another model that jointly fits with the CMB and other astronomical data (Model II; $H_0 = 71$ km s^{-1}Mpc^{-1}, $\Omega_{\mathrm{M}} = 0.315$, $\Omega_\Lambda = 0.696$; Wright 2007), where $H_0 = 100h$ km s^{-1}Mpc^{-1} is the Hubble constant, and Ω_I is the current density parameter of the FRW world model. The Hubble constant of Model I is quite lower than the popular value of Model II, but is allowed because the low Hubble constant has been reported from observations of Cepheids plus SNIa ($H_0 = 62.3 \pm 1.3 \pm 5.0$ km s^{-1}Mpc^{-1}; Sandage et al. 2006) and of eclipsing binaries ($H_0 = 61$ km s^{-1}Mpc^{-1}; Bonanos et al. 2006).[8]

Using the FRW model parameters as input, we calculate radius parameters a_{R0}, a_{M0}, and a_Λ from a formula $a_{I0} = (3/8\pi G\varepsilon_{I0})^{1/2} = H_0^{-1}\Omega_I^{-1/2}$. The radiation energy density has been calculated from $\varepsilon_{\mathrm{R0}} = \pi^2 k_{\mathrm{B}}^4 T_{\mathrm{cmb}}^4 / 15\hbar^3$ [see Sec. 4(f)] with the CMB temperature $T_{\mathrm{cmb}} = 2.725$ K (Mather et al. 1999). For simplicity, the neutrino contribution to the radiation energy density has been omitted. The present curvature radius a_0 has been obtained from the relation

$$H_0 = a_0^{-1}(da/d\tau)_0 = (a_{\mathrm{R0}}^{-2} + a_{\mathrm{M0}}^{-2} + a_\Lambda^{-2} - a_0^{-2})^{1/2}. \qquad (61)$$

The basic parameters characterizing the closed world model are the curvature radius of the universe (a_0) and the radius parameters (a_{R0}, a_{M0}, a_Λ) at the

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[8] The cosmological parameters of Model I are very similar to the recent values of the non-flat ΛCDM model constrained with Planck 2018 CMB data alone (omegak_plikHM_TTTEEE_lowl_lowE; Planck Collaboration 2018b): $H_0 = 54.4^{+3.3}_{-4.0}$ km s^{-1}Mpc^{-1}, $\Omega_{\mathrm{M}} = 0.485^{+0.058}_{-0.068}$, $\Omega_\Lambda = 0.560^{+0.050}_{-0.043}$ (68% confidence limits).
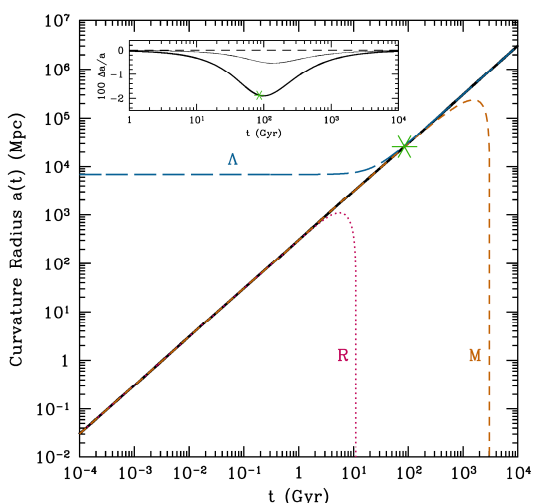
Fig. 3. Evolution of curvature radius a over the world time t (thick solid curve; Model I). For comparison, the curvature radius of radiation, matter, and Λ-universes are shown as dotted (R), dashed (M), and long-dashed (Λ) curves, respectively. The small panel shows the fractional difference of the curvature radius of our universe relative to $a=t$ ($\Delta a/a \equiv [a-t]/t$ in unit of percent; thick and thin solid curves for Model I and II, respectively). The stars denote quantities at the present time $t_0 = 85.3$ Gyr [Model I; see Sec. 4(g)]. A rough estimate of redshift is given by $z \approx t_0/t - 1$.

present time.

It is interesting to note that converting parameters of the flat FRW world model into those of the new closed model always gives limiting values of $a_0 = \infty$, $\dot{a}_0 = 1$, and $\ddot{a}_0 = 0$. The flat FRW world is a limiting case of the closed universe with infinite curvature radius. In such a limit, the proper time becomes frozen ($d\tau = 0$).

The evolution histories of the two closed world models are summarized in Figs. 3-6 below, where we have also plotted analytic solutions for radiation, matter, and Λ-universes ($t_i = 0$ is assumed for Λ-u). All the numerical values given in the text are based on Model I. The evolution of curvature radius of the closed world model is shown in Fig. 3, where we have performed a numerical integration of Eq. (39) to obtain $a(t)$. Note that the solution of H-universe with infinite energy density greatly approximates the evolution of curvature radius of our universe, which
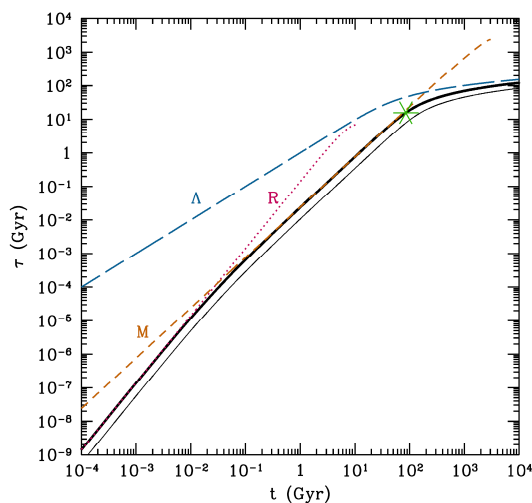

Fig. 4. The proper time of a comoving observer τ versus the world time t (thick and thin solid curves for Model I and II, respectively). The τ-t relations expected in radiation, matter, and Λ-universes are shown as dotted (R), dashed (M), and long-dashed (Λ) curves, respectively. The star indicates $t_0 = 85.3$ Gyr and $\tau_0 = 15.8$ Gyr.

differs from $a=t$ by maximally about 2% at $t \simeq 100$ Gyr, as shown in the small panel.

Figure 4 shows the relation between the proper time of a comoving observer and the world time, obtained by integrating Eq. (52). The total elapsed time until the present time, the age of the universe, is denoted as a star at $t_0 = 85.3$ Gyr and $\tau_0 = 15.8$ Gyr [see Sec. 4(g)]. The τ-t relations expected in radiation, matter, and Λ-universes are written in analytic forms as

$$\tau = b_R[1 - \cos(t/b_R)] \quad \text{(R-u)}, \tag{62}$$

$$\tau = b_M\left[\left(\frac{t-2b_M}{2b_M}\right)\sqrt{1 - \left(\frac{t-2b_M}{2b_M}\right)^2} + \frac{\pi}{2} + \arcsin\left(\frac{t-2b_M}{2b_M}\right)\right] \quad \text{(M-u)}, \tag{63}$$

and

$$\tau = a_\Lambda \ln\left[\frac{t-t_i}{a_\Lambda} + \sqrt{1 + \left(\frac{t-t_i}{a_\Lambda}\right)^2}\right] \quad (\Lambda\text{-u}), \tag{64}$$

