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NONABELIAN GROUP ACTIONS ON 3-DIMENSIONAL NILMANIFOLDS WITH THE FIRST HOMOLOGY $\mathbb{Z}^2 \oplus \mathbb{Z}_2$

Mina Han*, Daehwan Koo**, and Joonkook Shin***

Abstract. We study free actions of finite nonabelian groups on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_2$, up to topological conjugacy. We show that there exist three kinds of nonabelian group actions in π_1, two in π_2 or $\pi_{5,i} (i = 1, 2, 3)$, one in the other cases, and clarify what those groups are.

1. Introduction

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Let Γ be any lattice of $\mathcal{H}$ and $\mathcal{Z}(\mathcal{H})$ be the center of $\mathcal{H}$. Then $\mathbb{Z} = \Gamma \cap \mathcal{Z}(\mathcal{H})$ and $\Gamma \big/ \Gamma \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H} \big/ \mathcal{Z}(\mathcal{H})$, respectively. Therefore, the lattice Γ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is, there is an exact sequence:

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1.$$

Let a, b, and c be elements of Γ such that the images of a and b in $\mathbb{Z}^2$ generate $\mathbb{Z}^2$ and c generates the center $\mathbb{Z}$. Then it is known that such Γ is isomorphic to one of the following groups, for some p :

$$\Gamma_p = \langle a, b, c \ \mid \ [b, a] = c^p, \ [c, a] = [c, b] = 1 \rangle, \quad p \neq 0,$$

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*** The corresponding author.

where $[b,a]=b^{-1}a^{-1}ba$. This group is realized as a uniform lattice of $\mathcal{H}$ if one takes

$$a=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix},\ b=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\ c=\begin{bmatrix}1&0&\frac{1}{p}\\0&1&0\\0&0&1\end{bmatrix}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Remark that Γ_p is equal to Γ_{-p}. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p;\mathbb{Z})=\Gamma_p/[\Gamma_p,\Gamma_p]=\mathbb{Z}^2\oplus\mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds

$$\mathcal{N}_p=\mathcal{H}/\Gamma_p$$

covered by the standard nilmanifold $\mathcal{N}_1$.

The classification of finite group actions on a 3-dimensional nilmanifold can be understood by the works of Bieberbach, Heil and Waldhausen [6, 7, 12]. Free actions of cyclic, abelian and finite group on the 3-torus were studied in [8], [11] and [5], respectively. It is known ([4; Proposition 6.1.]) that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry. It is interesting ([3]) that if a finite group G acts freely on the (standard) 3-dimensional nilmanifold $\mathcal{N}_1$ with the first homology $\mathbb{Z}^2$, then either G is cyclic or there does not exist any finite group acting freely on the standard nilmanifold $\mathcal{N}_1$ which yields an infra-nilmanifold homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$. Free actions of finite abelian groups on the 3-dimensional nilmanifold $\mathcal{N}_p$ with the first homology $\mathbb{Z}^2\oplus\mathbb{Z}_p$ were classified in [1]. Recently, the results of [1] were generalized by changing the finite abelian conditions to finite group conditions in [10], where the authors classified the free actions of finite groups on 3-dimensional nilmanifolds $\mathcal{N}_p$ with the first homology $\mathbb{Z}^2\oplus\mathbb{Z}_p$ by using the method in [1], up to topological conjugacy. However, since the finite groups acting freely on $\mathcal{N}_p$ are represented by generators in [10], it is difficult to know exactly what those finite groups are.

In this paper we focus on the free actions of finite nonabelian groups on $\mathcal{N}_2$ with $\mathrm{H}_1(\mathcal{H}/\Gamma_2;\mathbb{Z})=\mathbb{Z}^2\oplus\mathbb{Z}_2$. Note that our results cannot be obtained directly from [10], because of many unknown variables. But when $p=2$ and $n=1$, we can find a necessary and sufficient conditions for being a normal nilpotent subgroup of an almost Bieberbach group, and classify those nonabelian groups. This classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

2. Criteria for affine conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_2$. The problem will be reduced to a purely group-theoretic one. We quote most of the Introduction and Section 2 of [1] in this section for the reader's convenience.

Note that if $M = \mathcal{H}/\pi$ is a 3-dimensional infra-nilmanifold, then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π', where π' is a subgroup of

$$\mathrm{Aff}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{R})$$

such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$.

The following is the list for 15 kinds of the 3-dimensional almost Bieberbach groups imbedded in $\mathrm{Aff}(\mathcal{N}) = \mathcal{N} \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}))$ ([10, p.1414]). We shall use

$$t_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{K} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

respectively, where I is the identity in $\mathrm{Aut}(\mathcal{N}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. In each presentation, n is any positive integer and t_3 is central except π_3 and π_4. Note that t_1 and t_2 are fixed, but K in t_3 varies for each $\pi_{i,j}$. For example, $K = n$ for π_1; $K = 2n$ for π_2, etc.

$\pi_1 = \langle\, t_1, t_2, t_3 \mid [t_2, t_1] = t_3^n \,\rangle$,

$\pi_2 = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n},\ \alpha^2 = t_3, \alpha t_1 \alpha^{-1} = t_1^{-1}, \alpha t_2 \alpha^{-1} = t_2^{-1} \,\rangle$,

$$\begin{aligned}\pi_3 = \langle\, t_1, t_2, t_3, \alpha \mid{} & [t_2, t_1] = t_3^{2n},\ [t_3, t_1] = [t_3, t_2] = 1,\ \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ & \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 = t_2^{-1} \alpha t_3^{-n},\ \ \alpha^2 = t_1 \,\rangle,\end{aligned}$$

$$\begin{aligned}\pi_4 = \langle\, t_1, t_2, t_3, \alpha, \beta \mid{} & [t_2, t_1] = t_3^{4n},\ [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1, \\ & \beta t_3 \beta^{-1} = t_3^{-1}, \alpha t_1 = t_1^{-1} \alpha t_3^{2n}, \alpha t_2 = t_2^{-1} \alpha t_3^{-2n}, \\ & \alpha^2 = t_3, \beta^2 = t_1, \beta t_1 \beta^{-1} = t_1, \beta t_2 = t_2^{-1} \beta t_3^{-2n}, \\ & \alpha\beta = t_1^{-1} t_2^{-1} \beta \alpha t_3^{-(2n+1)} \,\rangle,\end{aligned}$$

$$\pi_{5,1} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{4n-2},\ \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3 \,\rangle,$$
$$\pi_{5,2} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3^3 \,\rangle,$$
$$\pi_{5,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3 \,\rangle,$$
$$\pi_{6,1} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle,$$
$$\pi_{6,2} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle,$$
$$\pi_{6,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n-2}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle,$$
$$\pi_{6,4} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{3n-1}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle,$$
$$\pi_{7,1} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle,$$
$$\pi_{7,2} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n-2}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle,$$
$$\pi_{7,3} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle,$$
$$\pi_{7,4} = \langle\, t_1, t_2, t_3, \alpha |\, [t_2, t_1] = t_3^{6n-4},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle.$$

