

## VOLUMES OF GEODESIC BALLS IN HEISENBERG GROUPS $\mathbb{H}^5$

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ABSTRACT. Let  $\mathbb{H}^5$  be the 5-dimensional Heisenberg group equipped with a left-invariant metric. In this paper we calculate the volumes of geodesic balls in  $\mathbb{H}^5$ . Let  $B_e(R)$  be the geodesic ball with center  $e$  (the identity of  $\mathbb{H}^5$ ) and radius  $R$  in  $\mathbb{H}^5$ . Then, the volume of  $B_e(R)$  is given by

$$\begin{aligned} & Vol(B_e(R)) \\ &= \frac{4\pi^2}{6!} \left( p_1(R) + p_4(R) \sin R + p_5(R) \cos R + p_6(R) \int_0^R \frac{\sin t}{t} dt \right. \\ & \quad \left. + q_4(R) \sin(2R) + q_5(R) \cos(2R) + q_6(R) \int_0^{2R} \frac{\sin t}{t} dt \right) \end{aligned}$$

where  $p_n$  and  $q_n$  are polynomials with degree  $n$ .

### 1. Introduction

Let  $\mathcal{N}$  be a 2-step nilpotent Lie algebra with an inner product  $\langle, \rangle$  and  $N$  its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by  $\langle, \rangle$  on  $\mathcal{N}$ . Let  $\mathcal{Z}$  be the center of  $\mathcal{N}$ . Then  $\mathcal{N}$  is represented by the direct sum of  $\mathcal{Z}$  and its orthogonal complement  $\mathcal{Z}^\perp$ .

For each  $Z \in \mathcal{Z}$ , a skew symmetric linear transformation  $j(Z) : \mathcal{Z}^\perp \rightarrow \mathcal{Z}^\perp$  is defined by  $j(Z)X = (adX)^*Z$  for  $X \in \mathcal{Z}^\perp$ . Or, equivalently,

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$$

for all  $X, Y \in \mathcal{Z}^\perp$ .

A 2-step nilpotent Lie algebra  $\mathcal{N}$  is said to be *an algebra of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 id$$

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for all  $Z \in \mathcal{Z}$ . And a Lie group  $N$  is said to be a *group of Heisenberg type* if its Lie algebra  $\mathcal{N}$  is an algebra of Heisenberg type.

The Heisenberg groups are examples of Heisenberg type. That is, let  $n \geq 1$  be any integer and  $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$  a basis of  $R^{2n} = \mathcal{V}$ . Let  $\mathcal{Z}$  be a one dimensional vector space spanned by  $\{Z\}$ . Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any  $i = 1, 2, \dots, n$  with all other brackets are zero. Give on  $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$  the inner product such that the set of vectors  $\{X_i, Y_i, Z | i = 1, 2, \dots, n\}$  forms an orthonormal basis. Let  $N$  be the simply connected 2-step nilpotent group of Heisenberg type which is determined by  $\mathcal{N}$  and equipped with a left-invariant metric induced by the inner product in  $\mathcal{N}$ . The group  $N$  is called the  $(2n + 1)$ -dimensional Heisenberg group and denoted by  $\mathbb{H}^{2n+1}$ .

In this paper, we calculate the volumes of the geodesic balls in the Heisenberg group  $\mathbb{H}^5$ :

**MAIN THEOREM.** Let  $B_e(R)$  be the geodesic ball with center  $e$  (the identity of  $\mathbb{H}^5$ ) and radius  $R$  in  $\mathbb{H}^5$ . Then, the following holds.

$$\begin{aligned} & Vol(B_e(R)) \\ &= \frac{4\pi^2}{6!} \left( p_1(R) + p_4(R) \sin R + p_5(R) \cos R + p_6(R) \int_0^R \frac{\sin t}{t} dt \right. \\ & \quad \left. + q_4(R) \sin(2R) + q_5(R) \cos(2R) + q_6(R) \int_0^{2R} \frac{\sin t}{t} dt \right) \end{aligned}$$

where

$$\begin{aligned} p_1(R) &= 576R, & q_4(R) &= 30 + 54R^2 + 4R^4, \\ p_4(R) &= -240 - 108R^2 - 2R^4, & q_5(R) &= 132R + 116R^3 + 8R^5, \\ p_5(R) &= -528R - 116R^3 - 2R^5, & q_6(R) &= 360R^2 + 240R^4 + 16R^6, \\ p_6(R) &= -720R^2 - 120R^4 - 2R^6. \end{aligned}$$

For a Riemannian manifold  $M$  and  $p \in M$ , the volume growth,  $VG_p(M)$  of  $M$  at  $p$  is defined by

$$VG_p(M) = \inf\{x \in \mathbb{R} \mid \lim_{r \rightarrow \infty} \frac{Vol(B_p(r))}{r^x} = 0\}.$$

If  $M$  is a Lie group with a left invariant metric, then we see that  $VG_p(M) = VG_q(M)$  for any  $p, q \in M$ . In this case, it is denoted by  $VG(M)$ . For example, the volume growth of the Euclidean space  $\mathbb{R}^5$  is  $VG(\mathbb{R}^5) = 5$  since the volume of ball with radius  $r$  is  $\frac{8\pi^2 r^5}{15}$  in  $\mathbb{R}^5$  (see [12]).