respectively. Inserting Eq. (62) into (56) gives $a(\tau) = [\tau(2b_R - \tau)]^{1/2}$, which goes over into $a \propto \tau^{1/2}$ if $\tau \ll 2b_R$, the behavior of a scale factor in the radiation-dominated FRW universe.
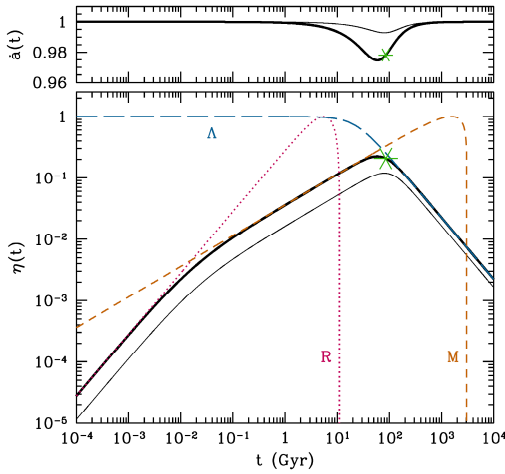
Fig. 5. Variation of (top) cosmic expansion speed $\dot{a}$ and (bottom) speed of light η over the world time t (in unit of the limiting speed of the special relativity; thick and thin solid curves for Model I and II, respectively). The relation $\eta^2 + \dot{a}^2 = 1$ holds. The speed of light in radiation, matter, and Λ-universes are shown as dotted (R), dashed (M), and long-dashed (Λ) curves. The stars denote quantities at the present time.

Figure 5 shows the time-variation of cosmic expansion speed (top) and speed of light (bottom). At the present time, the universe is expanding faster than the light by a factor of 4.7 with $\dot{a} = 0.978$ and $\eta = 0.208$ (denoted as stars). The speed of light in radiation, matter, and Λ-universes are written as

$$\eta(t) = \sin(t/b_R) = a/b_R \quad \text{(R-u)}, \tag{65}$$

$$\eta(t) = \sqrt{1 - \left(\frac{t - 2b_M}{2b_M}\right)^2} = (a/b_M)^{1/2} \quad \text{(M-u)}, \tag{66}$$

and

$$\eta(t) = \frac{a_\Lambda}{[a_\Lambda^2 + (t - t_i)^2]^{1/2}} = a_\Lambda/a \quad \text{(Λ-u)}, \tag{67}$$

respectively. The behavior of time-varying light speed implies that photons are frozen ($\eta = 0$) when the universe expands with the maximum speed, e.g., at the beginning or far in the future of the universe.

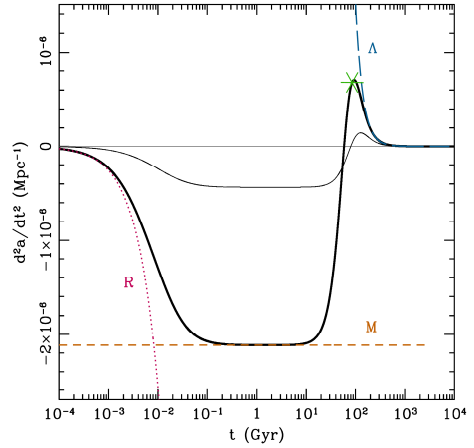Figure 6 shows the history of cosmic expansion acceleration calculated from

Fig. 6. Evolution of cosmic expansion acceleration $\ddot{a}$ in unit of Mpc^{-1} (thick and thin solid curves for Model I and II, respectively). The acceleration in radiation, matter, and Λ-universes are shown as dotted (R), dashed (M), and long-dashed (Λ) curves. The star denotes the acceleration at the present time.

$$\ddot{a} = -\frac{3(\varepsilon + 3P)}{16\pi G \varepsilon^2 a^3}, \tag{68}$$

which has been obtained by combining Eqs. (38) and (39). In the radiation-dominated era, the expansion acceleration had gradually decreased from zero, and became negatively constant during the matter-dominated era ($t = 0.1$-10 Gyr). Afterward, the universe has been decelerated until $t = 58.8$ Gyr ($\tau = 10.0$ Gyr, $z = 0.448$) from which it starts to accelerate positively. The transition epoch corresponds to the point of maximum (minimum) speed of light (expansion speed). The universe arrives at the maximum acceleration at $t = 93.5$ Gyr ($\tau = 17.5$ Gyr, $z = -0.088$). The present universe is on a stage before the maximum acceleration. The future universe will expand eternally with asymptotic acceleration of zero.

(c) Energy-momentum relation of particles

Now we define energy and momentum of a free particle with rest mass m in the closed universe. Here, we mean the rest mass by the intrinsic mass of the particle that is independent of its peculiar motion and the dynamics of the universe. Thus m

is the mass as measured in a locally inertial frame comoving with the particle. It is also equivalent to the rest mass in the stationary universe. The action for the free material particle moving along a trajectory with end points A and B has the form (Landau and Lifshitz 1975, §8)

$$S = -m \int_A^B ds = \int_A^B L \, dt, \tag{69}$$

where the Lagrangian

$$L = -m(1 - \dot{a}^2 - a^2 v^2)^{1/2} \tag{70}$$

goes over into $-m + ma^2v^2/2$ in the limit of $av \ll 1$ and $\dot{a} = 0$. The motion of the particle is determined from the principle of least action, $\delta S = -m\delta \int ds = 0$ (e.g., Landau and Lifshitz 1975, §87), resulting in the geodesic equation

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0. \tag{71}$$

From the Lagrangian, we calculate energy and momentum of the material particle. The 3-momentum of the particle is obtained from $p_\alpha = \partial L / \partial v^\alpha$, with individual components $p_1 = m\gamma v^1 a^2$, $p_2 = m\gamma v^2 a^2 \sin^2\chi$, and $p_3 = m\gamma v^3 a^2 \sin^2\chi \sin^2\theta$. The energy of the particle is given by

$$E_p = p_\alpha v^\alpha - L = \frac{m(1 - \dot{a}^2)}{(1 - \dot{a}^2 - a^2 v^2)^{1/2}}, \tag{72}$$

and the relativistic mass by

$$m_r = E_p / \eta^2 = m\gamma. \tag{73}$$

Both energy and mass of a particle are tightly related to the expansion speed of the universe. For a comoving particle with a fixed comoving coordinate ($v^\alpha = 0$), the energy and the relativistic mass become $E_p = m(1 - \dot{a}^2)^{1/2}$ and $m_r = m(1 - \dot{a}^2)^{-1/2}$, respectively.

Let us define the 4-momentum vector of a particle as

$$p^i = (m\gamma, m\gamma v\mathbf{n}) = \left(\frac{E_p}{1 - \dot{a}^2}, \frac{p}{a}\mathbf{n} \right), \tag{74}$$

where p is the magnitude of the proper momentum defined as $p = (-p^\alpha p_\alpha)^{1/2} = m\gamma a v$ and $\mathbf{n}$ is a unit vector indicating the direction of motion of the particle. According to this definition, the particle's energy is the time component of the covariant 4-momentum $p_k = mu_k = (E_p, -ap\mathbf{n})$. From the square of the 4-momentum

$$p^i p_i = m^2, \tag{75}$$

we obtain an energy-momentum relation

$$E_p^2 = (p^2 + m^2)(1 - \dot{a}^2), \tag{76}$$

which goes over into $E_p^2 = p^2 + m^2$ in the limit of $\dot{a} = 0$. One important expectation from Eq. (76) is that the energy of a material particle vanishes when $\dot{a} = 1$, e.g., at the beginning of the universe.

