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APPROXIMATE LINEAR MAPPING OF DERIVATION-TYPE ON BANACH $*$-ALGEBRA

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Abstract. We consider additive mappings similar to derivations on Banach $*$-algebras and we will first study the conditions for such additive mappings on Banach $*$-algebras. Then we prove some theorems concerning approximate linear mappings of derivation-type on Banach $*$-algebras. As an application, approximate linear mappings of derivation-type on C^*-algebra are characterized.

1. Introduction

The stability problem for derivations on Banach algebra was considered by authors in [3, 14]. Bourgin proved the superstability of homomorphism in [4]. In particular, Badora dealt with the stability of Bourgin-type for derivations in [3].

The study of stability problem has originally been formulated by Ulam [16] : *under what condition does there exist a homomorphism near an approximate homomorphism?* Hyers [8] had answered affirmatively the question of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] and for approximately linear mappings was presented by Rassias [15].

Since then, many interesting results of the stability problems to a number of functional equations and inequalities (or involving derivations) have been investigated (refer [11] and [12]). The reader is referred to the book [9] for many information of stability problem with a large variety of applications.

On the other hand, many authors (see, for example, [5]) have studied the additive mappings δ_1, δ_2 on $*$-rings $\mathcal{R}$ similar to derivations and

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Jordan derivations on $*$-rings. These mappings δ_1, δ_2 satisfy

$$\delta_1(xy) = x\delta_1(y) + \delta_1(x)y^* \text{ for all } x, y \in \mathcal{R}$$

and

$$\delta_2(x^2) = x\delta_2(x) + \delta_2(x)x^* \text{ for all } x \in \mathcal{R}.$$

The aim of this work is to establish some theorems for approximate linear mappings of derivation-type on Banach $*$-algebra related to the additive mappings mentioned in the above paragraph. Furthermore, the division of this work is devoted to the applications for such approximate linear mappings of derivation-type on C^*-algebra.

2. Main results

We first take into account the additive functional inequality which is needed in this work.

Lemma 2.1. *Let δ be a mapping from a vector space $\mathcal{A}$ to a normed space $\mathcal{B}$. Then it satisfies the inequality*

$$\|\delta(x) - \delta(y) - 2\delta(z)\| \leq \|\delta(x - y - 2z)\| \tag{2.1}$$

for all $x, y, z \in \mathcal{A}$ if and only if it is an additive mapping.

Proof. Suppose that a mapping δ satisfies the inequality (2.1). Letting $x = y = z = 0$ in (2.1), we get $\delta(0) = 0$. And by replacing x, y and z with $x + y, x - y$ and y, respectively, in (2.1), we obtain

$$\delta(x + y) - \delta(x - y) = 2\delta(y) \tag{2.2}$$

for all $x, y \in \mathcal{A}$. Also, by letting $x + y = u$ and $x - y = v$ in (2.2), we get

$$\delta(u) - \delta(v) = 2\delta\Big(\frac{u - v}{2}\Big) \tag{2.3}$$

for all $u, v \in \mathcal{A}$. Replacing v by $-u$ in (2.3), we have

$$\delta(-u) = -\delta(u) \tag{2.4}$$

for all $u \in \mathcal{A}$. Setting $u = 2y$ and $v = 0$ in (2.3), we arrive at $\delta(2y) = 2\delta(y)$. Setting $y = \frac{x}{2}$ in the last expression, we obtain $\delta(\frac{x}{2}) = \frac{1}{2}\delta(x)$. So the relation (2.3) can be written

$$\delta(u) - \delta(v) = \delta(u - v) \tag{2.5}$$

for all $u, v \in \mathcal{A}$. Letting $u = x$ and $v = -y$ in (2.5) and using (2.4), we yield that

$$\delta(x + y) = \delta(x) + \delta(y)$$

for all $x, y \in \mathcal{A}$, so that δ is additive.