COROLLARY. The volume growth,  $VG(\mathbb{H}^5)$  of  $\mathbb{H}^5$  is given as follows;

$$VG(\mathbb{H}^5) = 6.$$

## 2. Preliminaries

Let  $\mathcal{N}$  be a 2-step nilpotent Lie algebra with an inner product  $\langle, \rangle$  and  $N$  be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by  $\langle, \rangle$  on  $\mathcal{N}$ . The center of  $\mathcal{N}$  is denoted by  $\mathcal{Z}$ . Then  $\mathcal{N}$  can be expressed as the direct sum of  $\mathcal{Z}$  and its orthogonal complement  $\mathcal{Z}^\perp$ .

Recall that for  $Z \in \mathcal{Z}$ , a skew symmetric linear transformation  $j(Z) : \mathcal{Z}^\perp \rightarrow \mathcal{Z}^\perp$  is defined by  $j(Z)X = (\text{ad}X)^*Z$  for  $X \in \mathcal{Z}^\perp$ . Or, equivalently,

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$$

for  $X, Y \in \mathcal{Z}^\perp$ . A 2-step nilpotent Lie group  $N$  is said to be a *group of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 \text{id}$$

for all  $Z \in \mathcal{Z}$ .

Let  $\gamma(t)$  be a curve in  $N$  such that  $\gamma(0) = e$  (identity element in  $N$ ) and  $\gamma'(0) = X_0 + Z_0$  where  $X_0 \in \mathcal{Z}^\perp$  and  $Z_0 \in \mathcal{Z}$ . Since  $\exp : \mathcal{N} \rightarrow N$  is a diffeomorphism ([10]), the curve  $\gamma(t)$  can be expressed uniquely by  $\gamma(t) = \exp[X(t) + Z(t)]$  with

$$\begin{aligned} X(t) &\in \mathcal{Z}^\perp, & X'(0) &= X_0, & X(0) &= 0 \\ Z(t) &\in \mathcal{Z}, & Z'(0) &= Z_0, & Z(0) &= 0. \end{aligned}$$

A. Kaplan([8],[9]) shows that the curve  $\gamma(t)$  is a geodesic in  $N$  if and only if

$$\begin{aligned} X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0. \end{aligned}$$

The following Lemma is useful in the later.

LEMMA 2.1 ([2]). *Let  $N$  be a simply connected 2-step nilpotent Lie group with a left invariant metric and let  $\gamma(t)$  be a geodesic of  $N$  with  $\gamma(0) = e$  and  $\gamma'(0) = X_0 + Z_0$  where  $X_0 \in \mathcal{Z}^\perp$  and  $Z_0 \in \mathcal{Z}$ . Then, one has*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R$$

where  $X'(t) = e^{tj(Z_0)}X_0$  and  $l_{\gamma(t)}$  is the left translation by  $\gamma(t)$ .

Throughout this paper, different tangent spaces will be identified with  $\mathcal{N}$  via a left translation. So, in the above lemma, we can consider  $\gamma'(t)$  as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

Let  $\mathbb{H}^{2n+1}$  be the  $(2n+1)$ -dimensional Heisenberg group with a left invariant metric and  $\mathcal{N}$  its Lie algebra. Let  $\gamma(t)$  be a unit speed geodesic on  $\mathbb{H}^{2n+1}$  with  $\gamma(0) = e$  (the identity element of  $\mathbb{H}^{2n+1}$ ) and  $\gamma'(0) = X_0 + Z_0$  where  $X_0 \in \mathcal{Z}^\perp$  and  $Z_0 \in \mathcal{Z}$ . Assume that  $X_0 \neq 0$  and  $Z_0 \neq 0$ . Since

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}$$

is an orthonormal set in  $\mathcal{N}$ , we can obtain an orthonormal basis

$$\mathcal{B} = \{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0, \\ Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^\perp, k = 1, 2, \dots, n-1\}$$

by adding

$$\{Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^\perp, k = 1, 2, \dots, n-1\}$$

to

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}.$$

Let

$$e_1(t) = \frac{|Z_0|}{|X_0|}X'(t) - \frac{|X_0|}{|Z_0|}Z_0, \\ e_2(t) = \frac{1}{|Z_0||X_0|}j(Z_0)X'(t)$$

and let

$$e_{2k-1}(t) = e^{tj(Z_0)}Y_k, \\ e_{2k}(t) = \frac{1}{|Z_0|}e^{tj(Z_0)}j(Z_0)Y_k \text{ for each } k = 2, 3, \dots, n.$$

Then,  $\{\gamma'(t), e_{2k-1}(t), e_{2k}(t) | k = 1, 2, \dots, n\}$  is an orthonormal frame along  $\gamma(t)$  on  $\mathbb{H}^{2n+1}$  (see [5]). We start the following Proposition.