The equation of motion of a particle with small peculiar velocity ($v^\alpha \ll 1$) is obtained from the space component of Eq. (71) as

$$\frac{dv^\alpha}{dt} + \left(\frac{\dddot{a}\dot{a}}{1 - \dot{a}^2} + 2\frac{\dot{a}}{a} \right) v^\alpha = 0, \tag{77}$$

where any quadratic of v^α has been dropped. The solution to this equation

$$av^\alpha \propto \frac{\sqrt{1 - \dot{a}^2}}{a} \tag{78}$$

shows how the proper peculiar velocity of a particle evolves as a result of the cosmic expansion.

From Eqs. (74) and (76), the energy and the 4-momentum of a massless photon are

$$E_\gamma = p_\gamma (1 - \dot{a}^2)^{1/2}, \tag{79}$$

and

$$p^i = \left(\frac{p_\gamma}{\sqrt{1 - \dot{a}^2}}, \frac{p_\gamma}{a}\mathbf{n} \right), \tag{80}$$

where p_γ is the photon's proper spatial momentum. The photon's energy and spatial momentum are

usually expressed as photon's frequency and inverse wavelength multiplied by the Planck constant ($E_\gamma = h\nu$ and $p_\gamma = h/\lambda$). Therefore, Eq. (79) is equivalent to

$$\nu\lambda = \eta = (1 - \dot{a}^2)^{1/2}. \tag{81}$$

A comoving observer measures frequency and wavelength of the same photon as $\nu_c = \nu(1 - \dot{a}^2)^{-1/2}$ and $\lambda_c = \lambda$. Thus, $\nu_c\lambda_c = 1$ in the locally inertial frame. The subscript c denotes a quantity measured by the comoving observer.

(d) Doppler shift and cosmological redshift of photons

The stretch of photon's wavelength is induced by the receding motion of an observer from a light source (Doppler shift), or by the cosmic expansion (cosmological redshift).

First, let us consider the Doppler shift. For simplicity, the cosmic expansion speed is assumed to be fixed. Suppose that an observer with 4-velocity $u^i = (\gamma, \gamma v \mathbf{n}_1)$ is moving away from a light source emitting photons with 4-momentum $p^i = (p_{em}/\sqrt{1 - \dot{a}^2}, p_{em}\mathbf{n}_2/a)$, and is receiving photons from the source. The observed momentum of a photon is given by the inner product of u^i and p^i:

$$p_{ob} = p^i u_i = \frac{\sqrt{1 - \dot{a}^2} - av\cos\theta_{12}}{\sqrt{1 - \dot{a}^2 - a^2 v^2}} p_{em}, \tag{82}$$

where p_{em} and p_{ob} are the proper spatial momenta of emitted and observed photons, respectively, and $\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos\theta_{12}$ ($\theta_{12} = 0$ for receding and $\theta_{12} = \pi$ for approaching observers). The ratio of momenta (or energies) of observed to emitted photons for the longitudinal Doppler effect ($\theta_{12} = 0$) is

$$\frac{p_{ob}}{p_{em}} = \frac{E_{ob}}{E_{em}} = \sqrt{\frac{c_\chi - v}{c_\chi + v}}, \tag{83}$$

where $c_\chi \equiv \eta/a$ is the light speed in the comoving coordinate system. The ratio for the transversal Doppler effect ($\theta_{12} = \pi/2$) is

$$\frac{p_{ob}}{p_{em}} = \frac{E_{ob}}{E_{em}} = \frac{1}{\sqrt{1 - v^2/c_\chi^2}}. \tag{84}$$

The two formulas for the Doppler effect are similar in form to those in the special relativity.

Next, for the cosmological redshift, let us suppose that photons, emitted at world time t from a light source at comoving coordinate χ, have arrived at the origin at t_0. Using the photon's geodesic equation and assuming that photons have traveled radially by the symmetry of space, we get

$$\int_0^\chi d\chi' = -\int_{t_0}^t \frac{\sqrt{1 - \dot{a}^2}}{a} dt'$$
$$= \int_t^{t_0} c_\chi(t') dt' = \int_{t+\delta t}^{t_0 + \delta t_0} c_\chi(t') dt', \tag{85}$$

where the minus sign in front of the second integral indicates that photons have propagated from the distant source to the origin. The δt and δt_0 are the world time intervals during which a photon's wave crest propagates by the amount of its wavelength at the points of emission and observation, respectively (i.e., $\nu = \delta t^{-1}$). The third equality holds because the integral does not change after the infinitesimal time intervals δt and δt_0.

Manipulating Eq. (85) gives the frequency ratio of emitted to observed photons,

$$\frac{\nu}{\nu_0} = \frac{\delta t_0}{\delta t} = \frac{c_\chi(t)}{c_\chi(t_0)}$$
$$= \frac{a_0}{a}\left(\frac{1 - \dot{a}^2}{1 - \dot{a}_0^2}\right)^{1/2} = (1 + z)^2 \frac{A(0)}{A(z)} \equiv r(z). \tag{86}$$

The cosmological time dilation function $r(z)$ is useful for comparing physical quantities at past and present epochs. The variation of $r(z)$ is shown in Fig. 7. The corresponding function in the FRW world model, $1 + z$, has a similar value to $r(z)$ only at low redshift ($z \lesssim 1$), going to infinity at infinite redshift. The $r(z)$ is almost constant during the radiation-dominated era with the maximum value of

$$r(\infty) = [(\varepsilon_{R0} + \varepsilon_{M0} + \varepsilon_\Lambda)/\varepsilon_{R0}]^{1/2} = 113. \tag{87}$$

From Eqs. (81) and (86), the photon's frequency and wavelength vary as

$$\nu \propto \frac{\sqrt{1-\dot{a}^2}}{a} \quad \text{and} \quad \lambda \propto a. \tag{88}$$

Because the wavelength of a photon increases in proportional to a, the redshift is equal to the fractional difference between wavelengths at the points of observation and emission of the photon: $z = a_0/a - 1 = (\lambda_{\text{ob}} - \lambda_{\text{em}})/\lambda_{\text{em}}$. Besides, the photon's energy and spatial momentum vary as

$$E_\gamma \propto \frac{\sqrt{1-\dot{a}^2}}{a} \quad \text{and} \quad p_\gamma \propto \frac{1}{a}. \tag{89}$$

From Eqs. (56), (57), and (59), one finds that $E_\gamma = \text{const.}$ (R-u), $E_\gamma \propto a^{-1/2}$ (M-u), and $E_\gamma \propto a^{-2}$ (Λ-u). In the comoving observer's frame, both photon's energy and spatial momentum always vary as a^{-1}.

(e) Total energy in the Universe

The conservation of energy in the classical physics is closely related to the invariance of physical laws under a time-translation (Noether's theorem), which applies to the physics in the Minkowski space-time. In general relativity there is not necessarily a time coordinate with the translation-symmetry, so the conservation of energy is not generally expected. However, in an asymptotically flat region or in a locally inertial frame, it is possible to define the conserved energy. For this reason, it has been usually said that there is not a global but a local energy conservation law.