Conversely, if δ is an additive mapping, then it is easily proved that δ satisfies the inequality (2.1). $\square$

Now we assume that $\mathbb{T}_\varepsilon = \{e^{i\theta} : 0 \le \theta \le \varepsilon\}$. For any elements x, y in $*$-algebra $\mathcal{A}$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and let $Sym(\mathcal{A})$ be the set of self-adjoint elements in $\mathcal{A}$.

THEOREM 2.2. *Let $\mathcal{A}$ be a Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A}^3 \to [0, \infty)$ and $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfy the assumptions*

1. $\sum_{j=0}^{\infty} \frac{1}{2^j}\Phi(2^j x, 2^j y, 2^j z) < \infty \quad (x, y, z \in \mathcal{A})$,
2. $\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, y) = 0 \qquad (x, y \in \mathcal{A})$.

Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping subject to

$$\|\delta(tx) - t\delta(y) - 2\delta(z)\| \le \|\delta(x - y - 2z)\| + \Phi(x, y, z) \tag{2.6}$$

for all $x, y, z \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$ with

$$\|\delta(xy) - x\delta(y) - \delta(x)y^*\| \le \varphi(x, y) \tag{2.7}$$

for all $x \in Sym(\mathcal{A})$ and $y \in \mathcal{A}$. Then there exists a unique linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ such that

$$\mathcal{L}(xy) = x\mathcal{L}(y) + \mathcal{L}(x)y^* \text{ for all } x, y \in \mathcal{A} \tag{2.8}$$

and

$$\|\mathcal{L}(x) - \delta(x)\| \le \sigma(x) \quad \text{for all } x \in \mathcal{A}, \tag{2.9}$$

where

$$\sigma(x) = \sum_{j=0}^{\infty}\left[\frac{1}{2^{j+1}}\Phi(2^{j+1}x, 0, 2^j x)\right] + 2\Phi(0, 0, 0).$$

In this case, the mapping $\mathcal{L}$ satisfies the identity

$$\mathcal{L}(x)[y, z] = 0 \tag{2.10}$$

for all $x, y, z \in \mathcal{A}$.

Proof. We first consider $t = 1$ in (2.6). Then we have

$$\|\delta(x) - \delta(y) - 2\delta(z)\| \le \|\delta(x - y - 2z)\| + \Phi(x, y, z) \tag{2.11}$$

for all $x, y, z \in \mathcal{A}$. By letting $x = y = z = 0$ in (2.11), we get $\|\delta(0)\| \le \Phi(0, 0, 0)$. Setting $x = x + y$, $y = x - y$ and $z = y$ in (2.11) yield

$$\|\delta(x + y) - \delta(x - y) - 2\delta(y)\| \le \Phi(x + y, x - y, y) + \Phi(0, 0, 0) \tag{2.12}$$

for all $x, y \in \mathcal{A}$. Putting $y = x$ in (2.12) and dividing by 2, we arrive at

$$\left\|\delta(x) - \frac{\delta(2x)}{2}\right\| \le \frac{1}{2}\Phi(2x, 0, x) + \Phi(0, 0, 0) \tag{2.13}$$

for all $x \in \mathcal{A}$. Substituting $2^n x$ for x in (2.13) and dividing by 2^n, we obtain

$$\Big\|\frac{\delta(2^n x)}{2^n} - \frac{\delta(2^{n+1}x)}{2^{n+1}}\Big\| \le \frac{1}{2^{n+1}}\Phi(2^{n+1}x, 0, 2^n x) + \frac{1}{2^n}\Phi(0,0,0),$$

which implies that

(2.14)

$$\begin{aligned}\Big\|\frac{\delta(2^n x)}{2^n} - \frac{\delta(2^m x)}{2^m}\Big\| &\le \sum_{j=m}^{n-1}\Big\|\frac{\delta(2^j x)}{2^j} - \frac{\delta(2^{j+1}x)}{2^{j+1}}\Big\| \\ &\le \sum_{j=m}^{n-1}\Big[\frac{1}{2^{j+1}}\Phi(2^{j+1}x, 0, 2^j x) + \frac{1}{2^j}\Phi(0,0,0)\Big]\end{aligned}$$

for all $x \in \mathcal{A}$ and all nonnegative integers m, n with $n > m$. This means that $\{\frac{\delta(2^n x)}{2^n}\}$ is a Cauchy sequence. Hence the sequence $\{\frac{\delta(2^n x)}{2^n}\}$ converges. So one can define a mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}(x) = \lim_{n\to\infty}\frac{\delta(2^n x)}{2^n} \tag{2.15}$$

for all $x \in \mathcal{A}$. Letting $m = 0$ and $n \to \infty$ in (2.14), we arrive at (2.9).