PROPOSITION 2.2 ([5]). For each  $k = 1, 2, \dots, n$ , let  $J_{2k-1}(t)$  and  $J_{2k}(t)$  be the Jacobi fields with  $J_{2k-1}(0) = J_{2k}(0) = 0, J'_{2k-1}(0) = e_{2k-1}(0)$  and  $J'_{2k}(0) = e_{2k}(0)$ . Then, we have that

(1) for  $k = 1$ ,

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

where

$$B_1(t) = \frac{1}{|Z_0|^3} \begin{bmatrix} \sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t & |Z_0|(\cos(|Z_0|t) - 1) \\ |Z_0|(1 - \cos(|Z_0|t)) & |Z_0|^2 \sin(|Z_0|t) \end{bmatrix},$$

(2) for  $k = 2, 3, \dots, n$

$$\begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix} = B_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix}$$

where

$$B_k(t) = \begin{bmatrix} \frac{1}{|Z_0|} \sin(|Z_0|t) & |Z_0|(\cos(|Z_0|t) - 1) \\ \frac{1}{|Z_0|^3}(1 - \cos(|Z_0|t)) & \frac{1}{|Z_0|} \sin(|Z_0|t) \end{bmatrix}.$$

COROLLARY 2.3 ([1],[6]). Let  $\mathbb{H}^{2n+1}$  be the  $(2n + 1)$ -dimensional Heisenberg group and  $\mathcal{N}$  its Lie algebra. Let  $\gamma(t)$  be a unit speed geodesic on  $N$  with  $\gamma(0) = e$  (the identity element of  $N$ ) and  $\gamma'(0) = X_0 + Z_0$  where  $X_0 \in \mathcal{Z}^\perp$  and  $Z_0 \in \mathcal{Z}$ .

(1) If  $Z_0 \neq 0$ , then all the conjugate points along  $\gamma$  are at  $t \in \frac{2\pi}{|Z_0|} \mathbb{Z}^* \cup \mathbb{A}$  where

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \dots\}$$

and

$$\mathbb{A} = \{t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2) \frac{|Z_0|t}{2} = \tan \frac{|Z_0|t}{2}\}.$$

In particular,  $\frac{2\pi}{|Z_0|}$  is the first conjugate point of  $e$  along  $\gamma$ .

(2) If  $Z_0 = 0$ , then there are no conjugate points along  $\gamma$ .

For the conjugate points of another type of Heisenberg groups, Quaternionic Heisenberg groups  $\mathbb{H}^{4n+3}$ , see [4].

G.Walschap([11]) showed that the first conjugate loci and the cut loci are equal in the case of the groups of Heisenberg type or the 2-step nilpotent groups with a one-dimensional center. So, we consider the geodesic balls  $B_e(R)$  with the radius  $R \leq 2\pi$ . In this paper we calculate the volumes of geodesic balls in  $\mathbb{H}^5$ .

In ([5]), C. Jang, J. Park and K. Park obtained a formula of the volumes of geodesic balls in the Heisenberg group  $\mathbb{H}^3$  as the form of power series.

**THEOREM 2.4.** ([5] Theorem 3.8) *Let  $B_e(R)$  be the geodesic ball with center  $e$  and radius  $R$  in  $\mathbb{H}^3$ . Then, the following holds.*

$$\text{Vol}(B_e(R)) = 4\pi \left( \frac{R^3}{3} + 2 \sum_{n=2}^{\infty} (-1)^n \frac{R^{2n+1}}{(2n+1)!(2n-1)(2n-3)} \right).$$

Recently S. Jeong and K. Park ([7]) calculated the volumes of geodesic balls in the Heisenberg group  $\mathbb{H}^3$ .

**THEOREM 2.5.** ([7] Theorem 3.4) *Let  $0 \leq R \leq 2\pi$  and  $B_e(R)$  be the geodesic ball with center  $e$  (the identity of  $\mathbb{H}^3$ ) and radius  $R$  in  $\mathbb{H}^3$ . Then, the following holds.*

$$\begin{aligned} & \text{Vol}(B_e(R)) \\ &= \frac{\pi}{6} \left( -16R + (R^2 + 6) \sin R + (R^3 + 10R) \cos R + (R^4 + 12R^2) \int_0^R \frac{\sin t}{t} dt \right). \end{aligned}$$

### 3. Proof of main theorem

We start to prove Main Theorem. Note that

$$\det(B_1(t)) = \frac{1}{|Z_0|^4} \{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\}$$

and

$$\det(B_k(t)) = \frac{2}{|Z_0|^2} (1 - \cos(|Z_0|t))$$

for each  $k = 2, 3, \dots, n$ .