Let us estimate the total energy in the universe based on the definition of energy in Sec. 4(c). First, we need to define the volume element. The 4-dimensional volume element is given by

$$dV_4 = \sqrt{-g}\, dx^0 dx^1 dx^2 dx^3 = \sqrt{-g}\, dt\, d\chi\, d\theta\, d\phi, \tag{90}$$

where g is the determinant of the metric tensor g_{ik}. We obtain the 4-volume of the universe from $V_4(t) = \int dV_4 = 2\pi^2 \int_0^t (1-\dot{a}^2)^{1/2} a^3 dt'$. The proper
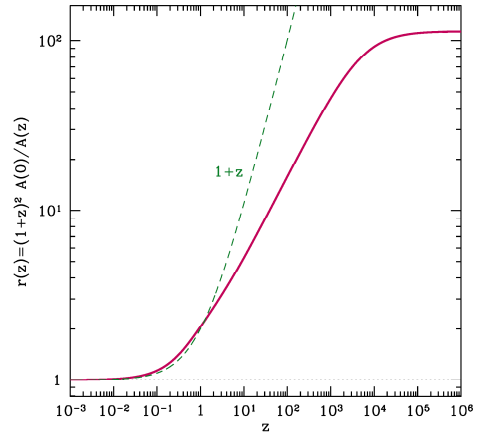

Fig. 7. Variation of $r(z)$ along with redshift (thick solid curve; Model I). The upper bound value of $r(z)$ at infinite redshift is 113. For comparison, the cosmological time dilation factor $1+z$ in the FRW world model is shown as a dashed curve.

3-volume of the universe is the time-derivative of V_4:

$$V(t) = \frac{dV_4}{dt} = 2\pi^2 (1-\dot{a}^2)^{1/2} a^3 = \frac{2\pi^2 a_0^3}{(1+z)^2 A(z)}. \tag{91}$$

The factor $(1-\dot{a}^2)^{1/2}$ appears as a natural contraction effect due to the expansion of the universe. At the present time, $V_0 = 6.94 \times 10^{13}\, \text{Mpc}^3$. One can verify that the 3-volume of the universe evolves as $V \propto a^4$ (R-u), $V \propto a^{7/2}$ (M-u), and $V \propto a^2$ (Λ-u). Note that $V_c = dV_4/d\tau \propto a^3$ in the comoving observer's frame and thus $V = (1-\dot{a}^2)^{1/2} V_c$.

If there are N comoving particles with rest mass m in the universe, then the matter energy density is

$$\varepsilon_{\text{M}} = \frac{NE_{\text{p}}}{V} = \frac{Nm(1-\dot{a}^2)^{1/2}}{(1-\dot{a}^2)^{1/2} V_c} = \frac{Nm}{V_c} = \varepsilon_{\text{Mc}}. \tag{92}$$

Therefore, matter energy densities both in the world reference and the comoving observer's frames are equivalent to each other, which also applies to the radiation energy density if m is replaced with p_γ in Eq. (92). The matter energy density is related to the matter density ρ_{M} by $\varepsilon_{\text{M}} = \rho_{\text{M}} \eta^2$ because $\rho_{\text{M}} = Nm_{\text{r}}/V = Nm/V_c(1-\dot{a}^2) = \varepsilon_{\text{M}}/(1-\dot{a}^2)$.

The total energy of each species I is calculated

from

$$E_I = \varepsilon_I V = 2\pi^2 a_0^3 \varepsilon_{I0} \frac{(1+z)^{1+3w_I}}{A(z)}. \qquad (93)$$

The present values of total radiation, matter, and dark energies (E_{R0}, E_{M0}, $E_{\Lambda0}$) are listed in Table 2. In radiation, matter, and Λ-universes, the total energy evolves as $E_R = \text{const.}$, $E_M \propto a^{1/2}$, and $E_\Lambda \propto a^2$, respectively. On the other hand, $E_{Rc} \propto a^{-1}$, $E_{Mc} = \text{const.}$, and $E_{\Lambda c} = a^3$ in the comoving observer's frame, implying that the radiation energy is infinite at the initial time, and the matter energy is always constant.

Figure 8 shows the variation of total radiation, matter, and dark energies during the history of the universe (thick solid curves), together with that of energy fraction parameters (D_I; thin solid curves). The total radiation energy remained constant in the radiation-dominated era ($t \lesssim 10^{-3}$ Gyr), and thereafter has decreased. The total matter energy was zero at $t = 0$, arrived at the maximum value at $t = 58.8$ Gyr ($\tau = 10.0$ Gyr), and then has decreased, going back to zero far in the future. The total dark energy has continuously risen up from zero as the universe expands. We can estimate the total rest mass in the universe by transforming the total matter energy into $M = E_{M0}(1 - \dot{a}_0^2)^{-1/2} \approx 10^{25} M_\odot$, which corresponds to about 10^{14} galaxies with a typical mass of $10^{11} M_\odot$.

From the definition of energy fraction parameters [Eqs. (42) and (43)], one finds that $\varepsilon V c_\chi$ is a conserved quantity such that

$$\varepsilon(t) V(t) c_\chi(t) = \varepsilon_0 V_0 c_{\chi 0} = \text{const.}, \qquad (94)$$

where ε is the total energy density. Thus, the ratio of present to past total energies in the universe is obtained as

$$\frac{E_0}{E(t)} = \frac{\varepsilon_0 V_0}{\varepsilon(t) V(t)} = \frac{c_\chi(t)}{c_\chi(t_0)} = r(z), \qquad (95)$$

where $E = \sum_I E_I$ (long dashed curve in Fig. 8) and $E_0 = 1.09 \times 10^{79}$ erg. The initial amount of total energy in the universe is $E(0) = E_0/r(\infty) = 9.62 \times 10^{76}$ erg.
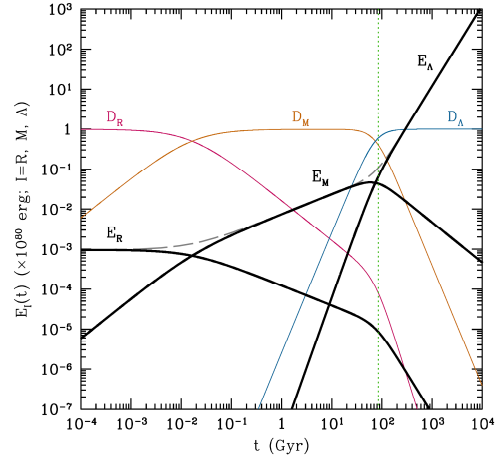

Fig. 8. Variation of total radiation (E_R), matter (E_M), and dark (E_Λ) energies in unit of 10^{80} erg over the world time t (thick solid curves; Model I), together with that of total energy $E = E_R + E_M + E_\Lambda$ (long dashed curve). Thin solid curves represent the variation of the corresponding energy fraction parameters (D_R, D_M, D_Λ; $D_R + D_M + D_\Lambda = 1$) with a dimensionless unit. The vertical dotted line indicates t_0.

Since only the radiation contributes to the total energy at $t = 0$, the same value is obtained from $E(0) = E_R(0) = E_{R0} r(\infty)$ with the help of $E_R(t) = E_{R0} r(z)$ deduced from Eq. (86).

According to the definition of energy in this paper, the total energy is not conserved in the expanding closed universe, but increases with time. Especially, the total energy is finite at the beginning of the universe.

(f) Energy density, pressure, and temperature in thermal equilibrium

We describe the evolution of energy density, pressure, and temperature of gas in the early universe (see Kolb and Turner 1990 for details). Our discussion is restricted to the relativistic gas particles in thermal equilibrium. The particles are assumed to have low rest mass compared to their kinetic energy.