Now we claim that the mapping $\mathcal{L}$ is linear. By (2.11), one notes that

$$\begin{aligned}&\|\mathcal{L}(x) - \mathcal{L}(y) - 2\mathcal{L}(z)\| = \lim_{n\to\infty}\frac{1}{2^n}\|\delta(2^n x) - \delta(2^n y) - 2\delta(2^n z)\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}[\,\|\delta(2^n(x - y - 2z))\| + \Phi(2^n x, 2^n y, 2^n z)\,] \\ &= \|\mathcal{L}(x - y - 2z)\|\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. According to Lemma 2.1, the mapping $\mathcal{L}$ is additive. Replacing x, y and z with $x + y, x - y$ and y, respectively, in (2.6), we have

$$\|\delta(t(x+y)) - t\delta(x-y) - 2\delta(y)\| \le \Phi(x+y, x-y, y) + \Phi(0,0,0) \tag{2.16}$$

for all $x, y \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$. Putting $y = 0$ in (2.16), we have

$$\|\delta(tx) - t\delta(x)\| \le \Phi(x, x, 0) + 3\Phi(0,0,0),$$

for all $x \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$, which gives that

$$\begin{aligned}\|\mathcal{L}(tx) - t\mathcal{L}(x)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|\delta(t\cdot 2^n x) - t\delta(2^n x)\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}[\Phi(2^n x, 2^n x, 0) + 3\Phi(0,0,0)] = 0.\end{aligned}$$

That is, we conclude that $\mathcal{L}(tx) = t\mathcal{L}(x)$ for all $x \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$. On account of Lemma in [7], we know that $\mathcal{L}$ is a linear.

Next we show that $\mathcal{L}$ satisfies the equation (2.8). It is easy to show that if $x \in Sym(\mathcal{A})$, then $2^n x \in Sym(\mathcal{A})$. We note from (2.7) that

$$\begin{aligned}\|\mathcal{L}(xy) - x\delta(y) - \mathcal{L}(x)y^*\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n xy) - 2^n x\delta(y) - \delta(2^n x)y^*\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, y) = 0\end{aligned}$$

for all $x \in Sym(\mathcal{A})$ and $y \in \mathcal{A}$. Thus we get

$$\mathcal{L}(xy) = x\delta(y) + \mathcal{L}(x)y^* \quad \text{for all } x \in Sym(\mathcal{A}) \text{ and } y \in \mathcal{A}.$$

Note that for elements $x \in \mathcal{A}$, we can write $x = x_1 + ix_2$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint. Thus we see that

$$\begin{aligned}\mathcal{L}(xy) &= \mathcal{L}((x_1 + ix_2)y) = \mathcal{L}(x_1 y) + i\mathcal{L}(x_2 y) \\ &= \big(x_1\delta(y) + \mathcal{L}(x_1)y^*\big) + i\big(x_2\delta(y) + \mathcal{L}(x_2)y^*\big) \\ &= (x_1 + ix_2)\delta(y) + \mathcal{L}(x_1 + ix_2)y^* \\ &= x\delta(y) + \mathcal{L}(x)y^*\end{aligned}$$

for all $x, y \in \mathcal{A}$. The equation guarantees that

$$2^n x\delta(y) + 2^n\mathcal{L}(x)y^* = 2^n\mathcal{L}(xy) = \mathcal{L}(x \cdot 2^n y) = x\delta(2^n y) + 2^n\mathcal{L}(x)y^*$$

for all $x, y \in \mathcal{A}$, which implies that $x\delta(y) = x\frac{\delta(2^n y)}{2^n}$. So, by (2.15), we have the identity (2.8).