**LEMMA 3.1** ([5]). *For  $t > 0$ , the following holds.*

$$\begin{aligned} & \det(B_1(t)B_2(t)) \\ &= \frac{1}{|Z_0|^4} \{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\} \frac{2}{|Z_0|^2} (1 - \cos(|Z_0|t)) \geq 0. \end{aligned}$$

LEMMA 3.2 ([7]). Let  $n$  be a natural number and  $f : [0, x] \rightarrow R$  has continuous  $n$ -th derivatives. Assume that for  $k = 0, 1, \dots, n-1$ , the  $\lim_{t \rightarrow 0^+} \frac{f^{(k)}(t)}{t^{n-k}}$  exists and  $\frac{f^{(n)}(t)}{t}$  is integrable on  $[0, x]$ . Then,  $\frac{f(t)}{t^{n+1}}$  is integrable on  $[0, x]$  and the following holds.

$$\int_0^x \frac{f(t)}{t^{n+1}} dt = - \sum_{k=0}^{n-1} \frac{1}{n(n-1) \cdots (n-k)} \left[ \frac{f^{(k)}(t)}{t^{n-k}} \right]_{0^+}^{x^-} + \frac{1}{n!} \int_0^x \frac{f^{(n)}(t)}{t} dt$$

where

$$\left[ \frac{f^{(k)}(t)}{t^{n-k}} \right]_{0^+}^{x^-} = \lim_{t \rightarrow x^-} \frac{f^{(k)}(t)}{t^{n-k}} - \lim_{t \rightarrow 0^+} \frac{f^{(k)}(t)}{t^{n-k}}.$$

We give a modification of Lemma 3.2 which is useful.

LEMMA 3.3. Let  $n$  be a natural number and  $f : [0, 1] \rightarrow R$  has continuous  $n$ -th derivatives. (i)  $\forall k = 0, 1, \dots, n-1$ ,  $f^{(k)}(0) = 0$  and (ii)  $\frac{f^{(n)}(t)}{t}$  is integrable on  $[0, 1]$ . Then,  $\frac{f(t)}{t^{n+1}}$  is integrable on  $[0, 1]$  and

$$\int_0^1 \frac{f(t)}{t^{n+1}} dt = \frac{1}{n!} \left( - \sum_{k=0}^{n-1} (n-k-1)! f^{(k)}(1) + f^{(n)}(0) \sum_{k=1}^n \frac{1}{k} + \int_0^1 \frac{f^{(n)}(t)}{t} dt \right).$$

*Proof.* We use the LHopitals theorem.

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} &= \lim_{t \rightarrow 0^+} \frac{f^{(1)}(t)}{nt^{n-1}} = \lim_{t \rightarrow 0^+} \frac{f^{(2)}(t)}{n(n-1)t^{n-2}} = \cdots \\ &= \lim_{t \rightarrow 0^+} \frac{f^{(n)}(t)}{n(n-1) \cdots (n-n+1)t^{n-n}}. \end{aligned}$$

We see that  $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \frac{f^{(n)}(0)}{n!}$ . Let  $\alpha = \frac{f^{(n)}(0)}{n!}$ , then

$$\lim_{t \rightarrow 0^+} \frac{f^{(k)}(t)}{t^{n-k}} = n(n-1) \cdots (n-k+1)\alpha.$$

By Lemma 3.2, we have that

$$\begin{aligned}
& \int_0^1 \frac{f(t)}{t^{n+1}} dt \\
&= - \sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left( \lim_{t \rightarrow 1^-} \frac{f^{(k)}(t)}{t^{n-k}} - \lim_{t \rightarrow 0^+} \frac{f^{(k)}(t)}{t^{n-k}} \right) \\
&\quad + \frac{1}{n!} \int_0^1 \frac{f^{(n)}(t)}{t} dt \\
&= - \sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left( f^{(k)}(1) - n(n-1)\cdots(n-k+1) \cdot \alpha \right) \\
&\quad + \frac{1}{n!} \int_0^1 \frac{f^{(n)}(t)}{t} dt \\
&= - \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{n(n-1)\cdots(n-k)} + \sum_{k=0}^{n-1} \frac{\alpha}{n-k} + \frac{1}{n!} \int_0^1 \frac{f^{(n)}(t)}{t} dt \\
&= - \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{n(n-1)\cdots(n-k)} + \frac{f^{(n)}(0)}{n!} \sum_{k=1}^n \frac{1}{k} + \frac{1}{n!} \int_0^1 \frac{f^{(n)}(t)}{t} dt \\
&= - \sum_{k=0}^{n-1} \frac{(n-k-1)! f^{(k)}(1)}{n(n-1)\cdots(n-k)(n-k-1)!} + \frac{f^{(n)}(0)}{n!} \sum_{k=1}^n \frac{1}{k} + \frac{1}{n!} \int_0^1 \frac{f^{(n)}(t)}{t} dt \\
&= \frac{1}{n!} \left( - \sum_{k=0}^{n-1} (n-k-1)! f^{(k)}(1) + f^{(n)}(0) \sum_{k=1}^n \frac{1}{k} + \int_0^1 \frac{f^{(n)}(t)}{t} dt \right).
\end{aligned}$$