The number of particles of species I per unit spatial volume dV per unit momentum volume dW can be expressed as

$$dN_I = \frac{g_I f_I(x^i, p_i)}{(2\pi\hbar)^3} dV dW, \qquad (96)$$

where g_I is the spin-degeneracy of the particle, f_I is the particle distribution function for species I, equivalent to the mean number of particles occupying a given quantum state, and $\hbar = h/2\pi$. Assuming zero chemical potential, we can write $f_I = [\exp(E_p/k_B T_I) \pm 1]^{-1}$, where plus and minus signs are for fermions (f) and bosons (b), respectively, k_B is the Boltzmann constant, and T_I is the thermodynamic temperature of species I. The proper spatial and momentum volume elements in the world reference frame are given by

$$dV = \sqrt{-g}\, d\chi\, d\theta\, d\phi = (1-\dot{a}^2)^{1/2} dV_c,$$
$$dW = \frac{1}{\sqrt{-g}} dp_\chi\, dp_\theta\, dp_\phi = (1-\dot{a}^2)^{-1/2} dW_c, \qquad (97)$$

where dV_c and dW_c are proper volume elements in the comoving observer's frame. Note that $dVdW = dV_c dW_c$. The momentum volume element is written as $dW = (1-\dot{a}^2)^{-1/2} 4\pi p^2 dp$ for isotropic gas particles with proper momentum p.

The energy density of the relativistic gas is obtained by integrating over the momentum space the particle's energy multiplied with its distribution function:

$$\varepsilon_I = \frac{g_I}{(2\pi\hbar)^3} \int E_p f_I dW = \frac{g_I}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^3 dp}{e^{E_p/k_B T_I} \pm 1}$$
$$= \frac{4\pi g_I k_B^4 T_I^4}{(2\pi\hbar)^3 (1-\dot{a}^2)^2} \int_0^\infty \frac{x^3}{e^x \pm 1} dx \qquad (98)$$
$$= \frac{7\pi^2 g_I k_B^4 T_I^4}{240 \hbar^3 (1-\dot{a}^2)^2} \text{ (f)} \quad \text{and} \quad \frac{\pi^2 g_I k_B^4 T_I^4}{30 \hbar^3 (1-\dot{a}^2)^2} \text{ (b)},$$

where $x \equiv E_p/k_B T_I$ and $E_p \simeq p(1-\dot{a}^2)^{1/2}$ for the relativistic gas.

The pressure of the relativistic gas is obtained in a similar way:

$$P_I = \frac{g_I}{(2\pi\hbar)^3} \int p_{1d} v_{1d} f_I dW$$
$$= \frac{g_I}{3(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^3 dp}{e^{E_p/k_B T_I} \pm 1} = \frac{\varepsilon_I}{3}, \qquad (99)$$

where p_{1d} and v_{1d} are proper momentum and velocity in one direction: $v_{1d} = (p_{1d}/E_p)(1-\dot{a}^2)$ and $p_{1d}^2 = p^2/3$ for isotropic gas. The relativistic gas acts like radiation, with an equation of state $w_I = 1/3$ and the energy density varying as $\varepsilon_I \propto a^{-4}$. Therefore, from Eq. (99) the thermodynamic temperature of the relativistic gas evolves as

$$T_I \propto \frac{\sqrt{1-\dot{a}^2}}{a}. \qquad (100)$$

In the comoving observer's frame, $T_{Ic} \propto a^{-1}$. From a formula of the entropy density $\sigma_I = (\varepsilon_I + P_I)/k_B T_I$, one can verifies that the total entropy $S_I = \sigma_I V$ of the relativistic gas is constant.

For photons, the quantity $x = h\nu/k_B T_R$ is invariant during the cosmic expansion history because the photon's frequency varies in the same way as the temperature does. Since x is also frame-independent ($x = x_c$), we have $T_R = T_{Rc}(\nu/\nu_c) = T_{Rc}(1-\dot{a}^2)^{1/2}$. The present CMB temperature in the world reference frame is $T_{R0} = T_{Rc0}(1-\dot{a}_0^2)^{1/2} = 0.57$ K ($T_{Rc0} = T_{cmb}$). From Eq. (100), the ratio of past to present radiation temperatures is

$$\frac{T_R}{T_{R0}} = \frac{a_0}{a}\left(\frac{1-\dot{a}^2}{1-\dot{a}_0^2}\right)^{1/2} = r(z), \qquad (101)$$

which enables us to estimate the radiation temperature at the past epoch. For example, at the beginning of the universe $T_R(0) = T_{R0} r(\infty) = 64.0$ K, while it is infinite in the comoving observer's frame. The behavior of $r(z)$ implies that $T_R = $ const. in the radiation-universe.

The epoch of radiation-matter equality is determined from the condition $\varepsilon_M = \varepsilon_R$:

$$1 + z_{eq} = \frac{\varepsilon_{M0}}{\varepsilon_{R0}} = \left(\frac{a_{R0}}{a_{M0}}\right)^2 = 5081. \qquad (102)$$

At this epoch ($t_{eq} = 1.65 \times 10^{-2}$ Gyr, $\tau_{eq} = 29700$ yr), the size of the universe was $a_{eq} = 5.05$ Mpc and the radiation temperature was $T_{R,eq} = T_{R0} r(z_{eq}) = 45.3$ K,

or $T_{\mathrm{Rc,eq}} = T_{\mathrm{Rc0}}(1+z_{\mathrm{eq}}) = 13850 \text{ K}.$

To summarize, as judged in the world reference frame, the early universe was cold and all the physical processes in it were extremely slow. Especially, the universe started from a regular (non-singular) point in the sense that physical quantities have finite values at the initial time. The singular nature of the FRW universe comes from the fact that the flow of the proper time was frozen ($d\tau = 0$) at $t = 0$.

(g) Cosmic distance and time scales

Lastly, we consider cosmic distance and time scales in the closed world model. As the most popular distance measure, the coordinate distance (d_{C}) to a galaxy at redshift z is obtained by integrating the photon's geodesic equation ($d\chi = dt\sqrt{1-\dot{a}^2}/a$),

$$d_{\mathrm{C}}(z) = a_0 \chi_{\mathrm{C}}(z) = a_0 \int_t^{t_0} \frac{\sqrt{1-\dot{a}^2}}{a} dt'$$
$$= \int_0^z \frac{a_0 dz'}{[A^2(z') - (1+z')^2]^{1/2}}. \tag{103}$$

Here $\chi_{\mathrm{C}}(z)$ is the comoving coordinate distance. For sufficiently large a_0, Eq. (103) goes over into $d_{\mathrm{C}}(z) \approx \int_0^z a_0 dz'/A(z')$, which is equivalent to the coordinate distance in the flat FRW world model. The coordinate distance in the matter-universe has an analytic form

$$d_{\mathrm{C}}(z) = 2a_0 \arctan\left[\left(\frac{a_0}{a_{\mathrm{M0}}}\right)^2 (1+z) - 1\right]^{1/2}$$
$$- 2a_0 \arctan\left[\left(\frac{a_0}{a_{\mathrm{M0}}}\right)^2 - 1\right]^{1/2} \quad (\text{M-u}). \tag{104}$$

