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CHARACTERIZATIONS OF SIMPLE BCK/BCI-ALGEBRAS

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Abstract. The notions of a fuzzy simple BCK/BCI-algebra and an $(\in, \in\vee q)$-fuzzy simple BCK/BCI-algebra are introduced. Using these notions, characterizations of a simple BCK/BCI-algebra are considered.

1. Introduction

Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold; as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty (see [15] for generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki (see [3, 4, 5, 6]) and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. After the introduction of fuzzy sets by Zadeh [14], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in\vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1] by using the combined notions of " elongingness" and " quasicoincidence"

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of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [13]. Jun [7, 8] introduced the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI-algebra and investigated related results.

In this paper, we introduce the notions of a fuzzy simple BCK/BCI-algebra and an $(\in, \in\vee\,\mathrm{q})$-fuzzy simple BCK/BCI-algebra. Using these notions, we provide characterizations of a simple BCK/BCI-algebra.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) if $x * y = 0$ and $y * x = 0$, then $x = y$

for all $x, y, z \in X$. If a BCI-algebra $(X; *, 0)$ satisfies the following identity:

(V) $0 * x = 0$

for all $x \in X$, then $(X; *, 0)$ is called a *BCK-algebra*. In the following, for notational convinience, we denote any BCK/BCI-algebra $(X; *, 0)$ by X. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $x * 0 = x$,
(a2) if $x * y = 0$, then $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$,
(a3) $(x * y) * z = (x * z) * y$,
(a4) $((x * z) * (y * z)) * (x * y) = 0$

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies, for all $x, y \in X$, the following conditions:

(b1) $0 \in A$,
(b2) if $x * y \in A$ and $y \in A$, then $x \in A$.

A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies:

(b3) $\mu(0) \geq \mu(x)$,
(b4) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all $x, y \in X$. A fuzzy subset μ of a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $[x;t]$.

For a fuzzy subset μ of a set X, we say that a fuzzy point $[x;t]$ is

(i) *contained* in μ, denoted by $[x;t] \in \mu$, ([13]) if $\mu(x) \geq t$.

(ii) *quasi-coincident* with μ, denoted by $[x;t] \, \mathrm{q} \, \mu$, ([13]) if $\mu(x) + t > 1$.

For a fuzzy point $[x;t]$ and a fuzzy subset μ of a set X, we say that

(iii) $[x;t] \in\!\vee\, \mathrm{q}\ \mu$ if $[x;t] \in \mu$ or $[x;t] \, \mathrm{q} \, \mu$.

(iv) $[x;t] \, \overline{\alpha} \, \mu$ if $[x;t] \, \alpha \, \mu$ does not hold for $\alpha \in \{\in, \mathrm{q}, \in\!\vee\, \mathrm{q}\}$.

We refer the reader to the books [2, 12] for further information regarding BCK/BCI-algebras.

3. Characterizations of simple BCK/BCI-algebras

For a fuzzy subset μ of a BCK/BCI-algebra X, consider the set

$$I_a := \{x \in X \mid \mu(x) \geq \mu(a)\}$$

where $a \in X$.

LEMMA 3.1. *If μ is a fuzzy ideal of a BCK/BCI-algebra X, then the set I_a is an ideal of X for all $a \in X$.*

Proof. Let μ be a fuzzy ideal of a BCK/BCI-algebra X. The condition (b3) implies that $0 \in I_a$ for all $a \in X$. Let $x, y \in X$ be such that $x*y \in I_a$ and $y \in I_a$ for all $a \in X$. Then $\mu(x*y) \geq \mu(a)$ and $\mu(y) \geq \mu(a)$. It follows from (b4) that $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \mu(a)$ so that $x \in I_a$. Therefore I_a is an ideal of X for all $a \in X$. □

LEMMA 3.2. *For any non-empty subset I of a BCK/BCI-algebra X, let μ_I be a fuzzy subset of X defined by*

$$\mu_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then I is an ideal of X if and only if μ_I is a fuzzy ideal of X.

Proof. Straightforward. □

DEFINITION 3.3. [12] A BCK/BCI-algebra X is said to be *simple* if it has no non-zero proper ideal.

EXAMPLE 3.4. *Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 1. Then $(X, *, 0)$ is a simple BCK-algebra.*

EXAMPLE 3.5. *Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 2. Then $(X, *, 0)$ is a simple BCI-algebra.*

Table 1: $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

Table 2: $*$-operation

$*$	0	a	b	c	d
0	0	d	c	b	a
a	a	0	d	c	b
b	b	a	0	d	c
c	c	b	a	0	d
d	d	c	b	a	0

Definition 3.6. A BCK/BCI-algebra X is said to be *fuzzy simple* if every fuzzy ideal of X is a constant function, that is, for every fuzzy ideal μ of X, $\mu(x) = \mu(y)$ for all $x, y \in X$.