To show uniqueness of $\mathcal{L}$, let us assume that $T : \mathcal{A} \to \mathcal{A}$ is another linear mapping satisfying (2.8) and (2.9). Then we have by (2.9)

$$\begin{aligned}\|\mathcal{L}(x) - T(x)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\mathcal{L}(2^n x) - T(2^n x)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}[\,\|\mathcal{L}(2^n x) - \delta(2^n x)\| + \|\delta(2^n x) - T(2^n x)\|\,] \\ &\le \lim_{n\to\infty} \frac{1}{2^{n-1}}\sigma(2^n x) = 0\end{aligned}$$

for all $x \in \mathcal{A}$, which means that $\mathcal{L} = T$.

On the other hand, in view of (2.8), observe that

$$\begin{aligned}xy\mathcal{L}(z) + x\mathcal{L}(y)z^* + \mathcal{L}(x)y^*z^* &= xy\mathcal{L}(z) + \mathcal{L}(xy)z^* \\ &= \mathcal{L}(xy \cdot z) = \mathcal{L}(x \cdot yz) \\ &= x\mathcal{L}(yz) + \mathcal{L}(x)(yz)^* \\ &= xy\mathcal{L}(z) + x\mathcal{L}(y)z^* + \mathcal{L}(x)z^*y^*.\end{aligned}$$

This implies that $\mathcal{L}(x)[y^*, z^*] = 0$ for all $x, y, z \in \mathcal{A}$. Replacing y by y^* and z by z^* in the previous relation, we get the identity (2.10), which completes the proof. □

THEOREM 2.3. *Let $\mathcal{A}$ be a Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A}^3 \to [0, \infty)$ and $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfy the assumptions*

1. $\rho(x) = \sum_{j=0}^{\infty} 2^j \Phi\big(\frac{x}{2^j}, 0, \frac{x}{2^{j+1}}\big) < \infty \quad (x \in \mathcal{A})$,
2. $\lim_{n\to\infty} 2^n \varphi\big(\frac{x}{2^n}, y\big) = 0 \quad (x, y \in \mathcal{A})$.

Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping subject to the inequalities (2.6) *and* (2.7). *Then there exists a unique linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ with the identity* (2.8) *and*

$$\|\mathcal{L}(x) - \delta(x)\| \leq \rho(x) \tag{2.17}$$

for all $x \in \mathcal{A}$. In this case, the mapping $\mathcal{L}$ satisfies the relation (2.10).

Proof. Letting $x = y = z = 0$ in (2.11), we get $\|\delta(0)\| \leq \Phi(0, 0, 0)$. By assumption of Φ, we should have $\Phi(0, 0, 0) = 0$. Thus $\delta(0) = 0$. Replacing x, y and z with $x + y, x - y$ and y, respectively, in (2.11), we arrive at

$$\|\delta(x + y) - \delta(x - y) - 2\delta(y)\| \leq \Phi(x + y, x - y, y)$$

for all $x, y \in \mathcal{A}$. Letting $x = \frac{u}{2}, y = \frac{u}{2}$ in the last expression, we get

$$\left\|\delta(u) - 2\delta\Big(\frac{u}{2}\Big)\right\| \leq \Phi\Big(u, 0, \frac{u}{2}\Big)$$

for all $u \in \mathcal{A}$.

The remainder of the proof can be carried out similarly as the corresponding part of Theorem 2.2. □

3. Applications

In this section, we write the unit element by e.

THEOREM 3.1. *If $\mathcal{A}$ is either a semiprime Banach $*$-algebra or a unital Banach $*$-algebra in Theorem* 2.2 (*resp, Theorem* 2.3), *then δ is a linear mapping with relations* (2.8) *and* (2.10). *In this case $\mathcal{A}$ is semiprime, δ is a central mapping.*

Proof. It follows by Theorem 2.2 (resp, Theorem 2.3) that there exists a unique linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ with properties (2.8) and (2.10). In particular, considering the proof of Theorem 2.2 (resp, Theorem 2.3), we see that $x\{\delta(y) - \mathcal{L}(y)\} = 0$ for all $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is unital, set $x = e$. Then $\delta = \mathcal{L}$.