Let $(G, \mathcal{N}_p)$ be a free affine action of a finite group G on the nilmanifold $\mathcal{N}_p$. Then $\mathcal{N}_p/G$ is an infra-nilmanifold. Let $\pi = \pi_1(\mathcal{N}_p/G)$, and $\Gamma_p = \pi_1(\mathcal{N}_p)$. Then π is an almost Bieberbach group. In fact, since the covering projection $\mathcal{N}_p \to \mathcal{N}_p/G$ is regular, Γ_p is a normal subgroup of π.

Definition 2.1. Let $\pi \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of π. We say that (N_1, π) is *affinely conjugate* to (N_2, π), denoted by $N_1 \sim N_2$, if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t, T)\pi(t, T)^{-1} = \pi$ and $(t, T)N_1(t, T)^{-1} = N_2$.

Suppose there are two normal subgroups N, N' of π. The two actions of $\pi/N, \pi/N'$ are *equivalent* if and only if there exists a homeomorphism f of $\mathcal{H}$ which conjugates the pair (N, π) into (N', π). Of course, such a conjugation is achieved by an affine map $f \in \mathrm{Aff}(\mathcal{H})$.

Our classification problem of free finite group actions $(G, \mathcal{N}_p)$ with

$$\pi_1(\mathcal{N}_p/G) \cong \pi$$

can be solved by finding all normal nilpotent subgroups N of π each of which is isomorphic to Γ_p, and classify (N, π) up to affine conjugacy. This procedure is a purely group-theoretic problem and can be handled by affine conjugacy.

The following proposition is a working criterion for determining all normal nilpotent subgroups of π isomorphic to Γ_p.

PROPOSITION 2.2 ([1, Proposition 3.1]). *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by a set of generators*

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle,$$

where d_1, d_2 are divisors of p; K is determined by $t_3^K = [t_2, t_1]$; $0 \leq m < d_2$, $0 \leq n_i < \frac{Kd_1d_2}{p}$ $(i = 1, 2)$.

3. Free actions of finite nonabelian groups on the nilmanifold $\mathcal{N}_2$

In this section, we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_2$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_i (i = 1, 2, 5, 6, 7)$. Note that we deal only with $n = 1$ to clarify those groups in this paper. Nonabelian groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_2$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$ were studied in [9] . This, as in other parts of calculations, was done by the program Mathematica [13] and hand-checked.

THEOREM 3.1. *(π_1) Suppose F is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_1$. Then F is isomorphic to either the dihedral group D_4 or the quaternion group Q_8.*

Proof. Note that $\pi_1 = \langle\, t_1, t_2, t_3 \mid [t_2, t_1] = t_3^n \,\rangle$.

Let $N = \langle\, t_1^{d_1} t_2^{m} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{nd_1d_2}{p}} \,\rangle$ be a normal nilpotent subgroup of π_1 and isomorphic to Γ_2. Take $n = 1$. Then $[\pi_1, \pi_1] = t_3$. Since d_1, d_2 are divisors of 2, $0 \leq m < \bar{d} = gcd(d_1, d_2)$, and $\frac{pm}{d_1d_2} \in \mathbb{Z}$ by proposition 2.2, we have the following three cases.

(i) When $d_1 = 1, d_2 = 2$: In this case, we have $N = \langle\, t_1,\ t_2^2,\ t_3 \,\rangle$ and $\pi_1/N \cong \mathbb{Z}_2$.

(ii) When $d_1 = 2, d_2 = 1$: There exists only one normal nilpotent subgroup $N' = \langle\, t_1^2,\ t_2,\ t_3 \,\rangle$. It is easily checked that $N' \sim N = \langle\, t_1,\ t_2^2,\ t_3 \,\rangle$.

(iii) When $d_1 = 2, d_2 = 2$: There exist 3 affinely non-conjuate normal nilpotent subgroups:

$$N_1 = \langle\, t_1^2,\ t_2^2,\ t_3^2 \,\rangle, \quad N_2 = \langle\, t_1^2,\ t_2^2 t_3,\ t_3^2 \,\rangle, \quad N_3 = \langle\, t_1^2 t_3,\ t_2^2 t_3,\ t_3^2 \,\rangle.$$

Thus we can obtain that $\pi_1/N_1 \cong D_4$, $\pi_1/N_2 \cong D_4$, and $\pi_1/N_3 \cong Q_8$. $\square$

The following lemma gives a necessary condition for being a normal nilpotent subgroup of an almost Bieberbach group which is isomorphic to Γ_p.

LEMMA 3.2 ([10, Lemma 3.1]). *Let N be a normal nilpotent subgroup of an almost Bieberbach group $\pi_2, \pi_{5,i} (i = 1, 2, 3)$ or $\pi_{7,j} (j = 1, 2, 3, 4)$ which is isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$$N_1 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \qquad N_2 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

$$N_3 = \langle t_1^{d_1} t_2^m t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \quad N_4 = \langle t_1^{d_1} t_2^m t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

where d_1, d_2 are divisors of p; $0 \le m < \bar{d} = gcd(d_1, d_2)$, $\frac{pm}{d_1d_2} \in \mathbb{Z}$ in the case of π_2,

$\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} \in \mathbb{Z}$ and d_1 is a common divisor of m and d_2 in the case of $\pi_{5,i}$,

$\frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}$ and d_1 is a common divisor of m and d_2 in the case of $\pi_{7,j}$.

The following proposition is a working criterion for affine conjugacy among normal nilpotent subgroups of π_2.

PROPOSITION 3.3 ([10, Proposition 3.3]). *Let N_i $(i = 1, 2, 3, 4)$ be a normal nilpotent subgroup of π_2 in Lemma 3.2 and isomorphic to Γ_p. Then we have the following:*

(1) *$N_1 \sim N_2$ if and only if $m = 0$, $d_1 = p$.*
(2) *$N_1 \sim N_3$ if and only if $m = 0$, $d_2 = p$.*
(3) *$N_1 \sim N_4$ if and only if $m = 0$, $d_1 = d_2 = p$.*
(4) *$N_2 \sim N_3$ if and only if $m = 0$, $d_1 = d_2$.*
(5) *$N_2 \sim N_4$ if and only if either $m = 0$, $d_2 = p$, or $2m = d_2$, $2d_1 = p$.*
(6) *$N_3 \sim N_4$ if and only if $m = 0$, $d_1 = p$.*

Now by using Lemma 3.2 and Proposition 3.3, we can obtain the following result.

