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PERTURBATIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. We show the boundedness and uniform Lipschitz stability for the solutions to the functional perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),$$

under perturbations. We impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))ds$, $h(t, y(t), T_2 y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t,y)$ using the notion of h-stability.

1. Introduction and Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

and the perturbed differential system of (1.1) including an operator T such that

(1.2)
$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),\ y(t_0) = y_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, $f(t,0) = 0$, $g(t,0,0) = h(t,0,0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators and $\mathbb{R}^n$ is an n-dimensional Euclidean space. We always assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. Let $x(t, t_0, x_0)$ denote the unique

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solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

Pachpatte [20,21] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T.

Pinto [22,23] introduced the notion of h-stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu and Koo [7,8] investigated h-stability of solutions for nonlinear perturbed systems. Goo [11,12] and Im [15] studied the boundedness of solutions for the nonlinear perturbed differential systems.

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [10]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asmptotic stability in variation of Brauer [4] and uniformly stability in variation of Brauer and Strauss [3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. Im and Goo [16-18] studied uniform Lipschitz stability and asymptotic properties of solutions for nonlinear perturbed systems.

In this paper, we investigate the boundedness and uniform Lipschitz stability for the solutions of the perturbed functional differential systems via t_∞-similarity.

Now, we recall some definitions of stability. The symbol $|\cdot|$ will be used to denote any convenient vector norm on $\mathbb{R}^n$.

Definition 1.1. [10] The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(S) *stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$.

Definition 1.2. [23] The system (1.1) (the zero solution $x = 0$ of (1.1)) is called (hS) *$h-$stable* if there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$, (here $h(t)^{-1} = \frac{1}{h(t)}$).

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [9].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [9,14].

Before proceeding to the statement of main results, we give some known results.

Lemma 1.4. [23] *The linear system*

$$x' = A(t)x,\ x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\Phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (1.6).

We consider Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. [2] *Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.6. [7] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

Theorem 1.7. [8] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

Lemma 1.8. ($Bihari - type\ inequality$) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],\ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{dom W^{-1}}\Big\}.$$

For the proof we need the following lemma and three corollaries.

LEMMA 1.9. [13] *Let $u, \lambda_i \in C(\mathbb{R}^+)$ for $0 \le i \le 10$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^t \lambda_0(s)u(s)ds + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s)\int_{t_0}^s \Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)u(r)dr + \lambda_7(\tau)\int_{t_0}^\tau \lambda_8(r)w(u(r))dr\Big)d\tau ds + \int_{t_0}^t \lambda_9(s)\int_{t_0}^s \lambda_{10}(\tau)u(\tau)d\tau ds.$$

Then, we have

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t \Big(\lambda_0(s) + \lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^\tau \lambda_8(r)dr)d\tau + \lambda_9(s)\int_{t_0}^s \lambda_{10}(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^t (\lambda_0(s) + \lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^\tau \lambda_8(r)dr)d\tau + \lambda_9(s)\int_{t_0}^s \lambda_{10}(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

COROLLARY 1.10. *Let $u, \lambda_i \in C(\mathbb{R}^+)$ for $0 \le i \le 8$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^t \lambda_0(s)u(s)ds + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)u(r)dr)d\tau ds + \int_{t_0}^t \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)u(\tau)d\tau ds.$$

Then, we have

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t \Big(\lambda_0(s) + \lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^{t} (\lambda_0(s) + \lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau + \lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

COROLLARY 1.11. *Let* $u, \lambda_i \in C(\mathbb{R}^+)$ *for* $0 \le i \le 6$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t} \lambda_0(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds \\ &+ \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) \\ &+ \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)u(r)dr)d\tau ds. \end{aligned}$$

Then, we have

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) &+ \int_{t_0}^{t} \Big(\lambda_0(s) + \lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \\ &+ \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau\Big)ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^{t} (\lambda_0(s) + \lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

COROLLARY 1.12. *Let* $u, \lambda_i \in C(\mathbb{R}^+)$ *for* $0 \le i \le 6$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \end{aligned}$$

$$0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau$$
$$+\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau$$
$$+\lambda_5(s)\textstyle\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main Results

In this section, we study the boundedness and uniform Lipschitz stability for solutions of the perturbed functional differential systems via t_∞-similarity.

To obtain bounded properties and ULS, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(H3) $w(u)$ is nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for all $v > 0$.

(H4) The solution $x = 0$ of (1.1) is ULSV.