If $\mathcal{A}$ is nonunital, then $\delta(y)-\mathcal{L}(y)$ lies in the right annihilator $ran(\mathcal{A})$ of $\mathcal{A}$. If $\mathcal{A}$ is semiprime, then $ran(\mathcal{A}) = \{0\}$, so that $\delta = \mathcal{L}$.

Furthermore, replacing y by $y\delta(x)$ in (2.10) and using it, we have

$$\delta(x)y[\delta(x), z] = 0 \tag{3.1}$$

for all $x, y, z \in \mathcal{A}$. Letting y by zy in (3.1), we get $\delta(x)zy[\delta(x), z] = 0$. Left multiplication in (3.1) by z, we arrive at $z\delta(x)y[\delta(x), z] = 0$. Combining the last two expressions, we obtain $[\delta(x), z]y[\delta(x), z] = 0$. The semiprimeness of $\mathcal{A}$ implies that $[\delta(x), z] = 0$ for all $x, z \in \mathcal{A}$. Therefore $\delta(x) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. This shows that δ maps $\mathcal{A}$ into its center $Z(\mathcal{A})$, which concludes the proof. □

Corollary 3.2. *If $\mathcal{A}$ is a C^*-algebra in Theorem* 2.2 (*resp, Theorem* 2.3), *then δ is a commuting linear mapping.*

Proof. Since a C^*-algebra is semiprime [2], we have from Theorem 3.1 that the linear mapping δ satisfies the condition $[\delta(x), x] = 0$ for all $x \in \mathcal{A}$. Thereby the proof is ended. □

Theorem 3.3. *If $\mathcal{A}$ is a noncommutative prime Banach $*$-algebra in Theorem* 2.2 (*resp, Theorem* 2.3), *then δ is identically zero.*

Proof. Note that a prime algebra is semiprime. According to Theorem 3.1, δ is a linear mapping with relations (2.8) and (2.10).

Since (2.10) holds and $\mathcal{A}$ is noncommutative, choose z that does not belong to the center of $\mathcal{A}$. Then it follows from [5, Lemma 1] that δ is identically zero, which ends the proof. □

Theorem 3.4. *If $\mathcal{A}$ is a semisimple Banach $*$-algebra in Theorem* 2.2 (*resp, Theorem* 2.3), *then δ is continuous linear mapping.*

Proof. Observe that a semisimple algebra is semiprime. In view of Theorem 3.1, we see that δ is a linear mapping with (2.8).

So the mapping δ satisfies the equation

$$\delta(x^2) = x\delta(x) + \delta(x)x^* \quad \text{for all} \quad x \in \mathcal{A}. \tag{3.2}$$

Since $\mathcal{A}$ is a semisimple, we have by [6, Corollarly 2.3] that δ is continuous, which completes the proof. □

It is well known that any primitive C^*-algebra is prime [13]. Then the previous theorem has the same result for a noncommutative primitive C^*-algebra.

Now we denote by $U(A)$ the set of all unitary elements in a unital C^*-algebra $\mathcal{A}$.

THEOREM 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra. Assume that mappings $\Phi : \mathcal{A}^3 \to [0, \infty)$ and $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfy the assumptions*

1. $\sum_{j=0}^{\infty} \frac{1}{2^j} \Phi(2^j x, 2^j y, 2^j z) < \infty \quad (x, y, z \in \mathcal{A})$,
2. $\lim_{n\to\infty} \frac{1}{2^n} \varphi(x, 2^n y) = 0 \qquad (x, y \in \mathcal{A})$.

Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping subject to (2.6) *with*

$$\|\delta(xy) - x\delta(y) - \delta(sx)y^*\| \leq \varphi(x, y) \tag{3.3}$$

for all $x \in U(A), y \in \mathcal{A}$ and $s \in \mathbb{R}$. Then there exists a unique linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying (2.8) *and* (2.9). *Moreover, the mapping $\mathcal{L}$ satisfies the identity* (2.10).