THEOREM 3.4. *(π_2) Suppose F is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_2$. Then F is isomorphic to either the quaternion group Q_8 or the central product group $C_8 \circ D_4$.*

Proof. For the case of $n = 1$, we have

$$\pi_2 = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^2,\ \alpha^2 = t_3,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_2 \alpha^{-1} = t_2^{-1}\,\rangle.$$

Let N be a normal nilpotent subgroup of π_2 which is isomorphic to Γ_2. Then by Lemma 3.2, N can be represented by one of the following sets of generators,

$$N_1 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2},\ t_3^{d_1 d_2} \rangle, \qquad N_2 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2} t_3^{\frac{d_1 d_2}{2}},\ t_3^{d_1 d_2} \rangle,$$

$$N_3 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{d_1 d_2}{2}},\ t_2^{d_2},\ t_3^{d_1 d_2} \rangle, \qquad N_4 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{d_1 d_2}{2}},\ t_2^{d_2} t_3^{\frac{d_1 d_2}{2}},\ t_3^{d_1 d_2} \rangle,$$

where d_1, d_2 are divisors of $p = 2$ and $0 \le m < gcd(d_1, d_2)$, $\frac{pm}{d_1 d_2} \in \mathbb{Z}$.

(i) When $d_1 = d_2 = 1$: $m = 0$. Since $\frac{d_1 d_2}{2} = \frac{1}{2} \notin \mathbb{Z}$, N_2, N_3, and N_4 do not occur. Thus we have only one normal subgroup $N = \langle\, t_1,\ t_2,\ t_3\,\rangle$ and $\pi_2/N = \langle\, \alpha N\,\rangle \cong \mathbb{Z}_2$.

(ii) When $d_1 = 1, d_2 = 2$: Since $0 \le m < 1$, we have $m = 0$. Thus $\frac{2m}{d_1 d_2} = 0 \in \mathbb{Z}$ and the possible normal nilpotent subgroups are

$$N_1 = \langle\, t_1,\ t_2^2,\ t_3^2\,\rangle, \qquad N_2 = \langle\, t_1,\ t_2^2 t_3,\ t_3^2\,\rangle,$$

$$N_3 = \langle\, t_1 t_3,\ t_2^2,\ t_3^2\,\rangle, \qquad N_4 = \langle\, t_1 t_3,\ t_2^2 t_3,\ t_3^2\,\rangle.$$

It is not hard to see that $N_1 \sim N_3$ and $N_2 \sim N_4$ by Proposition 3.3. The normality can be easily checked as follows:

$$\alpha t_2^2 \alpha^{-1} = t_2^{-2} \in N_1, \quad \alpha(t_1)\alpha^{-1} = t_1^{-1} \in N_1, \quad \alpha(t_2^2 t_3)\alpha^{-1} = t_2^{-2} t_3 \in N_2.$$

Since $N_1 \supset [\pi_2,\ \pi_2] = \langle t_1^2, t_2^2, t_3^2 \rangle$, we can conclude that π_2/N_1 is abelian and

$$\pi_2/N_1 = \langle\, t_1, t_2, t_3, \alpha \rangle / \langle\, t_1,\ t_2^2,\ t_3^2 \rangle\ = \langle\, \alpha N_1, t_2 N_1 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2.$$

Note that π_2/N_2 is nonabelian and

$$\pi_2/N_2 = \langle\, t_1, t_2, t_3, \alpha \rangle / \langle\, t_1,\ t_2^2 t_3,\ t_3^2 \rangle\ \cong \langle\, \alpha N_2, t_2 N_2 \rangle.$$

Since

$$(t_2 N_2)^2 = (t_2^2) N_2 = (t_2^2 t_3) t_3 N_2 = t_3 N_2 = \alpha^2 N_2,$$

$$(t_2 N_2)^4 = N_2, \quad (\alpha N_2)^4 = t_3^2 N_2 = N_2,$$

$$(\alpha N_2)(t_2 N_2)(\alpha N_2)^{-1} = (\alpha t_2 \alpha^{-1}) N_2 = t_2^{-1} N_2 = (t_2 N_2)^{-1},$$

we can conclude that

$$\pi_2/N_2 = \langle\, t_2 N_2,\ \alpha N_2 \mid (t_2 N_2)^4 = 1,\ (t_2 N_2)^2 = (\alpha N_2)^2,$$
$$(\alpha N_2)(t_2 N_2)(\alpha N_2)^{-1} = (t_2 N_2)^{-1} \rangle.$$

Let $a = t_2N_2$ and $b = \alpha\, N_2$. Then this group is isomorphic to the quaternion group

$$Q_8 = \langle\, a, b \mid a^4 = 1,\, b^2 = a^2,\, bab^{-1} = a^{-1}\rangle.$$

(iii) When $d_1 = 2, d_2 = 1$: There exist 2 affinely non-conjuate normal subgroups

$$N_1' = \langle t_1^2,\ t_2,\ t_3^2\rangle, \quad N_3' = \langle t_1^2 t_3,\ t_2,\ t_3^2\rangle.$$

It is easy to see that $N_1' \sim N_1$ and $N_3' \sim N_2$ as in the case (ii).