This completes the proof.  $\square$

We introduce the volume formula of geodesic balls in Riemannian manifolds, which is well-known. For example, see ([3]). Let  $M$  be a Riemannian manifold with a metric  $g$  and  $p \in M$ . Take an orthonormal basis  $\{u_1, u_2, \dots, u_n\}$  of  $T_p M$  and let  $(x_1, x_2, \dots, x_n)$  be the coordinates determined by  $\{u_1, u_2, \dots, u_n\}$ . This local coordinate system is called the normal coordinate system at  $p$ . It is easy to show that  $\frac{\partial}{\partial x_i m} = (d \exp_p)_{\sum_{i=1}^n x_i u_i}(u_i)$  where  $m = \exp_p(\sum_{i=1}^n x_i u_i)$ . Then, the volume form  $v_g$  on  $U_p$  is given by

$$v_g = \sqrt{\det \left( g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$



where  $g_{ij}$  is the metric coefficients of  $g$  in  $U_p$ . Therefore, the volume of the geodesic ball  $B_p(r)$  is given by

$$Vol(B_p(r)) = \int_{\exp_p^{-1}(B_p(r))} \exp_p^* v_g.$$

Let  $\gamma(t)$  be the unit speed geodesic in  $M$  with  $\gamma(0) = p$ ,  $\gamma'(0) = u_1$  and let  $J_i(t)$  be the Jacobi field with  $J_i(0) = 0$  and  $J'_i(0) = u_i$  for each  $i = 2, 3, \dots, n$ . Then we know that

$$(d \exp_p)_{tu_1} u_1 = \gamma'(t)$$

and

$$(d \exp_p)_{tu_1} u_i = \frac{1}{t} J_i(t)$$

for each  $i = 2, 3, \dots, n$ . So, we see that

$$\sqrt{\det \left( g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right)} = t^{-(n-1)} \sqrt{\det(g(J_i(t), J_j(t)))}.$$

Hence, we have that

$$\exp_p^* v_g = t^{-(n-1)} \sqrt{\det(g(J_i(t), J_j(t)))} dx_1 dx_2 \cdots dx_n = \sqrt{\det(g(J_i(t), J_j(t)))} dt du$$

where  $du$  denotes the canonical measure of the unit sphere  $S^{n-1}$ . Therefore, by Fubini's Theorem we get that

$$Vol(B_p(r)) = \int_{S^{n-1}} \int_0^r \sqrt{\det(g(J_i(t), J_j(t)))} dt du.$$

Using Proposition 2.2 for  $n=2$ , we obtain that

$$\begin{aligned} & \det(< J_i(t), J_j(t) >) \\ &= \det(J_i(t) \cdot J_j(t)) \\ &= \det \left( \begin{bmatrix} J_1(t) \\ J_2(t) \\ J_3(t) \\ J_4(t) \end{bmatrix} [J_1(t) \ J_2(t) \ J_3(t) \ J_4(t)] \right) \\ &= \det \left( B_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \cdot^t \left( B_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \right) \right) \det \left( B_2(t) \begin{bmatrix} e_3(t) \\ e_4(t) \end{bmatrix} \cdot^t \left( B_2(t) \begin{bmatrix} e_3(t) \\ e_4(t) \end{bmatrix} \right) \right) \\ &= \det(B_1(t) \cdot^t (B_1(t))) \det(B_2(t) \cdot^t (B_2(t))) \\ &= \left( \frac{1}{|Z_0|^4} \{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\} \left\{ \frac{2}{|Z_0|^2} (1 - \cos(|Z_0|t)) \right\} \right)^2. \end{aligned}$$

By Lemma 3.1, we have that

$$\begin{aligned} & \sqrt{\det \langle J_i(t), J_j(t) \rangle} \\ &= \frac{1}{|Z_0|^4} \{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\} \left\{ \frac{2}{|Z_0|^2} (1 - \cos(|Z_0|t)) \right\}. \end{aligned}$$

Let  $u = (x_1, x_2, x_3, x_4, x_5) \in S^4$  and  $|Z_0| = x_5$ , then

$$f(x_5, t) = \frac{1}{(x_5)^4} \{2(1 - \cos(x_5 t)) - (1 - (x_5)^2)x_5 t \sin(x_5 t)\} \left\{ \frac{2}{(x_5)^2} (1 - \cos(x_5 t)) \right\}.$$

