JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 2, May 2019
http://dx.doi.org/10.14403/jcms.2019.32.2.179

ON SYMMETRIC BI-GENERALIZED DERIVATIONS OF LATTICE IMPLICATION ALGEBRAS

Kyung Ho Kim

Abstract. In this paper, we introduce the notion of symmetric bi-generalized derivation of lattice implication algebra L and investigated some related properties. Also, we prove that a map $F : L \times L \to L$ is a symmetric bi-generalized derivation associated with symmetric bi-derivation D on L if and only if F is a symmetric map and it satisfies $F(x \to y, z) = x \to F(y, z)$ for all $x, y, z \in L$.

1. Introduction

In order to research a logical system whose propositional value is given in a lattice. Y. Xu [11] proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems. Also, in [12], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [13] introduced the notion of filters in a lattice implication, and investigated their properties. In this paper, we introduced the notion of derivation, and considered the properties of derivations of lattice implication algebras. In this paper, we introduce the notion of symmetric bi-generalized derivation of lattice implication algebra L and investigated some related properties. Also, we prove that A map $F : L \times L \to L$ is a symmetric bi-generalized derivation associated with symmetric bi-derivation D on L if and only if F is a symmetric map and it satisfies $F(x \to y, z) = x \to F(y, z)$ for all $x, y, z \in L$.

Received February 15, 2019; Accepted March 19, 2019.
2010 Mathematics Subject Classification: Primary 16Y30, 03G25.
Key words and phrases: Lattice implication algebra, symmetric bi-derivation, symmetric bi-generalized derivation, $Fix_F(L)$, $Kerd$..

2. Preliminary

A *lattice implication algebra* is an algebra $(L;\wedge,\vee,{}',\rightarrow,0,1)$ of type $(2,2,1,2,0,0)$, where $(L;\wedge,\vee,0,1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms, for all $x,y,z\in L$,

(I1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$,
(I2) $x\rightarrow x=1$,
(I3) $x\rightarrow y=y'\rightarrow x'$,
(I4) $x\rightarrow y=y\rightarrow x=1\Rightarrow x=y$,
(I5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$,
(L1) $(x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$,
(L2) $(x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$.

If L satisfies conditions (I1) – (I5), we say that L is a *quasi lattice implication algebra*. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x\vee y\vee((x\wedge y)\rightarrow z)=1$ for all $x,y,z\in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x\leq y$ if and only if $x\rightarrow y=1$.

In a lattice implication algebra L, the following hold (see [11]), for all $x,y,z\in L$,

(u1) $0\rightarrow x=1$, $1\rightarrow x=x$ and $x\rightarrow 1=1$.
(u2) $x\rightarrow y\leq(y\rightarrow z)\rightarrow(x\rightarrow z)$.
(u3) $x\leq y$ implies $y\rightarrow z\leq x\rightarrow z$ and $z\rightarrow x\leq z\rightarrow y$.
(u4) $x'=x\rightarrow 0$.
(u5) $x\vee y=(x\rightarrow y)\rightarrow y$.
(u6) $((y\rightarrow x)\rightarrow y')'=x\wedge y=((x\rightarrow y)\rightarrow x')'$.
(u7) $x\leq(x\rightarrow y)\rightarrow y$.

In a lattice H implication algebra L, the following hold, for all $x,y,z\in L$,

(u8) $x\rightarrow(x\rightarrow y)=x\rightarrow y$.
(u9) $x\rightarrow(y\rightarrow z)=(x\rightarrow y)\rightarrow(x\rightarrow z)$.

A subset F of a lattice implication algebra L is called a *filter* of L it satisfies,

(F1) $1\in F$,
(F2) $x\in F$ and $x\rightarrow y\in F$ imply $y\in F$ for all $x,y\in L$.

Definition 2.1. Let L be a lattice implication algebra. A mapping $D(.,.) : L \times L \to L$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in L$.