(iv) When $d_1 = d_2 = 2$: Since $\frac{pm}{d_1 d_2} = \frac{2m}{4} = \frac{m}{2} \in \mathbb{Z}$, we have $m = 0$. Since $N_1 \sim N_2$, $N_1 \sim N_3$, and $N_2 \sim N_4$ by Proposition 3.3, there exists only one normal nilpotent subgroup $N_1 = \langle t_1^2,\ t_2^2,\ t_3^4\rangle$ of π_2. Note that π_2/N_1 is nonabelian and the normality can be easily checked. Thus we have

$$\pi_2/N_1 = \langle\, t_1, t_2, t_3, \alpha\rangle/\langle\, t_1^2,\, t_2^2,\, t_3^4\rangle\ = \langle\, \alpha N_1, t_1 N_1, t_2 N_1\rangle.$$

From the following relations

$$(\alpha N_1)^2 = t_3 N_1,\ (\alpha N_1)^4 = t_3^2 N_1,\ (\alpha N_1)^8 = N_1,\ (t_1 N_1)^2 = (t_2 N_1)^2 = N_1,$$
$$(t_2N_1)(t_1N_1)(t_2N_1) = (t_2 t_1 t_2)N_1 = (t_1 t_2^2 t_3^2)N_1 = (t_1\alpha^4)N_1,$$

we have

$$\begin{aligned}\pi_2/N_2 =&\langle\, \alpha N_1, t_1N_1, t_2N_1 \mid (\alpha N_1)^8 = (t_1N_1)^2 = (t_2N_1)^2 = 1,\\ &(\alpha N_1)^4 = (t_3N_1)^2, (t_2N_1)(t_1N_1)(t_2N_1) = (\alpha N_1)^4(t_1N_1),\\ &(\alpha N_1)(t_1N_1) = (t_1N_1)(\alpha N_1), (\alpha N_1)(t_2N_1) = (t_2N_1)(\alpha N_1)\rangle.\end{aligned}$$

Let $a = \alpha\, N_1$, $b = t_1N_1$, and $c = t_2N_1$. Then this group is isomorphic to the central product group
$C_8 \circ D_4 = \langle\, a, b, c \,|\, a^8 = c^2 = 1,\, b^2 = a^4,\, ab = ba,\, ac = ca,\, cbc = a^4 b\rangle.$ □

The following proposition is a working criterion for affine conjugacy among normal nilpotent subgroups of $\pi_{5,i}(i = 1, 2, 3)$.

Proposition 3.5 ([10, Proposition 3.12]). *Let $N_j (j = 1, 2, 3, 4)$ be a normal nilpotent subgroup of $\pi_{5,i}(i = 1, 2, 3)$ in Lemma 3.2 and isomorphic to Γ_p. Then we have the following :*

(1) $N_1 \sim N_4$ if and only if $m = 0,\ d_1 = d_2 = p$.

(2) $N_2 \sim N_3$ if and only if $m = 0,\ d_1 = d_2$.

(3) $N_1 \nsim N_2,\ \ N_1 \nsim N_3,\ \ N_2 \nsim N_4\ \ N_3 \nsim N_4$.

Now by using Lemma 3.2 and Proposition 3.5, we can obtain the following result.

THEOREM 3.6. *($\pi_{5,i}$) Suppose F is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_{5,i}(i = 1, 2, 3)$. Then F is isomorphic to the modular maximal-cyclic group M_{16} or $D_4.C_8$(the non-split extension by D_4 of C_8 acting via $C_8/C_4 = C_2$) in the case $\pi_{5,1}$ and M_{32} or $D_4.C_{16}$(the non-split extension by D_4 of C_{16} acting via $C_{16}/C_4 = C_2$) in the case $\pi_{5,2}$ and $\pi_{5,3}$.*

Proof. We shall deal with the case $\pi_{5,2}$. Note that when $n = 1$, we have

$$\pi_{5,2} = \langle\, t_1, t_2, t_3, \alpha |\; [t_2, t_1] = t_3^4,\; \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3^3 \,\rangle.$$

Let N be a normal nilpotent subgroup of $\pi_{5,2}$ which is isomorphic to Γ_2. Then by Lemma 3.2, N can be represented by one of the following sets of generators

$$N_1 = \langle t_1^{d_1} t_2^m,\; t_2^{d_2},\; t_3^{2d_1d_2}\rangle, \qquad N_2 = \langle t_1^{d_1} t_2^m,\; t_2^{d_2} t_3^{d_1d_2},\; t_3^{2d_1d_2}\rangle,$$
$$N_3 = \langle t_1^{d_1} t_2^m t_3^{d_1d_2},\; t_2^{d_2},\; t_3^{2d_1d_2}\rangle, \qquad N_4 = \langle t_1^{d_1} t_2^m t_3^{d_1d_2},\; t_2^{d_2} t_3^{d_1d_2},\; t_3^{2d_1d_2}\rangle,$$

where d_1, d_2 are divisors of $p = 2$, $0 \le m < d_2$, $\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} \in \mathbb{Z}$, and d_1 is a common divisor of m and d_2. Hence there are three possibilities.

(i) When $d_1 = d_2 = 1$: $m = 0$. Then the possible normal nilpotent subgroups are

$$N_1 = \langle\, t_1,\, t_2,\, t_3^2 \,\rangle, \qquad N_2 = \langle\, t_1,\, t_2 t_3,\, t_3^2 \,\rangle,$$
$$N_3 = \langle\, t_1 t_3,\, t_2,\, t_3^2 \,\rangle, \qquad N_4 = \langle\, t_1 t_3,\, t_2 t_3,\, t_3^2 \,\rangle.$$

Since $\alpha(t_2t_3)\alpha^{-1} = t_1^{-1}t_3 \notin N_2$ and N_2 is affinely conjugate to N_3 by Proposition 3.5, we have two normal nilpotent subgroups N_1, N_4. Since $N_1 \supset [\pi_{5,2},\, \pi_{5,2}] = \langle t_1t_2, t_2^2, t_3^4\rangle$, $\pi_{5,2}/N_1$ is abelian and $\pi_{5,2}/N_1 = \langle\, \alpha N_1 \,\rangle \cong \mathbb{Z}_8$.

Similarly, we can get $\pi_{5,2}/N_4 = \langle\, \alpha N_4 \,\rangle \cong \mathbb{Z}_8$.

(ii) When $d_1 = 1, d_2 = 2$: Note that $0 \le m < d_2$. If $m = 0$, then $\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} = \frac{1}{2} \notin \mathbb{Z}$. Hence we must have $m = 1$ and the possible normal nilpotent subgroups are

$$N_1 = \langle\, t_1t_2,\, t_2^2,\, t_3^4 \,\rangle, \qquad N_2 = \langle\, t_1t_2,\, t_2^2 t_3^2,\, t_3^4 \,\rangle,$$

$$N_3 = \langle\, t_1t_2t_3^2,\, t_2^2,\, t_3^4 \,\rangle, \qquad N_4 = \langle\, t_1t_2t_3^2,\, t_2^2t_3^2,\, t_3^4 \,\rangle.$$

From the following relations

$\alpha t_2^2\alpha^{-1} = t_1^{-2} = t_2^2(t_1t_2)^{-2} \in N_1,\ \alpha(t_1t_2)\alpha^{-1} = t_2t_1^{-1} = t_2^2(t_1t_2)^{-1} \in N_1,$