Therefore, we see that volume of geodesic ball with center  $e$  (the identity of  $\mathbb{H}^5$ ) and radius  $R$  in  $\mathbb{H}^5$  is given as follows;

$$\text{Vol}(B_e(R)) = \int_{S^4} \int_0^R f(x_5, t) dt du.$$

Since the area element  $du$  on the sphere  $S^4$  is given by

$$du = \frac{1}{\sqrt{1 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}} dx_1 dx_2 dx_3 dx_4,$$

we have that

$$\text{Vol}(B_e(R)) = 2 \int_D \int_0^R f(\sqrt{1 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}, t) \frac{dt dx_1 dx_2 dx_3 dx_4}{\sqrt{1 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}}$$

where

$$D = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

Changing the coordinates on  $D$  to spherical coordinates, we have that

$$\begin{aligned} & \text{Vol}(B_e(R)) \\ &= 2 \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^1 \int_0^R f(\sqrt{1 - r^2}, t) \frac{1}{\sqrt{1 - r^2}} \cdot r^3 \sin^2 \theta_1 \sin \theta_2 dt dr d\theta_1 d\theta_2 d\theta_3 \\ &= 4\pi \prod_{k=1}^2 \int_0^\pi \sin^k \theta d\theta \int_0^1 \int_0^R \frac{r^3}{\sqrt{1 - r^2}} f(\sqrt{1 - r^2}, t) dt dr \\ &= 4\pi^2 \int_0^1 \int_0^R \frac{r^3}{\sqrt{1 - r^2}} f(\sqrt{1 - r^2}, t) dt dr. \end{aligned}$$

Replacing  $x = \sqrt{1 - r^2}$ , we see that

$$\text{Vol}(B_e(R)) = 4\pi^2 \int_0^1 (1 - x^2) \int_0^R f(x, t) dt dx$$

where  $f(x, t) = \frac{1}{x^4} \{2(1 - \cos(xt)) - (1 - x^2)xt \sin(xt)\} \{\frac{2}{x^2}(1 - \cos(xt))\}$ .  
Since

$$\begin{aligned} \int_0^R f(x, t) dt &= \frac{1}{x^6} \left( -R(1 - \cos(Rx))^2 + 5 \int_0^R (1 - \cos(xt))^2 dt \right) \\ &\quad + \frac{1}{x^4} \left( R(1 - \cos(Rx))^2 - \int_0^R (1 - \cos(xt))^2 dt \right). \end{aligned}$$

We have that

$$\begin{aligned} Vol(B_e(R)) &= 4\pi^2 \int_0^1 (1 - x^2) \int_0^R f(x, t) dt dx \\ &= 4\pi^2 \left[ \int_0^1 \frac{1 - x^2}{x^6} \left( -R(1 - \cos(Rx))^2 + 5 \int_0^R (1 - \cos(xt))^2 dt \right) dx \right. \\ &\quad \left. + \int_0^1 \frac{1 - x^2}{x^4} \left( R(1 - \cos(Rx))^2 - \int_0^R (1 - \cos(xt))^2 dt \right) dx \right]. \end{aligned}$$

By the following Lemma 3.4, we see that

$$\begin{aligned} Vol(B_e(R)) &= \frac{4\pi^2}{6!} \left( 576R + (-240 - 108R^2 - 2R^4) \sin R + (-528R - 116R^3 - 2R^5) \cos R \right. \\ &\quad + (30 + 54R^2 + 4R^4) \sin(2R) + (132R + 116R^3 + 8R^5) \cos(2R) \\ &\quad + (-720R^2 - 240R^4 + 2R^6) \int_0^R \frac{\sin t}{t} dt \\ &\quad \left. + (360R^2 + 480R^4 - 16R^6) \int_0^{2R} \frac{\sin t}{t} dt \right). \end{aligned}$$

We can rewrite the above equation as follows

$$\begin{aligned} Vol(B_e(R)) &= \frac{4\pi^2}{6!} \left( p_1(R) + p_4(R) \sin R + p_5(R) \cos R + p_6(R) \int_0^R \frac{\sin t}{t} dt \right. \\ &\quad \left. + q_4(R) \sin(2R) + q_5(R) \cos(2R) + q_6(R) \int_0^{2R} \frac{\sin t}{t} dt \right) \end{aligned}$$

where

$$\begin{aligned} p_1(R) &= 576R, & q_4(R) &= 30 + 54R^2 + 4R^4, \\ p_4(R) &= -240 - 108R^2 - 2R^4, & q_5(R) &= 132R + 116R^3 + 8R^5, \\ p_5(R) &= -528R - 116R^3 - 2R^5, & q_6(R) &= 360R^2 + 240R^4 + 16R^6, \\ p_6(R) &= -720R^2 - 120R^4 - 2R^6. \end{aligned}$$

Thus we finish the proof.