Proof. As in the proof of Theorem 2.2, we obtain

$$\mathcal{L}(xy) = x\mathcal{L}(y) + \delta(sx)y^* \text{ for all } x \in U(\mathcal{A}),\ y \in \mathcal{A} \text{ and } s \in \mathbb{R}. \tag{3.4}$$

We set $x = y = e$ in (3.4) and then $\delta(se) = 0$ for all $s \in \mathbb{R}$. In view of (2.15), we see that $\mathcal{L}(e) = 0$.

Considering $s = 1$ in (3.4), we have

$$\mathcal{L}(xy) = x\mathcal{L}(y) + \delta(x)y^* \text{ for all } x \in U(\mathcal{A}) \text{ and } y \in \mathcal{A}. \tag{3.5}$$

Setting $y = e$ in (3.5) yields $\mathcal{L}(x) = \delta(x)$ for all $x \in U(\mathcal{A})$. Since $\mathcal{L}$ is linear and $\mathcal{A}$ is the linear span of its unitary elements [10], i.e., $x = \sum_{j=1}^{m} \lambda_j v_j$, where $\lambda_j \in \mathbb{C}$ and $v_j \in U(\mathcal{A})$, we have from (3.5)

$$\begin{aligned}
\mathcal{L}(xy) &= \sum_{j=1}^{m} \lambda_j \mathcal{L}(v_j y) = \sum_{j=1}^{m} \lambda_j \big(v_j \mathcal{L}(y) + \delta(v_j)y^*\big) \\
&= \sum_{j=1}^{m} \lambda_j v_j \cdot \mathcal{L}(y) + \sum_{j=1}^{m} \lambda_j \mathcal{L}(v_j)y^* \\
&= x\mathcal{L}(y) + \mathcal{L}\Big(\sum_{j=1}^{m} \lambda_j v_j\Big)y^* = x\mathcal{L}(y) + \mathcal{L}(x)y^*
\end{aligned}$$

for all $x, y \in \mathcal{A}$. This completes the proof. □

We also have the following conclusion by using the same approach as in the proof of Theorem 3.5.

THEOREM 3.6. *Let $\mathcal{A}$ be a unital C^*-algebra. Assume that mappings $\Phi : \mathcal{A}^3 \to [0, \infty)$ and $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfy the assumptions*

1. $\rho(x) = \sum_{j=0}^{\infty} 2^j \Phi\big(\frac{x}{2^j}, 0, \frac{x}{2^{j+1}}\big) < \infty \quad (x \in \mathcal{A})$,
2. $\lim_{n\to\infty} 2^n \varphi\big(x, \frac{y}{2^n}\big) = 0 \quad (x, y \in \mathcal{A})$.

Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ *is a mapping subjected to the inequalities* (2.6) *and* (3.3)*. Then there exists a unique linear mapping* $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ *with the identity* (2.8) *and the inequality* (2.17)*. Moreover, the mapping* $\mathcal{L}$ *satisfies the relation* (2.10)*.*

Here we suppose that $S = \{1, i\}$, where $i \in \mathbb{C}$. The below theorems hold for a noncommutative primitive unital C^*-algebra.

THEOREM 3.7. *Let* $\mathcal{A}$ *be a noncommutative prime unital Banach* $*$*-algebra. Assume that mappings* $\Phi : \mathcal{A}^3 \to [0, \infty)$ *and* $\varphi : \mathcal{A}^2 \to [0, \infty)$ *satisfy the assumptions of Theorem* 2.2*. Suppose that* $\delta : \mathcal{A} \to \mathcal{A}$ *is a mapping subjected to*

$$\|\delta(tx) - t\delta(y) - 2\delta(z)\| \leq \|\delta(x - y - 2z)\| + \Phi(x, y, z) \tag{3.6}$$

for all $x, y, z \in \mathcal{A}$ *and* $t \in S$ *with*

$$\|\delta(xy + yx) - x\delta(y) - \delta(x)y^* - y\delta(x) - \delta(y)x^*\| \leq \varphi(x, y) \tag{3.7}$$