Theorem 3.7. *For a BCK/BCI-algebra X, the following are equivalent:*

(1) *X is simple.*
(2) *X is fuzzy simple.*

Proof. Suppose X is simple and let μ be a fuzzy ideal of X. Then the set $I_a := \{x \in X \mid \mu(x) \geq \mu(a)\}$ is an ideal of X for all $a \in X$ by Lemma 3.1. Since X is simple, we have $I_a = S$ which implies $y \in I_a$ for all $y \in X$. Hence $\mu(y) \geq \mu(a)$. Similarly, we get $\mu(y) \leq \mu(a)$ and so $\mu(y) = \mu(a)$ for all $a, y \in X$. Therefore X is fuzzy simple.

Conversely, assume that X is fuzzy simple. If there exists a non-zero proper ideal I of X, then μ_I is an ideal of X by Lemma 3.2. Let $x \in X$. Since X is fuzzy simple, the fuzzy ideal μ_I is a constant function. It follows that $\mu_I(x) = \mu_I(a) = 1$ for all $a \in I$ so that $x \in I$. Therefore $X = I$, a contradiction. Consequently, X is simple. □

We now consider a characterization of a simple BCK/BCI-algebra based on the notion of $(\in, \in\vee\mathrm{q})$-fuzzy ideals of BCK/BCI-algebras.

Definition 3.8. [7] A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy ideal* of X with type $(\in, \in\vee\mathrm{q})$ (briefly, $(\in, \in\vee\mathrm{q})$-*fuzzy ideal* of X) if it satisfies:

(c1) if $[x; t] \in \mu$, then $[0; t] \in\vee\mathrm{q}\, \mu$,
(c2) if $[x * y; t_1] \in \mu$ and $[y; t_2] \in \mu$, then $[x; \min\{t_1, t_2\}] \in\vee\mathrm{q}\, \mu$

for all $x, y, z \in X$ and $t, t_1, t_2 \in (0, 1]$.

Lemma 3.9. [7] *A fuzzy subset μ of a BCK/BCI-algebra X is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X if and only if it satisfies: for all $x, y \in X$,*

(1) $\mu(0) \geq \min\{\mu(x), 0.5\}$,
(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y), 0.5\}$.

For a fuzzy subset μ of a BCK/BCI-algebra X and $a \in X$, we consider the set $J_a := \{x \in X \mid \mu(x) \geq \min\{\mu(a), 0.5\}\}$.

THEOREM 3.10. *If μ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra X, then the set J_a is an ideal of X for all $a \in X$.*

Proof. Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra X and $a \in X$. Using Lemma 3.9(1), we have $0 \in J_a$. Let $x, y \in X$ be such that $x * y \in J_a$ and $y \in J_a$. Then $\mu(x * y) \geq \min\{\mu(a), 0.5\}$ and $\mu(y) \geq \min\{\mu(a), 0.5\}$. It follows from Lemma 3.9(2) that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y), 0.5\} \geq \min\{\mu(a), 0.5\}$$

so that $x \in J_a$. Hence J_a is an ideal of X for all $a \in X$. □

For any fuzzy subset μ and an ideal I of a BCK/BCI-algebra X, we define two fuzzy subsets μ^* and μ_I^- of X by $\mu^*(x) = \min\{\mu(x), 0.5\}$ and

$$\mu_I^-(x) := \begin{cases} 0.5 & \text{if } x \in I, \\ 0 & \text{otherwise}, \end{cases}$$

respectively, for all $x \in X$.

THEOREM 3.11. *If μ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra X, then μ^* is a fuzzy ideal of X.*

Proof. Let $x, y \in X$. Using Lemma 3.9, we have

$$\begin{aligned} \mu^*(0) &= \min\{\mu(0), 0.5\} \\ &\geq \min\{\min\{\mu(x), 0.5\}, 0.5\} = \min\{\mu(x), 0.5\} = \mu^*(x) \end{aligned}$$

and

$$\begin{aligned} \mu^*(x) &= \min\{\mu(x), 0.5\} \\ &\geq \min\{\min\{\mu(x * y), \mu(y), 0.5\}, 0.5\} \\ &= \min\{\min\{\mu(x * y), 0.5\}, \min\{\mu(y), 0.5\}\} \\ &= \min\{\mu^*(x * y), \mu^*(y)\}. \end{aligned}$$

Therefore μ^* is a fuzzy ideal of X. □

THEOREM 3.12. *For any fuzzy subset μ and a subset I of a BCK/BCI-algebra X, the fuzzy subset μ_I^- is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X if and only if I is an ideal of X.*

Proof. Let I be an ideal of X and $x, y \in X$. Then $\mu_I^-(0) = 0.5 \geq \min\{\mu_I^-(x), 0.5\}$. If $x \in I$, then $\mu_I^-(x) = 0.5 \geq \min\{\mu_I^-(x*y), \mu_I^-(y), 0.5\}$. Suppose $x \notin I$. Then $x * y \notin I$ or $y \notin I$ because I is an ideal of X. It follows that $\mu_I^-(x) = 0 = \min\{\mu_I^-(x*y), \mu_I^-(y), 0.5\}$. Using Lemma 3.9, we know that μ_I^- is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X.