THEOREM 2.1. *Let (H1)-(H3) be satisfied. Assume that g in (1.2) satisfies*

$$|g(t,y,T_1y)| \le a(t)|y(t)| + |T_1y(t)|, \tag{2.1}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t}k(s)|y(s)|ds, \tag{2.2}$$

(2.3)
$$|h(t,y(t),T_2y(t))| \le c(t)w(|y(t)|) + d(t)\int_{t_0}^{t}p(s)w(|y(s)|)ds + |T_2y(t)|,$$

and

$$|T_2y(t)| \le m(t)|y(t)| + n(t)\int_{t_0}^{t}q(s)|y(s)|ds, \tag{2.4}$$

where $a, b, c, d, k, m, n, p, q \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t) W^{-1}\Big[W(c) + c_2 \int_{t_0}^t \Big(c(s) + m(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + d(s)\int_{t_0}^s p(\tau)d\tau + n(s)\int_{t_0}^s q(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, $c = c_1|y_0|\, h(t_0)^{-1}$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \textstyle\int_{t_0}^t \Big(c(s) + m(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + d(s)\int_{t_0}^s p(\tau)d\tau + n(s)\int_{t_0}^s q(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, from (H1), the solution $z = 0$ of (1.4) is hS by Theorem 1.7. Using the nonlinear variation of constants formula due to Lemma 1.5, together with (2.1)-(2.4) and (H2), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))|\Big(\int_{t_0}^s |g(\tau, y(\tau), T_1 y(\tau))|d\tau \\
&\qquad + |h(s, y(s), T_2 y(s))|\Big)ds \\
&\le c_1|y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1}\Big(\int_{t_0}^s (a(\tau)|y(\tau)| \\
&\qquad + b(\tau)\int_{t_0}^\tau k(r)|y(r)|dr)d\tau + d(s)\int_{t_0}^s p(\tau)w(|y(\tau)|)d\tau \\
&\qquad + m(s)|y(s)| + c(s)w(|y(s)|) + n(s)\int_{t_0}^s q(\tau)|y(\tau)|d\tau\Big)ds.
\end{aligned}$$

By (H3), we obtain

$$
\begin{aligned}
|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} &+ \int_{t_0}^{t} c_2 h(t)\Big(m(s)\frac{|y(s)|}{h(s)} + c(s)w(\frac{|y(s)|}{h(s)}) \\
&+ \int_{t_0}^{s}(a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr)d\tau \\
&+ d(s)\int_{t_0}^{s} p(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + n(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.
\end{aligned}
$$

Define $u(t) = |y(t)|h(t)^{-1}$. Then, an application of Corollary 1.10, we yields

$$
\begin{aligned}
|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\Big(c(s) + m(s) + \int_{t_0}^{s}(a(\tau) \\
+ b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + d(s)\int_{t_0}^{s} p(\tau)d\tau + n(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big]
\end{aligned}
$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Hence the above estimation obtains the desired result since the function h is bounded, and so the proof is complete. □

Remark 2.2. Letting $b(t) = d(t) = m(t) = n(t) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.3 in [12].

Theorem 2.3. *Let $a, b, c, d, k, m, q \in C(\mathbb{R}^+)$ and let (H1)-(H3) be satisfied. Assume that g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1y(s))|ds \le a(t)|y(t)| + |T_1y(t)|, \tag{2.5}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds, \tag{2.6}$$

$$|h(t, y(t), T_2y(t))| \le b(t)w(|y(t)|) + b(t)\int_{t_0}^{t} c(s)|y(s)|ds + |T_2y(t)|, \tag{2.7}$$

and

$$|T_2y(t)| \le m(t)|y(t)| + d(t)\int_{t_0}^{t} q(s)w(|y(s)|)ds, \tag{2.8}$$

where $a, b, c, d, k, m, q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on

$[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(a(s) + b(s) + m(s) + b(s)\int_{t_0}^{s}(c(\tau) + k(\tau))d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \textstyle\int_{t_0}^{t}\Big(a(s) + b(s) + m(s) + b(s)\int_{t_0}^{s}(c(\tau) + k(\tau))d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula due to Lemma 1.5, together with (2.5)-(2.8) and (H2), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(a(s)|y(s)| + b(s)\int_{t_0}^{s}(c(\tau) + k(\tau))|y(\tau)|d\tau + d(s)\int_{t_0}^{s} q(\tau)w(|y(\tau)|)d\tau + m(s)|y(s)| + b(s)w(|y(s)|)\Big)ds.$$

It follows from (H3) that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big((a(s) + m(s))\frac{|y(s)|}{h(s)} + b(s)w(\frac{|y(s)|}{h(s)}) + b(s)\int_{t_0}^{s}(c(\tau) + k(\tau))\frac{|y(\tau)|}{h(\tau)}d\tau + d(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Corollay 1.12, we obtain

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(a(s) + b(s) + m(s) + b(s)\int_{t_0}^{s}(c(\tau) + k(\tau))d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big],$$

where $c = c_1|y_0|h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

REMARK 2.4. Letting $d(t) = m(t) = 0$ in Theorem 2.3, we obtain the similar result as that of Theorem 2.2 in [11].