Definition 2.2. Let L be a lattice implication algebra and $x \in L$. A mapping $d(x) = D(x, x)$ is called *trace* of $D(.,.)$, where $D(.,.) : L \times L \to L$ is a symmetric mapping on L.

Definition 2.3. Let L be a lattice implication algebra and $D : L \times L \to L$ be a symmetric mapping. We call D a *symmetric bi-derivation* on L if it satisfies the following condition

$$D(x \to y, z) = (x \to D(y, z)) \vee (D(x, z) \to y)$$

for all $x, y, z \in L$.

Lemma 2.4. *Let D be a symmetric bi-derivation of L and let d be a trace of D. Then the following identities hold:*
(1) $D(1, 1) = d(1) = 1$.
(2) $D(1, x) = D(x, 1) = 1$ *for every* $x \in L$.
(3) $x \le D(x, y)$ *and* $y \le D(x, y)$ *for every* $x, y \in L$.
(4) $x \le d(x)$ *for every* $x \in L$.

3. Symmetric bi-generalized derivations of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. Let L be a lattice implication algebra. A symmetric map $F : L \times L \to L$ is called a *symmetric bi-generalized derivation* of L if there exists a symmetric bi-derivation D such that

$$F(x \to y, z) = (x \to F(y, z)) \vee (D(x, z) \to y)$$

for all $x, y, z \in L$.

Example 3.2. Let $L := \{0, a, b, 1\}$ be a set with the Cayley table.

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \to 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \to y) \to y, \qquad x \wedge y = ((x' \to y') \to y')'.$$

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $D : L \times L \to L$ by

$$D(x,y) = \begin{cases} a & \text{if } (x,y) = (0,0) \\ b & \text{if } (x,y) = (0,a) \text{ or } (x,y) = (a,0) \\ 1, & \text{otherwise} \end{cases}$$

It is easy to check that D is a symmetric bi-derivation on L. Also, define a map $F : L \times L \to L$ by

$$F(x,y) = \begin{cases} a & \text{if } (x,y) = (0,0) \\ b & \text{if } (x,y) = (0,a) \text{ or } (x,y) = (a,0) \text{ or } (x,y) = (b,b) \\ 1, & \text{otherwise} \end{cases}$$

Then F is a symmetric bi-generalized derivation associated with D of L.

Proposition 3.3. *Let D be a symmetric bi-derivation of L. If F is a symmetric bi-generalized derivation associated with D of L, then $F(1,1) = 1$.*

Proof. Let F be a symmetric bi-generalized derivation associated with D of L. Then we have

$$\begin{aligned} F(1,1) &= F(1 \to 1, 1) \\ &= (1 \to F(1,1)) \vee (D(1,1) \to 1) \\ &= F(1,1) \vee 1 = 1 \end{aligned}$$

□

Proposition 3.4. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. Then the followings hold:*

(1) $F(1,x) = F(x,1) = 1$ *for all* $x \in L$,

(2) $d(1) = 1$.

Proof. (1) Let F be a symmetric bi-generalized derivation associated with D of L. Then we have

$$\begin{aligned} F(1,x) &= F(1 \to 1, x) \\ &= (1 \to F(1,x)) \vee (D(1,x) \to 1) \\ &= F(1,x) \vee 1 = 1 \end{aligned}$$

for every $x \in L$. Similarly, $F(x, 1) = 1$ for every $x \in L$.

(2) It is clear from (1). □

Proposition 3.5. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. Then we have $F(x, y) = F(x, y) \vee x$ for all $x, y \in L$.*

Proof. Let F be a symmetric bi-generalized derivation associated with D of L. Then we have

$$\begin{aligned} F(x, y) &= F(1 \to x, y) = (1 \to F(x, y)) \vee (D(1, y) \to x) \\ &= F(x, y) \vee (1 \to x) = F(x, y) \vee x \end{aligned}$$

for all $x, y \in L$. □

Proposition 3.6. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If d is a trace of F, then $d(x) = d(x) \vee x$ for all $x \in L$.*