We can derive other astronomical distances based on luminosity and angular size of distant sources. For the luminosity distance, let us imagine that a light source at redshift z has intrinsic bolometric luminosity L_{c} as measured at the source. Since both photon's energy and arrival rate vary in proportion

to a^{-1} in the locally inertial frame, the flux of the light source as measured by the present comoving observer can be written as

$$\frac{L_{\mathrm{c}}}{4\pi d_{\mathrm{L}}^2} = \frac{L_{\mathrm{c}}}{4\pi a_0^2 \sin^2 \chi_{\mathrm{C}}} \left(\frac{a}{a_0}\right)^2, \tag{105}$$

where d_{L} is the luminosity distance to the source[9],

$$d_{\mathrm{L}}(z) = a_0(1+z)\sin\chi_{\mathrm{C}}(z). \tag{106}$$

The angular size distance (d_{A}) to a galaxy with physical size r_{g} and angular size θ_{g} is given by

$$d_{\mathrm{A}}(z) = \frac{r_{\mathrm{g}}}{\theta_{\mathrm{g}}} = \frac{\sqrt{g_{22}}\,\theta_{\mathrm{g}}}{\theta_{\mathrm{g}}} = \frac{a_0 \sin\chi_{\mathrm{C}}(z)}{1+z}. \tag{107}$$

The recession velocity ($v_{\mathrm{rec}} = z$) of a galaxy has also been used as a distance measure in the local universe. By integrating $dz = -da(a_0/a^2)$ from the definition of redshift, we get

$$z = \int_t^{t_0} \left(\frac{a_0}{a}\right)\left(\frac{\dot{a}}{a}\right) dt' = a_0 \int_t^{t_0} H(\tau) \frac{\sqrt{1-\dot{a}^2}}{a} dt', \tag{108}$$

where $H(\tau) = a^{-1}(da/d\tau) = a_0^{-1}[A^2(z) - (1+z)^2]^{1/2}$ is the Hubble parameter. Since the Hubble parameter remains almost constant during the recent epoch ($z \lesssim 0.1$), the recession velocity is related to the coordinate distance by $v_{\mathrm{rec}} \approx H_0 d_{\mathrm{C}}(z)$, which is the Hubble's law.

The age of the universe or the lookback time have been used as a measure of cosmic time scales. The age of the universe measured in world time is calculated from

$$t(z) = \int_0^t dt' = \int_z^\infty \frac{a_0 dz'}{(1+z')^2} \sqrt{\frac{A^2(z')}{A^2(z') - (1+z')^2}}, \tag{109}$$

while the age measured in proper time of a comoving observer (us) is obtained by integrating Eq. (52):

$$\tau(z) = \int_0^t \sqrt{1-\dot{a}^2}\, dt'$$
$$= \int_z^\infty \frac{a_0 dz'}{(1+z')} \frac{1}{[A^2(z') - (1+z')^2]^{1/2}}. \tag{110}$$

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[9] If the flux of the source is measured in the world reference frame, the luminosity distance becomes $d_{\mathrm{L}}(z) = a_0 r(z) \sin\chi_{\mathrm{C}}(z)$.

The latter is equivalent to the age of the FRW universe. Note that $t \geq \tau$ (see Fig. 4 for τ-t relation). At the present time, $t_0 = 85.3$ Gyr and $\tau_0 = 15.8$ Gyr. The lookback time, the time measured back from the present to the past, is given by $t_0 - t(z)$ or $\tau_0 - \tau(z)$.

Figure 9 compares coordinate, luminosity, angular size distances (top) and cosmic ages (bottom) as a function of redshift in the closed world model with Model I parameters of Table 2. Also shown are the corresponding distances and age for the flat FRW world model that best fits with the WMAP CMB data ($\Omega_M h^2 = 0.1277$ and $h = 0.732$; Spergel et al. 2007). Distances of both world models agree with each other at high redshift ($z \gtrsim 10$), but the flat FRW world model underestimates distances to nearby galaxies than the closed world model.

5. Inflation

The FRW world model has been criticized because of two shortcomings, namely, flatness and horizon problems. As shown in Sec. 3, the universe with positive energy density is always spatially closed. If the present universe is traced back to the past, it would become a more curved hypersurface, a 3-sphere with smaller curvature radius. Therefore, the closed world model proposed in this paper is free from the flatness problem. Next, let us consider the horizon problem, which is stated as follows. The observed CMB temperature fluctuations separated by more than a degree are similar to each other over the whole sky. In the FRW world model, such an angle corresponds to a distance where the causal contact was impossible on the last scattering surface. The large-scale uniformity of the CMB anisotropy suggests that the observed regions must have been in causal contact in the past.

The inflation paradigm has offered a reasonable solution to the puzzle of the large-scale homogeneity of the observable universe by proposing that there was a period of rapid expansion of the universe with positive acceleration (Guth 1981). According to the
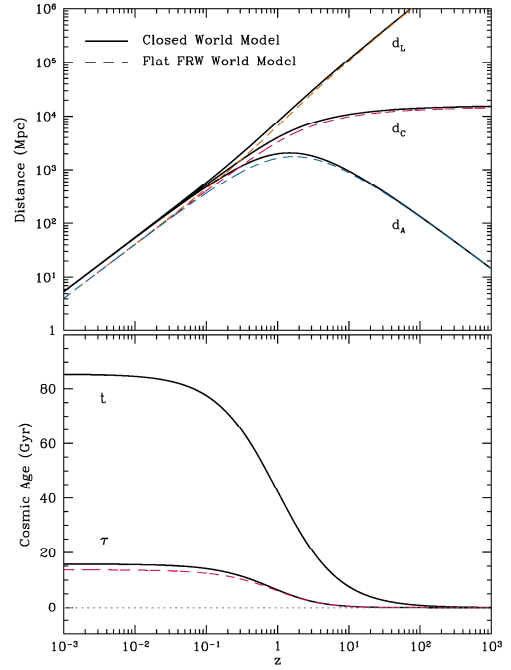

Fig. 9. (top) Coordinate (d_C), luminosity (d_L), and angular size (d_A) distances in the closed world model (solid curves; Model I). The coordinate distance to the big-bang is $d_C(\infty) = 15.7$ Gpc. (bottom) Cosmic ages measured in world time (t; top curve) and in proper time of a comoving observer (τ; bottom curve). The dashed curves are the corresponding distances and age for the flat FRW world model that best fits with the WMAP CMB data ($\Omega_M h^2 = 0.1277$ and $h = 0.732$; Spergel et al. 2007).

inflation theory, the inflation takes place due to the presence of a scalar field ϕ, whose energy density and pressure are given by $\varepsilon_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, and $P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$, respectively, where $V(\phi)$ is a potential of the scalar field. Here we assume that the dot over ϕ is the world-time derivative. From Eq. (38), for a universe dominated by the scalar field, the condition for the positive expansion acceleration is

$$\dot{\phi}^2 < V(\phi). \tag{111}$$

A distance scale of causally connected region (so called horizon size) is usually quantified by the Hubble radius and the particle horizon size. In the

FRW world model, one important implication of the positive expansion acceleration is that the comoving Hubble radius decreases with time, i.e., $d(aH)^{-1}/d\tau < 0$. The comoving Hubble radius is defined as the comoving distance at which the recession velocity as defined in the world reference frame is equal to the speed of light ($\dot{a}\chi_H = \eta$):

$$\chi_H(t) = \frac{\sqrt{1 - \dot{a}^2}}{\dot{a}} = \frac{1}{aH(\tau)} = \frac{1 + z}{[A^2(z) - (1+z)^2]^{1/2}}. \quad (112)$$

The comoving particle horizon size is the comoving distance a photon has travelled during the age of the universe:

$$\chi_P(t) = \int_0^t \frac{\sqrt{1 - \dot{a}^2}}{a} dt' = \int_z^\infty \frac{dz'}{[A^2(z') - (1+z')^2]^{1/2}}. \quad (113)$$

The proper Hubble radius and particle horizon size are given by $d_H(t) = a(t)\chi_H(t) = 1/H(\tau)$ and $d_P(t) = a(t)\chi_P(t)$, respectively.