LEMMA 3.4. *For  $R > 0$ , the followings hold.*

$$\begin{aligned} (1) \quad & \int_0^1 \frac{1-x^2}{x^6} \left( -R(1-\cos(Rx))^2 + 5 \int_0^R (1-\cos(xt))^2 dt \right) dx \\ &= \frac{1}{6!} \left( 576R + (-600 + 72R^2 - 2R^4) \sin R + (-168R + 64R^3 - 2R^5) \cos R \right. \\ &\quad + (75 - 36R^2 + 4R^4) \sin(2R) + (42R - 64R^3 + 8R^5) \cos(2R) \\ &\quad \left. + \int_0^1 \frac{(60R^4 - 2R^6) \sin(Rx) + (-120R^4 + 16R^6) \sin(2Rx)}{x} dx \right). \\ (2) \quad & \int_0^1 \frac{1-x^2}{x^4} \left( R(1-\cos(Rx))^2 - \int_0^R (1-\cos(xt))^2 dt \right) dx \\ &= \frac{1}{4!} \left( (12 - 6R^2) \sin R + (-12R - 6R^3) \cos R \right. \\ &\quad + \left( -\frac{3}{2} + 3R^2 \right) \sin(2R) + (3R + 6R^3) \cos(2R) \\ &\quad \left. + \int_0^1 \frac{(-24R^2 - 6R^4) \sin(Rx) + (12R^2 + 12R^4) \sin(2Rx)}{x} dx \right). \end{aligned}$$

*Proof.* Let  $h(x) = \int_0^{Rx} (1 - \cos t)^2 dt$ . Then, we have that

$$\begin{aligned} h'(x) &= R(1 - \cos(Rx))^2, & h''(x) &= 2R^2 \sin(Rx) - R^2 \sin(2Rx), \\ h^{(3)}(x) &= 2R^3 \cos(Rx) - 2R^3 \cos(2Rx), & h^{(4)}(x) &= -2R^4 \sin(Rx) + 4R^4 \sin(2Rx), \\ h^{(5)}(x) &= -2R^5 \cos(Rx) + 8R^5 \cos(2Rx), & h^{(6)}(x) &= 2R^6 \sin(Rx) - 16R^6 \sin(2Rx). \end{aligned}$$

So, we can rewrite that

$$\begin{aligned} & \int_0^1 \frac{1-x^2}{x^6} \left( -R(1-\cos(Rx))^2 + 5 \int_0^R (1-\cos(xt))^2 dt \right) dx \\ &= \int_0^1 \frac{1-x^2}{x^7} (-xh'(x) + 5h(x)) dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \frac{1-x^2}{x^4} \left( R(1-\cos(Rx))^2 - \int_0^R (1-\cos(xt))^2 dt \right) dx \\ &= \int_0^1 \frac{1-x^2}{x^5} (xh'(x) - h(x)) dx. \end{aligned}$$

To compute the above integral, the following derivatives are useful . Let  $m(x) = (1-x^2)(-xh'(x) + 5h(x))$ . Then, derivatives of  $m(x)$  give that

$$\begin{aligned} m'(x) &= -2x(-xh'(x) + 5h(x)) + (1-x^2)(-xh''(x) + 4h'(x)) \\ m''(x) &= -2(-xh'(x) + 5h(x)) - 4x(-xh''(x) + 4h'(x)) \\ &\quad + (1-x^2)(-xh^{(3)}(x) + 3h''(x)) \\ m^{(3)}(x) &= -6(-xh''(x) + 4h'(x)) - 6x(-xh^{(3)}(x) + 3h''(x)) \\ &\quad + (1-x^2)(-xh^{(4)}(x) + 2h^{(3)}(x)) \\ m^{(4)}(x) &= -12(-xh^{(3)}(x) + 3h''(x)) - 8x(-xh^{(4)}(x) + 2h^{(3)}(x)) \\ &\quad + (1-x^2)(-xh^{(5)}(x) + h^{(4)}(x)) \\ m^{(5)}(x) &= -20(-xh^{(4)}(x) + 2h^{(3)}(x)) - 10x(-xh^{(5)}(x) + h^{(4)}(x)) \\ &\quad + (1-x^2)(-xh^{(6)}(x)) \\ m^{(6)}(x) &= -30(-xh^{(5)}(x) + h^{(4)}(x)) - 12x(-xh^{(6)}(x)) \\ &\quad + (1-x^2)(-xh^{(7)}(x) - h^{(6)}(x)). \end{aligned}$$

Let  $q(x) = (1-x^2)(xh'(x) - h(x))$ . Then, derivatives of  $q(x)$  give that

$$\begin{aligned} q'(x) &= -2x(xh'(x) - h(x)) + (1-x^2)(xh''(x)) \\ q''(x) &= -2(xh'(x) - h(x)) - 4x(xh''(x)) + (1-x^2)(h''(x) + xh^{(3)}(x)) \\ q^{(3)}(x) &= -6(xh''(x)) - 6x(h''(x) + xh^{(3)}(x)) + (1-x^2)(2h^{(3)}(x) + xh^{(4)}(x)) \\ q^{(4)}(x) &= -12(h''(x) + xh^{(3)}(x)) - 8x(2h^{(3)}(x) + xh^{(4)}(x)) \\ &\quad + (1-x^2)(3h^{(4)}(x) + xh^{(5)}(x)). \end{aligned}$$