Conversely, assume that μ_I^- is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X and let $x, y \in X$. Suppose $0 \notin I$. Then $\mu_I^-(0) = 0$, and so $[0; 0.5] \overline{\in} \mu_I^-$. If $x \in I$, then $\mu_I^-(x) = 0.5$ and thus $[x; 0.5] \in \mu_I^-$. Hence $[0; 0.5] \in\vee\mathrm{q}\, \mu_I^-$ by (c1), and so $[0; 0.5]\, \mathrm{q}\, \mu_I^-$, i.e., $\mu_I^-(0) + 0.5 > 1$. This is a contradiction, and so $0 \in I$. Assume that $x*y \in I$ and $y \in I$. Then $\mu_I^-(x*y) = 0.5 = \mu_I^-(y)$, and so $[x*y; 0.5] \in \mu_I^-$ and $[y; 0.5] \in \mu_I^-$. Since μ_I^- is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X, it follows that $[x; 0.5] \in\vee\mathrm{q}\, \mu_I^-$ so that $[x; 0.5] \in \mu_I^-$ or $[x; 0.5]\, \mathrm{q}\, \mu_I^-$. Hence $\mu_I^-(x) \geq 0.5$ or $\mu_I^-(x) + 0.5 > 1$, which implies that $\mu_I^-(x) = 0.5$. Hence $x \in I$. Therefore I is an ideal of X. □

DEFINITION 3.13. A BCK/BCI-algebra X is said to be $(\in, \in\vee\mathrm{q})$-*fuzzy simple* if for any $(\in, \in\vee\mathrm{q})$-fuzzy ideal μ of X, μ^* is a constant function.

THEOREM 3.14. *For a BCK/BCI-algebra X, the following are equivalent:*

(1) *X is simple.*
(2) *X is $(\in, \in\vee\mathrm{q})$-fuzzy simple.*

Proof. (1) ⇒ (2). Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X. Then the set J_a is an ideal of X for all $a \in X$ (see Theorem 3.10). Since X is simple, we have $J_a = X$ and so $x \in J_a$ for all $x \in X$. Thus $\mu(x) \geq \min\{\mu(a), 0.5\}$, which implies that

$$\begin{aligned}\mu^*(x) &= \min\{\mu(x), 0.5\} \\ &\geq \min\{\min\{\mu(a), 0.5\}, 0.5\} = \min\{\mu(a), 0.5\} = \mu^*(a).\end{aligned}$$

Similarly, $\mu^*(x) \leq \mu^*(a)$. Therefore $\mu^*(x) = \mu^*(a)$ for all $a, x \in X$, that is, μ^* is a constant function. Consequently, X is $(\in, \in\vee\mathrm{q})$-fuzzy simple.

(2) ⇒ (1). Assume that X is not simple. Then there exists a non-zero proper ideal I of X. Using Theorem 3.12, μ_I^- is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X. Since X is $(\in, \in\vee\mathrm{q})$-fuzzy simple, $(\mu_I^-)^*$ is a constant function. Hence for every $a \in I$ and $x \in X$, we have

$$\min\{\mu_I^-(x), 0.5\} = (\mu_I^-)^*(x) = (\mu_I^-)^*(a) = \min\{\mu_I^-(a), 0.5\} = 0.5,$$

and so $\mu_I^-(x) = 0.5$. This shows that $x \in I$ so that $I = X$. This is a contradiction, and therefore X is simple. □

4. Conclusions

BCK-algebras [12] have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. Soft sets are deeply related to fuzzy sets and rough sets. The soft set theory is applied to BCK/BCI-algebras by the first author (see [9]), and applications of soft sets in ideal theory of BCK/BCI-algebras are carried out by Jun and Park [10]. Jun and Song [11] studied soft subalgebras and soft ideals of BCK/BCI-algebras related to fuzzy set theory. To investigate the structure of an algebraic system, it is clear that ideals with special properties play an important role. In this paper, we considered the notions of a fuzzy simple BCK/BCI-algebra and an $(\in, \in\vee\mathrm{q})$-fuzzy simple BCK/BCI-algebra. Using these notions, we provided characterizations of a simple BCK/BCI-algebra. It is our hope that this work would serve as a foundation for further study of the theory of BCK/BCI-algebras. Moreover, we would like to apply the notions/results in this article to other algebraic structures, and we will try to get another characterization of a simple BCK/BCI-algebra by using soft set theory.

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