THEOREM 2.5. *Let (H3) and (H4) be satisfied. Assume that the perturbing term g in (1.2) satisfies*

$$|g(t, y, T_1 y)| \le a(t)|y(t)| + |T_1 y(t)|, \tag{2.9}$$

$$|T_1 y(t)| \le b(t) \int_{t_0}^{t} k(s)|y(s)|ds, \tag{2.10}$$

$$|h(t, y(t), T_2 y(t))| \le b(t)w(|y(t)|) + n(t)|y(t)| + |T_2 y(t)|, \tag{2.11}$$

and

$$|T_2 y(t)| \le \int_{t_0}^{t} d(s)|y(s)|ds + \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{2.12}$$

where $a, b, c, d, k, n \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 are continuous operators, and

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \Big(b(s) + n(s) \\ &+ \int_{t_0}^{s} (a(\tau) + c(\tau) + d(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau\Big)ds\Big], \end{aligned} \tag{2.13}$$

where $M(t_0) < \infty$ and $b_1 = \infty$, $t_0 \le t < b_1$, and W, W^{-1} are the same functions as in Lemma 1.8. Then the zero solution of (1.2) is ULS.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By (H4), it is ULS. Applying the nonlinear variation of constants formula due to Lemma 1.5, together with (H3), (2.9)-(2.12), we obtain

$$\begin{aligned} |y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\Big(\int_{t_0}^{s} ((a(\tau) + d(\tau))\frac{|y(\tau)|}{|y_0|} + c(\tau)w(\frac{|y(\tau)|}{|y_0|}) \\ +b(\tau) \int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|}dr)d\tau + n(s)\frac{|y(s)|}{|y_0|} + b(s)w(\frac{|y(s)|}{|y_0|})\Big)ds. \end{aligned}$$

Let $u(t) = |y(t)||y_0|^{-1}$. Then, by Corollary 1.11, we have

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M\int_{t_0}^{t}\Big(b(s) + n(s) + \int_{t_0}^{s}(a(\tau) + c(\tau) + d(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau\Big)ds\Big].$$

Thus, by (2.13), we obtain $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

REMARK 2.6. Letting $b(t) = c(t) = d(t) = n(t) = 0$ in Theorem 2.5, we obtain the similar result as that of Corollary 3.2 in [5].

THEOREM 2.7. *Let (H3) and (H4) be satisfied. Assume that the perturbing term g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \le a(t)w(|y(t)|) + |T_1 y(t)|, \tag{2.14}$$

$$|T_1 y(t)| \le m(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.15}$$

$$|h(t, y(t), T_2 y(t))| \le c(t)|y(t)| + d(t)w(|y(t)|) + |T_2 y(t)|, \tag{2.16}$$

and

$$|T_2 y(t)| \le b(t)\int_{t_0}^{t} q(s)|y(s)|ds + m(t)\int_{t_0}^{t} n(s)w(|y(s)|)ds, \tag{2.17}$$

where $a, b, c, d, m, n, p, q \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 *are continuous operators, and*

$$\begin{aligned} M(t_0) = &W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}\Big(a(s) + c(s) + d(s) \\ &+ b(s)\int_{t_0}^{s} q(\tau)d\tau + m(s)\int_{t_0}^{s}(n(\tau) + p(\tau))d\tau\Big)ds\Big], \end{aligned} \tag{2.18}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$, $t_0 \le t < b_1$, *and* W, W^{-1} *are the same functions as in Lemma 1.8. Then the zero solution of (1.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By (H4), it is ULS. Using Lemma 1.5, together with (H3) and (2.14)-(2.17), we obtain

$$|y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\Big(c(s)\frac{|y(s)|}{|y_0|} + (a(s)+d(s))w(\frac{|y(s)|}{|y_0|})$$
$$+b(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{|y_0|}d\tau + m(s)\int_{t_0}^{s}(n(\tau)+p(\tau))w(\frac{|y(\tau)|}{|y_0|})d\tau\Big)ds.$$

Define $u(t) = |y(t)||y_0|^{-1}$. Then, from an application of Corollary 1.12, we have

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t}\Big(a(s)+c(s)+d(s)$$
$$+b(s)\int_{t_0}^{s} q(\tau)d\tau + m(s)\int_{t_0}^{s}(n(\tau)+p(\tau))d\tau\Big)ds\Big].$$

Hence, by (2.18), we obtain $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. Thus the theorem is proved. □

REMARK 2.8. Letting $b(t) = c(t) = n(t) = 0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.6 in [6].

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