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DECOMPOSITION PROPERTY FOR C^1 GENERIC DIFFEOMORPHISMS

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Abstract. C^1 generically, if a diffeomorphism $f : M \to M$ of a closed smooth Riemannian manifold M has the asymptotic average shadowing property or the average shadowing property then f has a decomposition property.

1. Introduction

Let M be a closed smooth Riemannian manifold without boundary and let Diff(M) be the space of C^1 diffeomorphisms of M. A closed f-invariant set Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. A closed f-invariant set $\Lambda \subset M$ is called *basic* if Λ is transitive, and the periodic points are dense. We say that f has a *decomposition property* if the nonwandering set

$$M = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n,$$

where each Λ_i is a basic sets.

A closed f-invariant set Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then we say that f is Anosov. We say that f satisfies *Axiom A* if the nonwandering set $\Omega(f) = \overline{P(f)}$ is hyperbolic, where $P(f)$ is the set of periodic points of f. Smale [21] proved that if a diffeomorphim f satisfies Axiom A then f admits a decomposition, that is, the nonwandering set $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$, where each Λ_i is basic sets. In dynamical systems, the shadowing

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properties and decomposition theorem are important concepts. For a homeomorphism $f : M \to M$ of a compact metric space, Aoki [2] proved that if the nonwandering set $\Omega(f)$ is expansive and has the shadowing property then it admits a decomposition theorem. In the paper, we consider that a types of the shadowing properties (asymptotic average shadowing property, or the average shadowing property).

For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is a *δ-average pseudo orbit* of f if there is $K = K_\delta > 0$ such that for all $n \geq K$ and $k \in \mathbb{Z}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average shadowing property* if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any δ-average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset M$ there is $z \in M$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon \text{ and } \limsup_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Blank [3] studied the average shadowing property, and many authors published in [8, 9, 13, 15, 17, 18, 19, 20]. For instance, Sakai [20] proved that for a surface, if a diffeomorphism f has the robustly average shadowing property then it is Anosov. Lee and Wen [18] proved that if a diffeomorphism f has the robustly average shadowing property on transitive sets then it admits a dominated splitting.

A sequence $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is an *asymptotically average pseudo orbit* of f if

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} d(f(x_i), x_{i+1}) = 0.$$

We say that f has the *asymptotic average shadowing property* if for any asymptotically average pseudo orbit $\{x_i\} \subset M$ of f there is $z \in M$ such that

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} d(f^i(z), x_i) = 0.$$

Gu [5] studied the asymptotically average shadowing property, and many authors published in [6, 7, 10, 11, 12, 13, 15, 16, 17]. For instance, Honary and Bahabadi [6] proved that for a surface, if a diffeomorphism f has the robustly asymptotically average shadowing property then it is Anosov. Lee [12] proved that if a diffeomorphism f has the robustly asymptotically average shadowing property on a chain transitive set then it admits a dominated splitting.

We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case $\mathcal{G}$ is dense in Diff(M). A property "P" is said to be *generic* if "P" holds for all diffeomorphisms which belong to some residual subset of Diff(M). Lee [14] proved that C^1 generically, if a diffeomorphism f of a three dimensional manifold M has the asymptotic average shadowing property or average shadowing property then it is Anosov. About the results, we consider the average shadowing property (the average shadowing property) and the decomposition property, for a C^1 generic diffeomorphism. The following is a main theorem of this paper.

Theorem A For C^1 generic $f \in \text{Diff}(M)$, if f has the following properties,

(a) asymptotic average shadowing property;
(b) average shadowing property,

then f has a decomposition property.

2. Proof of Theorem A

An invariant closed set $C \subset M$ is called a *chain transitive* if for any $\delta > 0$ and $x, y \in C$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^{n}(n \geq 1) \subset C$ such that $x_0 = x$ and $x_n = y$. We say that f is *chain transitive* if $C = M$. It is clear that the transitive set Λ is the chain transitive set C, but the converse is not true. Gu [5] proved that if a diffeomorphism f has the asymptotic average shadowing property then it is chain transitive. Park and Yong [19] proved that if a diffeomorphism f has the average shadowing property then it is chain transitive.

LEMMA 2.1. [19, 5] *If f has the asymptotic average shadowing property or the average shadowing property then it is chain transitive.*

A periodic point p of f is a *sink* if all its eigenvalues have modulus less than 1, and p is a *source* if all its eigenvalues have modulus greater than 1.