By the above substitutions, we can obtain the following results. We can rewrite that

$$\int_0^1 \frac{1-x^2}{x^7} (-xh'(x) + 5h(x)) dx = \int_0^1 \frac{m(x)}{x^7} dx$$

and

$$\int_0^1 \frac{1-x^2}{x^5} (xh'(x) - h(x)) dx = \int_0^1 \frac{q(x)}{x^5} dx.$$

Using Lemma 3.3, we have that

$$\begin{aligned} & \int_0^1 \frac{m(x)}{x^7} dx \\ &= \frac{1}{6!} \left( -\sum_{k=0}^5 (5-k)! m^{(k)}(1) + m^{(6)}(0) \sum_{k=1}^6 \frac{1}{k} + \int_0^1 \frac{m^{(6)}(t)}{t} dt \right) \\ &= \frac{1}{6!} \left( 300h(1) + 84h^{(1)}(1) + 36h^{(2)}(1) + 32h^{(3)}(1) + h^{(4)}(1) + h^{(5)}(1) \right. \\ & \quad \left. + \int_0^1 \frac{-30h^{(4)}(x) - h^{(6)}(x)}{x} dx \right) \\ &= \frac{1}{6!} \left( 576R + (-600 + 72R^2 - 2R^4) \sin R + (-168R + 64R^3 - 2R^5) \cos R \right. \\ & \quad \left. + (75 - 36R^2 + 4R^4) \sin(2R) + (42R - 64R^3 + 8R^5) \cos(2R) \right. \\ & \quad \left. + \int_0^1 \frac{(60R^4 - 2R^6) \sin(Rx) + (-120R^4 + 16R^6) \sin(2Rx)}{x} dx \right). \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \frac{q(x)}{x^5} dx \\ &= \frac{1}{4!} \left( -\sum_{k=0}^3 (3-k)! q^{(k)}(1) + q^{(4)}(0) \sum_{k=1}^4 \frac{1}{k} + \int_0^1 \frac{q^{(4)}(t)}{t} dt \right) \\ &= \frac{1}{4!} \left( -6h(1) + 6h^{(1)}(1) - 3h^{(2)}(1) - 3h^{(3)}(1) + \int_0^1 \frac{-12h^{(2)}(x) + 3h^{(4)}(x)}{x} dx \right) \\ &= \frac{1}{4!} \left( (12 - 6R^2) \sin R + (-12R - 6R^3) \cos R \right. \\ & \quad \left. + \left(-\frac{3}{2} + 3R^2\right) \sin(2R) + (3R + 6R^3) \cos(2R) \right. \\ & \quad \left. + \int_0^1 \frac{(-24R^2 - 6R^4) \sin(Rx) + (12R^2 + 12R^4) \sin(2Rx)}{x} dx \right). \end{aligned}$$

This completes the proof.  $\square$

## References

- [1] J. Berndt, F. Tricerri, and L. Vanhecke, *Geometry of generalized Heisenberg groups and their Damak-Ricci harmonic extensions*, Lecture Notes in Mathematics, **1598** (1995), 51-68.

- [2] P. Eberlein, *Geometry of 2-step Nilpotent Lie groups with a left invariant metric*, Ann. Scient. Ecole Normale Sup., **27** (1994), no. 5, 611-660.
- [3] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, Berlin, 1990.
- [4] C. Jang, J. Kim, Y. Kim, and K. Park, *Conjugate points on the Quaternionic Heisenberg group*, Jour. of Korean Math. Soc., **40** (2003), 61-72.
- [5] C. Jang, J. Park, and K. Park, *Geodesic Spheres and Balls of the Heisenberg groups*, Commun. Korean Math. Soc., **25** (2010), 83-96.
- [6] C. Jang and K. Park, *Conjugate Points on 2-step Nilpotent Groups*, Geom. Dedicata, **79** (2000), 65-80.
- [7] S. Jeong and K. Park, *Volume of Geodesic Balls in Heisenberg groups*, Jour. of the Chungcheong Math. Soc., **31** (2018), no. 4, 369-379.
- [8] A. Kaplan, *Riemannian Nilmanifolds attached to Clifford modules*, Geom. Dedicata, **11** (1981), 127-136.
- [9] A. Kaplan, *On the geometry of groups of Heisenberg Type*, Bull. London Math. Soc., **15** (1983), 35-42.
- [10] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer-Verlag, Berlin, 1972.
- [11] G. Walschap, *Cut and Conjugate Loci in two-step Nilpotent Lie groups*, Jour. of Geometric Analysis, **7** (1997), 343-355.
- [12] Wikipedia the Free Encyclopedia, *Volume of an n-ball*, [https://en.wikipedia.org/wiki/Volume\\_of\\_an\\_n-ball](https://en.wikipedia.org/wiki/Volume_of_an_n-ball).

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