$\alpha(t_1t_2)\alpha^{-1} = t_2t_1^{-1} = (t_1t_2)^{-1}t_2^2 \notin N_2,$

$\alpha(t_1t_2t_3^2)\alpha^{-1} = \alpha(t_1t_2)\alpha^{-1}t_3^2 = t_2t_1^{-1}t_3^2 = (t_1t_2t_3)^{-1}t_2^2t_3^4 \in N_3,$

$\alpha(t_1t_2t_3^2)\alpha^{-1} = t_2\alpha t_2\alpha^{-1}t_3^2 = t_2t_1^{-1}t_3^2 = (t_1t_2t_3)^{-1}t_2^2t_3^4 \notin N_4,$

we can conclude that there exist two normal nilpotent subgroups N_1, N_3 of $\pi_{5,2}$. Since $N_1 \supset [\pi_{5,2},\, \pi_{5,2}] = \langle t_1t_2, t_2^2, t_3^4\rangle$, we obtain that $\pi_{5,2}/N_1$ is abelian and

$$\pi_{5,2}/N_1 = \langle\, t_1, t_2, t_3, \alpha\rangle/\langle\, t_1t_2,\, t_2^2,\, t_3^4\rangle \;= \langle\, \alpha N_1, t_2N_1\rangle \cong \mathbb{Z}_{16}\times\mathbb{Z}_2.$$

Note that $\pi_{5,2}/N_3$ is nonabelian and

$$\pi_{5,2}/N_3 = \langle\, t_1, t_2, t_3, \alpha\rangle/\langle\, t_1t_2t_3^2,\, t_2^2,\, t_3^4\rangle \;= \langle\, \alpha N_3, t_2N_3\rangle.$$

Since

$$\begin{aligned} t_1N_3 &= (t_1t_2t_3^2)t_3^{-2}t_2^{-1}N_3 = t_3^{-2}t_2^{-1}N_3 = t_3^{-2}t_2^{-1}t_3^4N_3 = t_3^2t_2^{-1}N_3 \\ &= \alpha^8t_2^{-1}N_3, \end{aligned}$$

$(t_2N_3)(\alpha N_3) = (\alpha t_1)N_3 = \alpha(\alpha^8t_2^{-1})N_3 = \alpha^9t_2^{-1}N_3 = (\alpha N_3)^9(t_2N_3)^{-1},$

we have

$$\begin{aligned} \pi_{5,2}/N_3 = \langle\, \alpha N_3, t_2N_3 \mid (\alpha N_3)^{16} &= (t_2N_3)^2 = 1, \\ (t_2N_3)(\alpha N_3) &= (\alpha N_3)^9(t_2N_3)^{-1}\rangle. \end{aligned}$$

Let $a = \alpha\, N_3$ and $b = t_2N_3$. Then this group is isomorphic to the modular maximal-cyclic group

$$M_{32} = M_5(2) = \langle\, a, b \mid a^{16} = b^2 = 1,\, bab = a^9\rangle.$$

(iii) When $d_1 = d_2 = 2$: If $m = 1$, then $\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} = 1 + \frac{1}{4} \notin \mathbb{Z}$. Thus there does not exist any normal nilpotent subgroup. If $m = 0$, then N_1 is affinely conjugate to N_4 and N_2 is affinely conjugate to N_3 by Proposition 3.5. Hence we have the following two possible normal nilpotent subgroups

$$N_1 = \langle\, t_1^2,\, t_2^2,\, t_3^8\,\rangle,\quad N_2 = \langle\, t_1^2,\, t_2^2t_3^2,\, t_3^8\,\rangle.$$

Since

$$\alpha(t_1^2)\alpha^{-1} = t_2^2 \in N_1,\quad \alpha(t_1^2)\alpha^{-1} = t_2^2 \notin N_2,\quad \alpha(t_2^2)\alpha^{-1} = t_1^{-2} \in N_1,$$

we can conclude that N_2 is not a normal subgroup of $\pi_{5,2}$ and there exists only one normal nilpotent subgroup N_1 of $\pi_{5,2}$. Let $w_1 = t_1t_3^2N_1,\ w_2 = t_2N_1,\ w_3 = t_3N_1$, and $\beta = \alpha N_1$. From the following relations

$$ord(w_1) = 4,\ ord(w_2) = 2,\ ord(w_3) = 8 \quad and \quad (w_1)^2 = (w_3)^4,$$

$$w_2 w_1 w_2 = t_2(t_1 t_3^2) t_2 N_1 = (t_1 t_2 t_3^4) t_2 t_3^2 N_1 = (t_1 t_3^2)^3 t_2^2 t_1^{-2} N_1 = w_1^3,$$

we can obtain that

$$\begin{aligned} F &= \pi_{5,2}/N_1 = (\Gamma_2/N_1)\tilde{\times}\mathbb{Z}_4 \cong (C_8 \circ D_4)\tilde{\times}\mathbb{Z}_4 \\ &= \langle w_1, w_2, w_3, \beta \mid w_2^2 = w_3^8 = 1,\ w_1^2 = w_3^4,\ \beta^4 = w_3^3,\ w_2 w_1 w_2 = {w_3}^4 w_1, \\ &\qquad \beta w_1 \beta^{-1} = w_2 w_3^2,\ \beta w_2 \beta^{-1} = w_1^{-1} w_3^2,\ w_1 w_3 = w_3 w_1,\ w_2 w_3 = w_3 w_2 \rangle. \\ &\cong D_4.C_{16}, \end{aligned}$$

where $D_4.C_{16}$ is the non-split extension by D_4 of C_{16} acting via $C_{16}/C_8 = C_2$.

The other cases can be done similarly. $\square$

The following propositions are a working criterion for affine conjugacy among normal nilpotent subgroups of π_6.

Proposition 3.7 ([10, Proposition 3.16]). *Let $N_j (j = 1, 2, 3)$ be a normal nilpotent subgroup of $\pi_{6,i}(i = 1, 2)$ and isomorphic to Γ_p. Then we have the following:*

(1) $N_1^{\ell,r} \sim N_1^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0),\ (\frac{Kd_2}{3}, \frac{Kd_1}{3}),\ (\frac{Kd_2}{3}, -\frac{2Kd_1}{3}), (\frac{2Kd_2}{3}, \frac{2Kd_1}{3}), (\frac{2Kd_2}{3}, -\frac{Kd_1}{3})$.