Proof. Let d be a trace of symmetric bi-generalized derivation F associated with D of L. Then we have

$$\begin{aligned} d(x) &= F(x, x) = F(1 \to x, x) \\ &= (1 \to F(x, x)) \vee (D(1, x) \to x) \\ &= F(x, x) \vee (1 \to x) = d(x) \vee x \end{aligned}$$

for all $x \in L$. This completes the proof. □

Corollary 3.7. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If d is a trace of F, then $x \leq d(x)$ for all $x \in L$.*

Theorem 3.8. *Let $F : L \times L \to L$ be a symmetric map defined by $F(x \to y, z) = x \to F(y, z)$ on L. If D is a symmetric bi-derivation of L, then F is a symmetric bi-generalized derivation of L.*

Proof. For any $y \in L$, we have $F(1, y) = F(F(1, y) \to 1, y) = F(1, y) \to F(1, y) = 1$. Hence it follows that

$$x \to F(x, y) = F(x \to x, y) = F(1, y) = 1$$

for all $x, y \in L$. Hence $x \leq F(x, y)$ for all $x, y \in L$. Since $x \leq D(x, z)$, we have

$$D(x, z) \to y \leq x \to y \leq x \to F(y, z)$$

for all $x, y, z \in L$. Hence $F(x \to y, z) = x \to F(y, z) = (x \to F(y, z)) \vee (D(x, z) \to y)$ for all $x, y, z \in L$, which implies that F is a symmetric bi-generalized derivation associated with D on L. □

Theorem 3.9. *Let D be a symmetric bi-derivation of L and let $F : L \times L \to L$ be a symmetric bi-generalized derivation associated with D on L. Then F satisfies $F(x \to y, z) = x \to F(y, z)$ for all $x, y, z \in L$.*

Proof. Let F be a symmetric bi-generalized derivation of L and $x, y, z \in L$. Since $y \leq F(y, z)$ and $x \leq D(x, z)$, we have

$$D(x, z) \to y \leq x \to y \leq x \to F(y, z)$$

for all $x, y, z \in L$. Hence $F(x \to y, z) = (x \to F(y, z)) \vee (D(x, z) \to y) = x \to F(y, z)$ for all $x, y, z \in L$. □

As a consequence of Proposition 3.8 and 3.9, we get the following theorem.

Theorem 3.10. *Let D be a symmetric bi-derivation of L. A map $F : L \times L \to L$ is a symmetric bi-generalized derivation associated with D on L if and only if F is a symmetric map and it satisfies $F(x \to y, z) = x \to F(y, z)$ for all $x, y, z \in L$.*

Proposition 3.11. *Let D be a symmetric bi-derivation of L and let $F : L \times L \to L$ be a symmetric bi-generalized derivation associated with D on L. Then F satisfies $F(x, y \to z) = y \to F(x, z)$ for all $x, y, z \in L$.*

Proof. Since F is symmetric, by Theorem 3.9, we have

$$\begin{aligned} F(x, y \to z) &= F(y \to z, x) = y \to F(z, x) \\ &= y \to F(x, z) \end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

Proposition 3.12. *Let D be a symmetric bi-derivation of L and let $F : L \times L \to L$ be a symmetric bi-generalized derivation associated with D on L. Then F satisfies $F(x, y) = x' \to (y' \to F(0, 0))$ for all $x, y \in L$. That is, the value of F is determined by $F(0, 0)$.*

Proof. For every $x, y \in L$, we have

$$\begin{aligned} F(x, y) &= F(x'', y'') = F(x' \to 0, y' \to 0) \\ &= x' \to F(0, y' \to 0) = x' \to F(y' \to 0, 0) \\ &= x' \to (y' \to F(0, 0)). \end{aligned}$$

This completes the proof. □

Proposition 3.13. *Let D be a symmetric bi-derivation of L and let d be a trace of a symmetric bi-generalized derivation F associated with D of L. Then $d(x \to y) = x \to (x \to d(y))$ for all $x, y \in L$.*