LEMMA 2.2. [14, Lemma 2.1] *If f is chain transitive then f has neither sinks nor sources.*

Let p be a hyperbolic periodic point of f. Denoted by

$$H(p) = \overline{W^s(p) \pitchfork W^u(p)},$$

where $W^s(p)$ is the stable manifold of f and $W^u(p)$ is the unstable manifold of p. A point $x \in M$ is called *chain recurrent* if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1)$ such that $x_0 = x$ and $x_n = x$. Denote by $\mathcal{CR}(f)$ the set of all chain recurrent points of f. We define a relation $\leftrightsquigarrow$ on $\mathcal{CR}(f)$ by $x \leftrightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^n$ such that $x_0 = x$ and $x_n = y$ and a finite δ-pseudo orbit $\{w_i\}_{i=0}^n$ such that $w_0 = y$ and $w_n = x$. Then we know that the relation $\leftrightsquigarrow$ is an equivalence relation on $\mathcal{CR}(f)$. The equivalence classes are called the *chain recurrence classes* of f, denote by C. Note that if the class C have a hyperbolic periodic point p then we denote as $C(p)$. It is known that $H(p) \subset C(p)$. Bonatti and Crovisier [4] proved the following lemma.

LEMMA 2.3. *There is a residual set $\mathcal{G}_1 \subset \text{Diff}(M)$ such that (i) $\overline{P(f)} = \mathcal{CR}(f)$, and (ii) a homoclinic class $H(p)$ which contains a hyperbolic periodic point p is a chain recurrence class $C(p)$ which contains a hyperbolic periodic point p.*

For $f \in$ Diff(M), we say that a compact f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$||Df^n|_{E(x)}|| \cdot ||Df^{-n}|_{F(f^n(x))}|| \leq C\lambda^n.$$

THEOREM 2.4. [1, Theorem 2.2] *There is a residual set $\mathcal{G}_2 \subset \text{Diff}(M)$ such that either (a) or (b) holds:*

(a) *the nonwandering set $\Omega(f)$ admits a decomposition*

$$\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \Lambda_k,$$

where the sets Λ_i are pairwise disjoint compact f-invariant, each of which is the union of chain recurrence classes and admits a some dominated splitting;

(b) *there are infinitely many periodic sinks or sources of f.*

Proof of Theorem A (a). Let $f \in \mathcal{G}_1 \cap \mathcal{G}_2$ have the asymptotic average shadowing property. Then by Lemma 2.1, f is chain transitive and so, $\mathcal{CR}(f) = M$. Since f is chain transitive, according to Lemma 2.2, f does not contains sinks or sources. Thus by Theorem 2.4 (a), $\Omega(f)$ admits a decomposition, that is, $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \Lambda_k$, where the sets Λ_i are pairwise disjoint compact f-invariant, each of which is the union of chain recurrence classes. Since $f \in \mathcal{G}_1$ and Λ_i is a chain recurrence class which contains a hyperbolic periodic point p, Λ_i is a homoclinic class $H(p)$ which contains a hyperbolic periodic point p. By Smale's homoclinic transverse theorem, $H(p) = \overline{\{q \in P(f) : q \sim p\}}$, where $q \sim p$ means that $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. Since $H(p)$ is closed, transitive and f-invariant set, we know that each Λ_i is a basic set. Thus if $f \in \mathcal{G}_1 \cap \mathcal{G}_2$ have the asymptotic average shadowing property then $\Omega(f) = \mathcal{CR}(f) = M$ and so, f has a decomposition property. □

Proof of Theorem A (b). Let$f \in \mathcal{G}_1 \cap \mathcal{G}_2$ have the average shadowing property. Then by Lemma 2.1, f is chain transitive. Since f is chain transitive, according to Lemma 2.2, f does not contains sinks or sources. Thus by Theorem 2.4 (a), $\Omega(f)$ admits a decomposition theorem. As in proof of the previous case, we have that f has a decomposition property. □

For a C^1 generic $f \in \mathrm{Diff}(M)$, a transitive f is not expansive, if f has the shadowing property then f has the decomposition property. It is the following.

Corollary 2.5. *For C^1 generic $f \in \mathrm{Diff}(M)$, if f is transitive and has the shadowing property then f has a decomposition property.*

Proof. Let $f \in \mathcal{G}_1 \cap \mathcal{G}_2$ have the shadowing property. Since f is transitive, f has neither sinks nor sources. Then the nonwandering set $\Omega(f)$ admits a decomposition

$$\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \Lambda_k.$$

Here each sets Λ_i are basic sets. Since f is transitive, $\Omega(f) = M$ and so, f has a decomposition property. □

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