(2) $N_2^{\ell,r} \sim N_2^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0),\ (\frac{Kd_2}{3}, \frac{2Kd_1}{3}),\ (\frac{Kd_2}{3}, -\frac{Kd_1}{3}), (\frac{2Kd_2}{3}, \frac{Kd_1}{3}), (\frac{2Kd_2}{3}, -\frac{2Kd_1}{3})$.

(3) $N_3^{\ell,r} \sim N_3^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0), (\frac{Kd_2}{3}, 0), (\frac{2Kd_2}{3}, 0)$.

(4) $N_1 \nsim N_2,\quad N_1 \nsim N_3,\quad N_2 \nsim N_3$.

Proposition 3.8 ([10, Proposition 3.17]). *Let $N_j (j = 1, 2, 3)$ be a normal nilpotent subgroup of $\pi_{6,i}(i = 3, 4)$ and isomorphic to Γ_p. If $r \neq r'$ or $\ell \neq \ell'$, then*

$$N_i^{\ell,r} \nsim N_j^{\ell',r'} \quad (1 \leq i, \quad j \leq 3).$$

Theorem 3.9. *($\pi_{6,i}$) Suppose F is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_{6,i}$. Then F is isomorphic to the $Q_8 \rtimes C_9$ (the semidirect product of Q_8 and C_9 acting via $C_9/C_3 = C_3$) in the case $\pi_{6,1}$ and $\pi_{6,2}$, $SL_2(\mathbb{F}_3)$ (the special linear group on $\mathbb{F}_3^2$) in the case $\pi_{6,3}$, and $C_4.A_4$ (the central extension by C_4 of A_4) in the case $\pi_{6,4}$.*

Proof. When $n = 1$, we have

$\pi_{6,1} = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^3,\ \alpha t_1 \alpha^{-1} = t_2,\, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\, \alpha^3 = t_3 \,\rangle.$

Let N be a normal nilpotent subgroup of $\pi_{6,1}$ and isomorphic to Γ_2. Then

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{3d_1d_2}{p}} \,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \tfrac{3d_1d_2}{p}\right),$$

where d_1 and d_2 are divisors of $p = 2$, $\frac{d_1}{d_2} - \frac{m}{d_2} + \frac{m^2}{d_1 d_2} \in \mathbb{Z}$.

It is not hard to induce that

$$d_2 = (2s-1)d_1 (s \in \mathbb{N}),\ \ m = 0,\ \ d_1 = d_2 = 2.$$

By the normality of $N = \langle\, t_1^2 t_3^\ell,\ t_2^2 t_3^r,\ t_3^6 \,\rangle$, we have the following relations:

$$\begin{aligned} &\alpha (t_1^2 t_3^\ell) \alpha^{-1} = t_2^2 t_3^\ell = (t_2^2 t_3^r) t_3^{\ell - r} \in\ N, \quad \ell = r.\\ &\alpha (t_2^2 t_3^\ell) \alpha^{-1} = t_1^{-1} t_2^{-1} t_1^{-1} t_2^{-1} t_3^\ell = t_1^{-1} (t_1^{-1} t_2^{-1} t_3^3) t_2^{-1} t_3^\ell = t_1^{-2} t_2^{-2} t_3^{3+\ell}\\ &= (t_1^2 t_3^\ell)^{-1} (t_2^2 t_3^\ell)^{-1} t_3^{3\ell+3} \in N, \quad \ell = r = 1, 3, 5. \end{aligned}$$

Hence there exist three possible normal nilpotent subgroups of $\pi_{6,1}$:

$$N_1 = \langle\, t_1^2 t_3,\ t_2^2 t_3,\ t_3^6 \,\rangle, \quad N_3 = \langle\, t_1^2 t_3^3,\ t_2^2 t_3^3,\ t_3^6 \,\rangle, \quad N_5 = \langle\, t_1^2 t_3^5,\ t_2^2 t_3^5,\ t_3^6 \,\rangle.$$

Since N_1 is affinely conjugate to N_3 and N_5 by Proposition 3.7, we have only one normal nilpotent subgroup N_1 of $\pi_{6,1}$. Note that $\pi_{6,1}/N$ is abelian if and only if

$$N \supset [\pi_{6,1}, \pi_{6,1}] = \langle t_2 t_1^{-1},\ t_1^{-1} t_2^{-2},\ t_3^3 \rangle = \langle t_1 t_2^{-1},\ t_2^3,\ t_3^3 \,\rangle.$$

Thus we obtain that $\pi_{6,1}/N_1$ is nonabelian.

Let $w_1 = t_1 t_3^2 N_1$, $w_2 = t_2 t_3^2 N_1$, $w_3 = t_3^2 N_1$, and $\beta = \alpha N_1$.
From the following relations

$$ord(w_1) = 4,\ \ ord(w_2) = 4,\ \ ord(w_3) = 3,\ \ (w_1)^2 = (w_2)^2 = t_3^3 N_1 = \beta^9,$$

$$\begin{aligned} w_2 w_1 w_2^{-1} &= (t_2 t_3^2)(t_1 t_3^2)(t_2 t_3^2)^{-1} N_1 = (t_1 t_2 t_3^3) t_2^{-1} t_3^2 N_1 = (t_1 t_3^5) N_1\\ &= (t_1 t_3^5)(t_1^2 t_3) N_1 = w_1^3, \end{aligned}$$

we conclude that

$$\begin{aligned} F_1 = \pi_{6,1}/N_1 &= (\Gamma_2/N_1) \tilde{\times} \mathbb{Z}_3 \cong\ (C_3 \times Q_8) \tilde{\times} \mathbb{Z}_3\\ &= \langle w_1, w_2, w_3, \beta \,|\, w_2^4 = w_3^3 = 1, w_1^2 = w_2^2 = \beta^9,\, \beta^6 = w_3, w_2 w_1 w_2^{-1} = w_1^{-1},\\ &\quad \beta w_1 \beta^{-1} = w_2,\ \beta w_2 \beta^{-1} = w_1^{-1} w_2^{-1},\ \ w_1 w_3 = w_3 w_1,\ \ w_2 w_3 = w_3 w_2 \rangle,\\ &\cong\ Q_8 \rtimes C_9. \end{aligned}$$

It is not hard to get the results for the other cases. $\square$

The following proposition is a working criterion for affine conjugacy among normal nilpotent subgroups of π_7.

PROPOSITION 3.10 ([10, Proposition 3.19]). *Let $N_j (j = 1,2,3,4)$ be a normal nilpotent subgroup of $\pi_{7,i}(i = 1,2,3,4)$ in Lemma 3.2 and isomorphic to Γ_p. Then we have the following:*

(1) $N_2 \sim N_3$ if and only if $m = 0$, $d_1 = d_2$.