Proof. Let d be a trace of symmetric bi-generalized derivation F associated with D on L. Then, by Theorem 3.9, we have

$$\begin{aligned} d(x \to y) &= F(x \to y, x \to y) = x \to F(y, x \to y) \\ &= x \to F(x \to y, y) = x \to (x \to F(y, y)) \\ &= x \to (x \to d(y)) \end{aligned}$$

for all $x, y \in L$. This completes the proof. □

Corollary 3.14. *Let L be a lattice H implication algebra and let D be a symmetric bi-derivation of L. If d is a trace of a symmetric bi-generalized derivation F associated with D of L, then $d(x \to y) = x \to d(y)$ for all $x, y \in L$.*

Proposition 3.15. *Let d be a trace of a symmetric bi-generalized derivation F associated with D of L. Then $d(x \vee y) = (x \to y) \to ((x \to y) \to d(y))$ for all $x, y \in L$.*

Proof. Let $x, y \in L$. Then we obtain

$$\begin{aligned} d(x \vee y) &= F(x \vee y, x \vee y) = F((x \to y) \to y, (x \to y) \to y) \\ &= (x \to y) \to F(y, (x \to y) \to y) = (x \to y) \to ((x \to y) \to F(y, y)) \\ &= (x \to y) \to ((x \to y) \to d(y)). \end{aligned}$$

This completes the proof. □

Corollary 3.16. *Let let L be a lattice H implication algebra and let D be a symmetric bi-derivation of L. If d is a trace of a symmetric bi-generalized derivation F associated with D of L, then $d(x \vee y) = (x \to y) \to d(y)$ for all $x, y \in L$.*

Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with symmetric bi-derivation D of L. For a fixed element $a \in L$, let us define a map $d_a : L \to L$ such that $d_a(x) = F(x, a)$ for every $x \in L$.

Proposition 3.17. *Let F be a symmetric bi-generalized derivation associated with D of L. Then $d_a(x \to y) = x \to d_a(y)$ for all $x, y \in L$.*

Proof. Let $x, y \in L$. Then we obtain

$$\begin{aligned} d_a(x \to y) &= F(x \to y, a) = x \to F(y, a) \\ &= x \to d_a(y). \end{aligned}$$

This completes the proof. □

Corollary 3.18. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. Then* $d_a(x \vee y) = (x \to y) \to d_a(y)$ *for all* $x, y \in L$.

Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L and let d be a trace of F. Define a set $Fix_d(L)$ by

$$Fix_d(L) = \{x \in L \mid d(x) = x\}.$$

Proposition 3.19. *Let D be a symmetric bi-derivation of L and let L be a lattice H implication algebra and let F be a symmetric bi-generalized derivation associated with D of L. If $x \in L$ and $y \in Fix_d(L)$, then $x \to y \in Fix_d(L)$.*

Proof. Let $x \in L$ and $y \in Fix_d(L)$. Then we obtain

$$d(x \to y) = x \to d(y) = x \to y$$

by Corollary 3.14. This completes the proof. □

Proposition 3.20. *Let L be a lattice H implication algebra and let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If $x \in L$ and $y \in Fix_d(L)$, then $x \vee y \in Fix_d(L)$.*

Proof. Let $x \in L$ and $y \in Fix_d(L)$. Then we obtain

$$d(x \vee y) = (x \to y) \to d(y) = (x \to y) \to y = x \vee y$$

by Corollary 3.16. This completes the proof. □

Proposition 3.21. *Let L be a lattice H implication algebra and let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If $x \leq y$ and $x \in Fix_d(L)$, then $y \in Fix_d(L)$.*