for all $x, y \in \mathcal{A}$*. Then* δ *is a linear mapping with* (3.2)*.*

Proof. We first let $t = 1$ in (3.6). By applying the same method as in the proof of Theorem 2.2, we find that there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying (2.9) and (2.15). Secondly, we take into account $t = i$ in (3.6). Employing the same fashion as in the proof of Theorem 2.2, we see that $\mathcal{L}(ix) = i\mathcal{L}(x)$ for all $x \in \mathcal{A}$ and $i \in \mathbb{C}$.

Now we prove that δ satisfies the equation (3.2). We have by (3.7) that

$$\begin{aligned}
&\|\mathcal{L}(xy + yx) - x\delta(y) - \mathcal{L}(x)y^* - y\mathcal{L}(x) - \delta(y)x^*\| \\
&= \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n(xy + yx)) - 2^n x\delta(y) - \delta(2^n x)y^* - y\delta(2^n x) \\
&- 2^n\delta(y)x^*\| \leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, y) = 0,
\end{aligned}$$

which means that

$$\mathcal{L}(xy + yx) = x\delta(y) + \mathcal{L}(x)y^* + y\mathcal{L}(x) + \delta(y)x^* \text{ for all } x, y \in \mathcal{A}. \tag{3.8}$$

This leads to

$$\begin{aligned}
&x\delta(2^n y) + 2^n\mathcal{L}(x)y^* + 2^n y\mathcal{L}(x) + \delta(2^n y)x^* = \mathcal{L}(x \cdot 2^n y + 2^n y \cdot x) \\
&= 2^n\mathcal{L}(xy + yx) = 2^n(x\delta(y) + \mathcal{L}(x)y^* + y\mathcal{L}(x) + \delta(y)x^*)
\end{aligned}$$

for all $x, y \in \mathcal{A}$, which implies that

$$x\frac{\delta(2^n y)}{2^n} + \frac{\delta(2^n y)}{2^n}x^* = x\delta(y) + \delta(y)x^*.$$

It follows from (2.15) that

$$x\mathcal{L}(y) + \mathcal{L}(y)x^* = x\delta(y) + \delta(y)x^*$$

for all $x, y \in \mathcal{A}$. Setting $x = e$ in the last expression, we get $\mathcal{L} = \delta$. So the property (3.8) is as follows:

$$\delta(xy + yx) = x\delta(y) + \delta(x)y^* + y\delta(x) + \delta(y)x^* \tag{3.9}$$

for all $x, y \in \mathcal{A}$. Considering $y = x$ in (3.9), we see that δ satisfies the equation (3.2).

It remains to show that δ is a linear mapping. Now replacing y by se in (3.9), we get

$$2\delta(sx) = x\delta(se) + 2s\delta(x) + \delta(se)x^* \tag{3.10}$$

for all $x \in \mathcal{A}$ and $s \in \mathbb{R}$. On the other hand, we note from [5, Theorem 2] that $\delta(se) = 0$. So we have by (3.10) that $\delta(sx) = s\delta(x)$ for all $x \in \mathcal{A}$ and $s \in \mathbb{R}$. In particular, we know that $\delta(ix) = i\delta(x)$ for all $x \in \mathcal{A}$ and $i \in \mathbb{C}$. Hence we yield that

$$\delta(\lambda x) = \delta((s_1 + s_2 i)x) = s_1\delta(x) + s_2 i\delta(x) = (s_1 + s_2 i)\delta(x) = \lambda\delta(x)$$

for all $x \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Thus δ is linear mapping and so the theorem is proved. $\square$

As in the proof of Theorem 3.7, we arrive at the following.

THEOREM 3.8. *Let $\mathcal{A}$ be a noncommutative prime unital Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A}^3 \to [0, \infty)$ and $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfy the assumptions of Theorem* 2.3. *Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping subject to the conditions* (3.6) *and* (3.7). *Then δ is a linear mapping satisfying* (3.2).

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