(2) $N_1 \nsim N_2$, $N_1 \nsim N_3$, $N_1 \nsim N_4$, $N_2 \nsim N_4$, $N_3 \nsim N_4$.

THEOREM 3.11. *($\pi_{7,i}$) Suppose F is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_{7,i}$. Then F is isomorphic to the $Q_8.C_{36}$ (the non-split extension by Q_8 of C_{36} acting via $C_{36}/C_{12} = C_3$) in the case $\pi_{7,1}$ and $\pi_{7,3}$, $C_{16}.A_4$(the central extension by C_{16} of A_4) in the case $\pi_{7,2}$, and $C_8.A_4$ (the central extension by C_8 of A_4) in the case $\pi_{7,4}$.*

Proof. Note that when $n = 1$, we have

$$\pi_{7,1} = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^6,\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle.$$

Let N be a normal nilpotent subgroup of $\pi_{7,1}$ which is isomorphic to Γ_2. Then by Lemma 3.2, N can be represented by one of the following sets of generators,

$$N_1 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2},\ t_3^{3d_1d_2}\rangle, \qquad N_2 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2} t_3^{\frac{3d_1d_2}{2}},\ t_3^{3d_1d_2}\rangle,$$

$$N_3 = \langle t_1^{d_1} t_2^m t_3^{\frac{3d_1d_2}{2}},\ t_2^{d_2},\ t_3^{3d_1d_2}\rangle, \qquad N_4 = \langle t_1^{d_1} t_2^m t_3^{\frac{3d_1d_2}{2}},\ t_2^{d_2} t_3^{\frac{3d_1d_2}{2}},\ t_3^{3d_1d_2}\rangle,$$

where d_1, d_2 are divisors of $p = 2$; $\frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}$ and d_1 is a common divisor of m and d_2. From these relations, there are two possibilities.

(i) When $d_1 = d_2 = 1$: $m = 0$. Since $\frac{3d_1d_2}{2} = \frac{3}{2} \notin \mathbb{Z}$, N_2, N_3, N_4 do not occur. Thus we have only one normal nilpotent subgroup $N_1 = \langle\, t_1,\ t_2,\ t_3^3 \,\rangle$ and $\pi_{7,1}/N_1 = \langle\, \alpha N \,\rangle \cong \mathbb{Z}_{18}$.

(ii) When $d_1 = d_2 = 2$: $m = 0$. Since N_2 is affinely conjugate to N_3 by Proposition 3.10, there exist three possible normal nilpotent subgroups:

$$N_1 = \langle\, t_1^2,\ t_2^2,\ t_3^{12} \,\rangle, \quad N_2 = \langle\, t_1^2,\ t_2^2 t_3^6,\ t_3^{12} \,\rangle, \quad N_4 = \langle\, t_1^2 t_3^6,\ t_2^2 t_3^6,\ t_3^{12} \,\rangle.$$

From the following relations

$$\alpha(t_1^2)\alpha^{-1} = t_1 t_2 t_1 t_2 = t_1^2 t_2^2 t_3^6 \notin N_1, \quad \alpha(t_2^2 t_3^6)\alpha^{-1} = t_1^{-2} t_3^6 \notin N_2,$$

$$\alpha(t_1^2 t_3^6)\alpha^{-1} = t_1^2 t_2^2 t_3^{12} = t_1^2 t_3^6 t_2^2 t_3^6 \in N_4,$$

$$\alpha(t_2^2 t_3^6)\alpha^{-1} = t_1^{-2} t_3^6 = (t_1^2 t_3^6)^{-1} t_3^{12} \in N_4,$$

we can conclude that there exists only one normal nilpotent subgroup N_4 of $\pi_{7,1}$. Since $N_4 \not\supseteq [\pi_{7,1}, \pi_{7,1}] = \langle t_1,\ t_2,\ t_3^6 \rangle$, we know that $\pi_{7,1}/N_4$ is nonabelian.

Let $w_1 = t_1N_4,\ \ w_2 = t_2N_4,\ \ w_3 = t_3N_4$, and $\beta = \alpha N_4$. Since $ord(w_1) = 4,\ \ ord(w_2) = 4,\ \ ord(w_3) = 12,\ \ (w_1)^2 = (w_2)^2 = (w_3)^6$,
$$w_2w_1w_2^{-1} = (t_2t_1t_2^{-1})N_4 = (t_1t_2t_3^6)t_2^{-1}N_4 = (t_1t_3^6)N_4 = (t_1N_4)(t_1N_4)^2 = w_1^3 = w_1^{-1},$$
we can obtain that

$$\begin{aligned} F &= \pi_{7,1}/N_4 \\ &= \langle w_1, w_2, w_3, \beta \mid w_1^4 = w_3^{12} = 1,\ w_1^2 = w_2^2 = w_3^6,\ \beta^6 = w_3,\ w_2w_1w_2 = {w_1}^{-1}, \\ &\qquad \beta w_1\beta^{-1} = w_1w_2,\ \beta w_2\beta^{-1} = w_1^{-1},\ w_1w_3 = w_3w_1,\ w_2w_3 = w_3w_2 \rangle \\ &\cong\ Q_8.C_{36}, \end{aligned}$$

where $Q_8.C_{36}$ is the non-split extension by Q_8 of C_{36} acting via $C_{36}/C_{12} = C_3$.

The other cases can be obtained similarly. □

We can obtain the following corollary by summarizing the results from Theorem 3.1 through Theorem 3.11 and using the results in [9]. In [1, 2, 3], finite groups acting freely on the nilmanifold $\mathcal{N}_p$ are abelian. However, as we can see in the following corollary if a finite group acts freely on $\mathcal{N}_2$ with $n = 1$, there exist nonabelian groups which yield orbit manifolds homeomorphic to $\mathcal{N}/\pi_i$ for all i.