Figure 10 compares proper (top) and comoving (bottom) horizon sizes as a function of world time. It is important to note that comoving horizon sizes were zeros at the beginning of the universe, and then have increased until the recent epoch. As shown in Sec. 4(b), the universe was expanding with the limiting speed and zero acceleration at the initial time (Figs. 5 and 6). Therefore, the positive expansion acceleration or the decrease in the comoving Hubble radius is not allowed at the early stage of the closed universe. The decrease is only possible in later Λ-dominated universe ($t \gtrsim 100$ Gyr; Fig. 10, bottom).

Actually, the scalar field ϕ is not essential for driving the rapid expansion of the universe. Even if the scalar field is dominant, the condition (111) is not satisfied. We can only expect that $\dot{\phi}^2 = V(\phi)$ from the constraint of zero acceleration, obtaining an equation of state

$$P_\phi = -\frac{\varepsilon_\phi}{3}. \quad (114)$$

Thus the curvature radius increases as $a \propto t$ in the universe dominated by the scalar field. Further
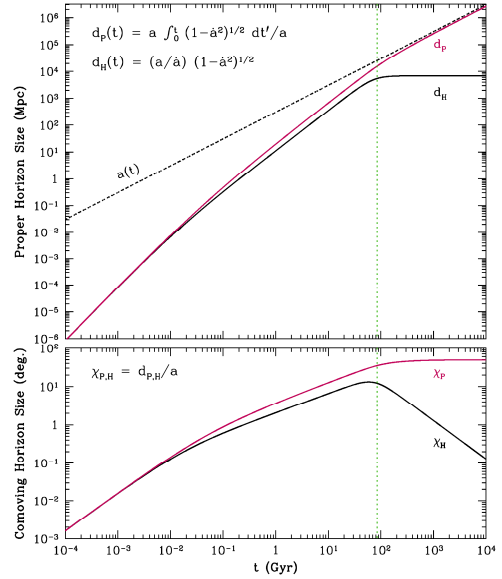

Fig. 10. Time-variation of proper (top) and comoving (bottom) horizon sizes in the closed universe (Model I), namely, the Hubble radius (d_H, χ_H) and the particle horizon size (d_P, χ_P). For comparison, the evolution of curvature radius $a(t)$ is shown as dashed curve in the top panel. The proper Hubble radius has an asymptotic maximum $d_H(\infty) = a_\Lambda = 6867$ Mpc. The comoving horizon size is shown in unit of degree, with an asymptotic maximum $\chi_P(\infty) = 49.4°$. The vertical dotted lines indicate the present epoch.

constraining the universe to expand with the limiting speed demands that the energy density of the scalar field should be infinite, as in the extreme case of H-universe [Sec. 4(b)]. However, the radiation-universe with finite total energy provides a far simpler expansion history $a \simeq t$ for $t \ll b_R$ [Eq. (56)], which demonstrates the sufficiency of the radiation in driving the rapid expansion and the needlessness of the scalar field. In conclusion, it is improbable that the inflation with positive acceleration occurred in the early universe.

If the universe expands with the limiting speed, the peculiar velocity of a particle vanishes as implied by Eq. (34): the matter and radiation were frozen with zero propagation speed at the earliest epoch. Besides, information at one region could not be easily transferred to other regions due to the

small horizon size. Thus, physical information sharing through the causal contact during the expansion of the universe is not an efficient way to explain for the large-scale homogeneity of the universe. The spherically symmetric and uniform distribution of supernova remnants (e.g., Tycho's supernova 1572; Warren et al. 2005) driven by a strong shock into the ambient interstellar medium shows that the large-scale homogeneity can be generated from the ballistic explosion at the single point, without the causal contact during the expansion. At the beginning of the universe, everything was on the single point so that every information such as temperature and energy density could be shared in full and uniform contact. Therefore, if the initial condition was properly set at the creation of the universe, e.g., by the quantum processes at $t \lesssim t_P$ (Planck time), which is out of the scope of the classical physics, the observed uniformity of density distributions at super-horizon scales may be explained.

6. Conclusion

In this paper, the general world model for homogeneous and isotropic universe has been proposed. By introducing the world reference frame as a global and fiducial system of reference, we have defined the line element so that the effect of cosmic expansion on the physical space-time separation can be correctly included in the metric. With this framework, we have demonstrated theoretically that the flat universe is equivalent to the Minkowski space-time and that the universe with positive energy density is always spatially closed and finite. The open universe is unrealistic because it cannot accommodate positive energy density. Therefore, in the world of ordinary materials, only the spatially closed universe is possible to exist.

The naturalness of the finite world with positive energy density comes from the Mach's principle that the motion of a mass particle depends on the mass distribution of the entire world. The principle is consistent only with the finite world because the dynamics of a reference frame cannot be defined in the infinite, empty world. The closed world model satisfies the Mach's principle and supports Einstein's perspective on the physical universe.

We have reconstructed evolution histories of the closed world models that are consistent with the recent astronomical observations, based on the nearly flat FRW world models (Model I and II; Sec. 4). The present curvature radius of the universe is $a_0 = 25.7$ Gpc ($a_0 = 40.2$ Gpc) for Model I (Model II). The expansion histories of both models imply that the closed universe dominated by dark energy expands eternally. However, the currently favored flat FRW world exists only as a limiting case of the closed universe with infinite curvature radius that is expanding with the maximum speed ($\dot{a}_0 = 1$, $\ddot{a}_0 = 0$).

From the local nature of the FRW metric (Sec. 2) and of the proper time of a comoving observer [Sec. 4(e)], it is clear that the FRW world model describes the local universe as observed by the comoving observer. Since the Newton's gravitation law can be derived from the Einstein's field equations in the weak field and the small velocity limits, the gravitational action at a distance usually holds at a local region of space on scales far smaller than the Hubble horizon size (e.g., Peebles 1980). The proper Hubble radius d_H (Fig. 10, top) may provide a reasonable estimate of the characteristic distance scale where the Newton's gravity applies. The cosmic structures simulated by the Newton's gravity-based N-body method will significantly deviate from the real structures on scales comparable to d_H. The variation of the comoving Hubble radius $\chi_H = d_H/a$ also implies that in the past (future) the Newtonian dynamics was (will be) applicable on smaller region of space compared to the size of the universe (Fig. 10, bottom).

In this paper, the history of the universe has been tentatively reconstructed based on cosmological parameters of non-flat FRW world models. The more general cosmological perturbation theory and parameter

estimation are essential for accurate reconstruction of the cosmic history.

Acknowledgments

References

Bonanos, A.Z., et al. 2006, The First DIRECT Distance Determination to a Detached Eclipsing Binary in M33, Astrophys. J., 652, 313-322.

Clocchiatti, A., et al. 2006, Hubble Space Telescope and Ground-based Observations of Type Ia Supernovae at Redshift 0.5: Cosmological Implications, Astrophys. J., 642, 1-21.

Colberg, J.M., et al. 2000, Clustering of galaxy clusters in cold dark matter universes, Mon. Not. R. Astron. Soc., 319, 209-214.

Cole, S., et al. 2005, The 2dF Galaxy Redshift Survey: power-spectrum analysis of the final data set and cosmological implications, Mon. Not. R. Astron. Soc., 362, 505-534.