Proof. Let $x \leq y$ and $x \in Fix_d(L)$. Then we obtain

$$\begin{aligned} d(y) &= d(1 \to y) = d((x \to y) \to y) \\ &= d((y \to x) \to x) = d(y \vee x) \\ &= y \vee x \end{aligned}$$

by Proposition 3.16. Hence

$$d(y) = y \vee x = (y \to x) \to x = (x \to y) \to y = 1 \to y = y,$$

which implies that $y \in Fix_d(L)$. This completes the proof. □

Proposition 3.22. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. Then $F(x \vee y, z) = F(x, z) \vee F(y, z)$ and $F(x \wedge y, z) = F(x, z) \wedge F(y, z)$ for all $x, y, z \in L$.*

Proof. Let $x, y, z \in L$. Then we have

$$\begin{aligned} F(x \vee y, z) &= F(x'' \vee y'', z) = F((x' \wedge y')', z)) \\ &= F((x' \wedge y') \to 0, z) = (x' \wedge y') \to F(0, z) \\ &= (x' \to F(0, z)) \vee (y' \to F(0, z)) = F(x'', z) \vee F(y'', z) \\ &= F(x, z) \vee F(y, z). \end{aligned}$$

Similarly, we can prove that $F(x \wedge y, z) = F(x, z) \wedge F(y, z)$ for all $x, y, z \in L$. This completes the proof. □

Proposition 3.23. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. Then $F(x', x) = F(x, x') = 1$ for all $x \in L$.*

Proof. For every $x \in L$. Then we have

$$\begin{aligned} F(x', x) &= F(x \to 0, x) = x \to F(0, x) \\ &= x \to F(x, 0) = F(x \to x, 0) \\ &= F(1, 0) = 1. \end{aligned}$$

by Proposition 3.4. □

Proposition 3.24. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If $x' \le y$ for every $x, y \in L$, then $F(y, x) = 1$.*

Proof. For every $x, y \in L$, we know that $x' \le y$ implies $x' \vee y = y$. Hence

$$\begin{aligned} F(y, x) &= F(x' \vee y, x) = F(x', x) \vee F(y, x) \\ &= F(x \to 0, x) \vee F(y, x) = x \to F(0, x) \vee F(y, x) \\ &= x \to F(x, 0) \vee F(y, x) = F(x \to x, 0) \vee F(y, x) \\ &= F(1, 0) \vee F(y, x) = 1 \vee F(y, x) = 1. \end{aligned}$$

This completes the proof. □

Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L and let d be a trace of F. Define a set $Kerd$ by

$$Kerd = \{x \in L \mid F(x, x) = d(x) = 1\}.$$

Proposition 3.25. *Let L be a lattice H implication algebra and D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L. If $x \in L$ and $y \in Kerd$, then $x \to y \in Kerd$.*

Proof. Let $x \in L$ and $y \in Kerd$. Then we obtain

$$d(x \to y) = x \to d(y) = x \to 1 = 1$$

by Corollary 3.14. This implies that $x \to y \in Kerd$. □

Proposition 3.26. *Let D be a symmetric bi-derivation of L and let F be a symmetric bi-generalized derivation associated with D of L If $x \in L$ and $y \in Fix_d(L)$, then $x \vee y \in Fix_d(L)$.*

Proof. Let $x \in L$ and $y \in Kerd$. Then we have

$$\begin{aligned} F(y, x \vee y) &= F(x \vee y, y) \\ &= F(((x \to y) \to y, y)) \\ &= (x \to y) \to F(y, y) = (x \to y) \to 1) \\ &= 1, \end{aligned}$$

which implies that $F(y, x \vee y) = 1$. Hence we have

$$\begin{aligned} d(x \vee y) &= F(x \vee y, x \vee y) \\ &= F(((x \to y) \to y, x \vee y)) \\ &= (x \to y) \to F(y, x \vee y) = (x \to y) \to 1 \\ &= 1, \end{aligned}$$

This implies that $x \vee y \in Kerd$ for all $x \in L$. □

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Department of Mathematics,
Korea National University of Transportation
Chungju 27469, Republic of Korea
E-mail: ghkim@ut.ac.kr