Corollary 3.12. *The following table gives a complete list of all free actions(up to topological conjugacy) of finite groups G on $\mathcal{N}_2$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi$.*

π	$G = \pi/N$	*AC classes of normal nilpotent subgroups N*	*Group type*
π_1	$\mathbb{Z}_2$	$N = \langle t_1, t_2^2, t_3 \rangle$	*abelian*
	D_4	$N_1 = \langle t_1^2, t_2^2, t_3^2 \rangle$	*nonabelian*
	D_4	$N_2 = \langle t_1^2, t_2^2t_3, t_3^2 \rangle$	*nonabelian*
	Q_8	$N_3 = \langle t_1^2t_3, t_2^2t_3, t_3^2 \rangle$	*nonabelian*
π_2	$\mathbb{Z}_2$	$N = \langle t_1, t_2, t_3 \rangle$	*abelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$N_1 = \langle t_1^2, t_2, t_3^2 \rangle$	*abelian*
	Q_8	$N_2 = \langle t_1^2t_3, t_2, t_3^2 \rangle$	*nonabelian*
	$C_8 \circ D_4$	$N_3 = \langle t_1^2, t_2^2, t_3^4 \rangle$	*nonabelian*

π	$G = \pi/N$	*AC classes of normal nilpotent subgroups N*	*Group type*
π_3	$\mathbb{Z}_2$	$N = \langle t_1, t_2, t_3 \rangle$	*abelian*
	D_4	$N_1 = \langle t_1, t_2^2, t_3^2 \rangle$	*nonabelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$N_2 = \langle t_1, t_2^2 t_3, t_3^2 \rangle$	*abelian*
	$\mathbb{Z}_8$	$N_3 = \langle t_1 t_2, t_2^2 t_3, t_3^2 \rangle$	*abelian*
π_4	D_4	$N = \langle t_1, t_2, t_3^2 \rangle$	*nonabelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$N_1 = \langle t_1, t_2 t_3, t_3^2 \rangle$	*abelian*
	Q_8	$N_2 = \langle t_1 t_3, t_2 t_3, t_3^2 \rangle$	*nonabelian*
	$C_8.C_4$	$N_3 = \langle t_1 t_2, t_2^2 t_3^2, t_3^4 \rangle$	*nonabelian*
$\pi_{5,1}$	$\mathbb{Z}_4$	$N = \langle t_1, t_2, t_3 \rangle$	*abelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$N_1 = \langle t_1 t_2, t_2^2, t_3^2 \rangle$	*abelian*
	M_{16}	$N_2 = \langle t_1 t_2 t_3, t_2^2, t_3^2 \rangle$	*nonabelian*
	$D_4.C_8$	$N_3 = \langle t_1^2, t_2^2, t_3^4 \rangle$	*nonabelian*
$\pi_{5,2}$	$\mathbb{Z}_8$	$N = \langle t_1, t_2, t_3^2 \rangle$	*abelian*
	$\mathbb{Z}_8$	$N_1 = \langle t_1 t_3, t_2 t_3, t_3^2 \rangle$	*abelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	$N_2 = \langle t_1 t_2, t_2^2, t_3^4 \rangle$	*abelian*
	M_{32}	$N_3 = \langle t_1 t_2 t_3^2, t_2^2, t_3^4 \rangle$	*nonabelian*
	$D_4.C_{16}$	$N_4 = \langle t_1^2, t_2^2, t_3^8 \rangle$	*nonabelian*
$\pi_{5,3}$	$\mathbb{Z}_8$	$N = \langle t_1, t_2, t_3^2 \rangle$	*abelian*
	$\mathbb{Z}_8$	$N_1 = \langle t_1 t_3, t_2 t_3, t_3^2 \rangle$	*abelian*
	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	$N_2 = \langle t_1 t_2, t_2^2, t_3^4 \rangle$	*abelian*
	M_{32}	$N_3 = \langle t_1 t_2 t_3^2, t_2^2, t_3^4 \rangle$	*nonabelian*
	$D_4.C_{16}$	$N_4 = \langle t_1^2, t_2^2, t_3^8 \rangle$	*nonabelian*
$\pi_{6,1}$	$Q_8 \rtimes C_9$	$N_1 = \langle t_1^2 t_3, t_2^2 t_3, t_3^6 \rangle$	*nonabelian*
$\pi_{6,2}$	$Q_8 \rtimes C_9$	$N_1 = \langle t_1^2 t_3, t_2^2 t_3, t_3^6 \rangle$	*nonabelian*
$\pi_{6,3}$	$SL_2(\mathbb{F}_3)$	$N_1 = \langle t_1^2 t_3, t_2^2 t_3, t_3^2 \rangle$	*nonabelian*
$\pi_{6,4}$	$\mathbb{Z}_3$	$N = \langle t_1, t_2, t_3 \rangle$	*abelian*
	$C_4.A_4$	$N_1 = \langle t_1^2 t_3^2, t_2^2 t_3^2, t_3^4 \rangle$	*nonabelian*
$\pi_{7,1}$	$\mathbb{Z}_{18}$	$N = \langle t_1, t_2, t_3^3 \rangle$	*abelian*
	$Q_8.C_{36}$	$N_1 = \langle t_1^2 t_3^6, t_2^2 t_3^6, t_3^{12} \rangle$	*nonabelian*
$\pi_{7,2}$	$\mathbb{Z}_{12}$	$N = \langle t_1, t_2, t_3^2 \rangle$	*abelian*
	$C_{16}.A_4$	$N_1 = \langle t_1^2 t_3^4, t_2^2 t_3^4, t_3^8 \rangle$	*nonabelian*
$\pi_{7,3}$	$\mathbb{Z}_{18}$	$N = \langle t_1, t_2, t_3^3 \rangle$	*abelian*
	$Q_8.C_{36}$	$N_1 = \langle t_1^2 t_3^6, t_2^2 t_3^6, t_3^{12} \rangle$	*nonabelian*
$\pi_{7,4}$	$\mathbb{Z}_6$	$N = \langle t_1, t_2, t_3 \rangle$	*abelian*
	$C_8.A_4$	$N_1 = \langle t_1^2 t_3^2, t_2^2 t_3^2, t_3^4 \rangle$	*nonabelian*

EXAMPLE 3.13. Let G be a finite group of order 32 acting freely on $\mathcal{N}_2$. Then G is one of the following four groups:

$\mathbb{Z}_2 \times \mathbb{Z}_{16}$, modular group M_{32}, central product $C_8 \circ D_4$, 1^{st} non-split extension $C_8.C_4$.
In each case, non-affinely conjugate actions are as follows.

- $\mathbb{Z}_2 \times \mathbb{Z}_{16}$: one in $\pi_{5,i}(i = 2, 3)$
- M_{32}: one in $\pi_{5,i}(i = 2, 3)$
- $C_8 \circ D_4$: one in π_2
- $C_8.C_4$: one in π_4.

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*
Cheonan Wolbong Middle School
Cheonan 31169, Republic of Korea
E-mail: rano153@naver.com

**
Daejeon Science High School for the Gifted
Daejeon 34142, Republic of Korea
E-mail: pi3014@hanmail.net

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Department of Mathematics Education
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: jkshin@cnu.ac.kr