Dubrovin, B.A., Fomenko, A.T., and Novikov, S.P. 1984, Modern Geometry - Methods and Applications. Part I. The Geometry of Surfaces, Transformation Groups, and Fields (Springer-Verlag, New York).

Dunkley, J., et al. 2009, Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Likelihoods and Parameters from the WMAP data, Astrophys. J. Suppl., 180, 306-329.

Einstein, A. 1917, Kosmologische Betrachtungen zur allgemeinen Relativittstheorie, Sitzungsberichte der Preussischen Akad. d. Wiss., 1917, 142-152.

Einstein, A. 1922, The Meaning of Relativity (Princeton University Press), 98-108.

Eisenstein, D.J., et al. 2005, Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies, Astrophys. J., 633, 560-574.

Freedman, W.L., et al. 2001, Final Results from the Hubble Space Telescope Key Project to Measure the Hubble Constant, Astrophys. J., 553, 47-72.

Friedmann, A.A. 1922, Über die Krümmung des Raumes, Z. Phys., 10, 377-386.

Friedmann, A.A. 1924, Über die Möglichkeit einer Welt mit konstanter negativer Krmmung des Raumes, Z. Phys., 21, 326-332.

Guth, A.H. 1981, Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D, 23, 347-356.

Hinshaw, G., et al. 2007, Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Temperature Analysis, Astrophys. J. Suppl., 170, 288-334.

Hinshaw, G., et al. 2009, Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Data Processing, Sky Maps, and Basic Results, Astrophys. J. Suppl., 180, 225-245.

Hwang, J.-c., and Noh, H. 2006, Why Newtonian gravity is reliable in large-scale cosmological simulations, Mon. Not. R. Astron. Soc., 367, 1515-1520.

Jackson, J.C., and Jannetta, A.L. 2006, Legacy data and cosmological constraints from the angular-size/redshift relation for ultracompact radio sources, Journal of Cosmology and Astroparticle Physics, 0611, 002.

Kolb, E.W. 1989, A coasting cosmology, Astrophys. J., 344, 543-550.

Kolb, E.W., and Turner, M.S. 1990, The Early Universe (Addison-Wesley Publishing Company), §3.3.

Landau, L.D., and Lifshitz, E.M. 1975, The Classical Theory of Fields (4th ed., Pergamon Press Ltd).

Mach, E. 1893, The Science of Mechanics (6th ed., The Open Court Publishing Company), Chapter II.

Macri, L.M., Stanek, K.Z., Bersier, D., Greenhill, L.J., and Reid, M.J. 2006, A New Cepheid Distance to the Maser-Host Galaxy NGC 4258 and Its Implications for the Hubble Constant, Astrophys. J., 652, 1133-1149.

Mather, J.C., Fixsen, D.J., Shafer, R.A., Mosier, C., and Wilkinson, D.T. 1999, Calibrator Design for the COBE Far-Infrared Absolute Spectrophotometer (FIRAS), Astrophys. J., 512, 511-520.

McCrea, W.H., and Milne, E.A. 1934, Newtonian Universes and the Curvature of Space, Quart. J. Math., 5, 73-80.

Milne, E.A. 1934, A Newtonian Expanding Universe, Quart. J. Math., 5, 64-72.

Misner, C.W., Thorne, K.S., and Wheeler, J.A. 1973, Gravitation (W.H. Freeman and Company, San Francisco).

Page, L., et al. 2007, Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Polarization Analysis, Astrophys. J. Suppl., 170, 335-376.

Park, C., Kim, J., and Gott, J.R. 2005, Effects of Gravitational Evolution, Biasing, and Redshift Space Distortion on Topology, Astrophys. J., 633, 1-10.

Park, C.-G., 2012, Introduction to Relativistic Cosmology, Journal of Science Education Chonbuk National University, 37, 57-71.

Park, C.-G. and Ratra, B., 2018a, Observational constraints

on the tilted spatially-flat and the untilted nonflat ϕCDM dynamical dark energy inflation models, Astrophys. J. 868, 83 [arXiv:1807.07421 [astro-ph.CO]].

Park, C.-G. and Ratra, B., 2019a, Using the tilted flat-ΛCDM and the non-flat ΛCDM inflation models to measure cosmological parameters from a compilation of observational data, Astrophys. J. in press [arXiv:1801.00213 [astro-ph.CO]].

Park, C.-G. and Ratra, B., 2019b, Observational constraints on the tilted flat-XCDM and the untilted nonflat XCDM dynamical dark energy inflation parameterizations, Astrophys. Space Sci. 364, 82 [arXiv:1803.05522 [astro-ph.CO]].

Park, C.-G. and Ratra, B., 2019c, Measuring the Hubble constant and spatial curvature from supernova apparent magnitude, baryon acoustic oscillation, and Hubble parameter data, Astrophys. Space Sci. 364, 134 [arXiv:1809.03598 [astro-ph.CO]].

Peebles, P.J.E. 1980, The Large-Scale Structure of the Universe (Princeton University Press), §6.

Planck Collaboration, 2018a, Planck 2018 results. VI. Cosmological parameters, arXiv:1807.06209 [astro-ph.CO].

Planck Collaboration, 2018b, Planck 2018 results: Cosmological Parameter Tables.

Riess, A.G., et al. 2007, New Hubble Space Telescope Discoveries of Type Ia Supernovae at z ≥ 1: Narrowing Constraints on the Early Behavior of Dark Energy, Astrophys. J., 659, 98-121.

Robertson, H.P. 1929, On the Foundations of Relativistic Cosmology, Proc. Natl. Acad. Sci. U. S. A., 15, 822-829.

Sandage, A., Tammann, G.A., Saha, A., Reindl, B., Macchetto, F.D., and Panagia, N. 2006, The Hubble Constant: A Summary of the Hubble Space Telescope Program for the Luminosity Calibration of Type Ia Supernovae by Means of Cepheids, Astrophys. J., 653, 843-860.

Spergel, D.N., et al. 2007, Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Implications for Cosmology, Astrophys. J. Suppl., 170, 377-408.

Tegmark, M., et al. 2004, The Three-Dimensional Power Spectrum of Galaxies from the Sloan Digital Sky Survey, Astrophys. J., 606, 702-740.

Tegmark, M., et al. 2006, Cosmological constraints from the SDSS luminous red galaxies, Phys. Rev. D, 74, 123507.

Walker, A.G. 1935, On Riemannian Spaces with Spherical Symmetry about a Line and the Conditions for Isotropy in General Relativity, Quart. J. Math., 6, 81-93.

Warren, J.S., et al. 2005, Cosmic-Ray Acceleration at the Forward Shock in Tycho's Supernova Remnant: Evidence from Chandra X-Ray Observations, Astrophys. J., 634, 376-389.

Weinberg, S. 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (John Wiley & Sons, Inc.), Chapter 13.

Weyl, H. 1923, Zur allgemeinen Relativittstheorie, Z. Phys., 24, 230-232.

Wood-Vasey, M., et al. 2007, Observational Constraints on the Nature of Dark Energy: First Cosmological Results from the ESSENCE Supernova Survey, Astrophys. J., 666, 694-715.

Wright, E.L. 2007, Constraints on Dark Energy from Supernovae, Gamma-Ray Bursts, Acoustic Oscillations, Nucleosynthesis, Large-Scale Structure, and the Hubble Constant, Astrophys. J., 664, 633-639.