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A FIXED POINT APPROACH TO THE STABILITY OF THE QUADRATIC AND QUARTIC TYPE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of the quadratic and quartic type functional equations

$$\begin{aligned}
&f(kx+y)+f(kx-y)-k^2f(x+y)-k^2f(x-y)-2f(kx)\\
&\qquad +2k^2f(x)+2(k^2-1)f(y)=0,\\
&f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)+5f(x+y)\\
&\qquad -f(-x)=0,\\
&f(kx+y)+f(kx-y)-k^2f(x+y)-k^2f(x-y)\\
&\qquad -\frac{k^2(k^2-1)}{6}[f(2x)-4f(x)]+2(k^2-1)f(y)=0
\end{aligned}$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

Throughout this paper, let V and W be real vector spaces, Y a real Banach space, and k a fixed nonzero real number such that $|k| \neq 1$. For a given mapping $f : V \to W$, we use the following abbreviations

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$$
\begin{aligned}
f_e(x) &:= \frac{f(x)+f(-x)}{2},\\
D_1 f(x,y) &= f(kx+y)+f(kx-y)-k^2f(x+y)-k^2f(x-y)-2f(kx)\\
&\quad +2k^2f(x)+2(k^2-1)f(y),\\
D_2 f(x,y) &= f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)\\
&\quad +5f(x+y)-f(-x),\\
D_3 f(x,y) &= f(kx+y)+f(kx-y)-k^2f(x+y)-k^2f(x-y)\\
&\quad -\frac{k^2(k^2-1)}{6}[f(2x)-4f(x)]+2(k^2-1)f(y),
\end{aligned}
$$

for all $x, y \in V$. In 1940, the problem for the stability of group homomorphism was first raised by S. M. Ulam [14]. In the next year, D. H. Hyers [10] gave a partial solution to Ulam's question for the case of additive mappings. Hyers' result has greatly influenced the study of the stability problem of the functional equation. His result was generalized by Th. M. Rassias [13] and P. Găvruta [6].

In 2003, L. Cădariu and V. Radu [3] proved the stability of *the quadratic functional equation:*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1.1}$$

by using the fixed point method [4]. We call a solution of (1.1) *a quadratic mapping.* Notice that a mapping $f : V \to W$ is called a quartic mapping if f is a solution of the quartic functional equation

$$f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-24f(y)=0. \tag{1.2}$$

A mapping f is a quadratic and quartic mapping if f is represented by sum of a quadratic mapping and a quartic mapping. A functional equation is called a quadratic and quartic type functional equation provided that each solution of that equation is a quadratic and quartic mapping. Many mathematicians investigated the stability of the quadratic and quartic type functional equations [1, 7, 9, 8, 11, 15, 16].

In this paper, we will show that the functional equations $D_1f(x,y) = 0$, $D_2f(x,y) = 0$, $D_3f(x,y) = 0$ are quadratic and quartic type functional equations. Many mathematicians proved the stability of the quadratic and quartic functional equations by handling the quadratic part and the quartic part of the given mapping f, respectively. In this paper, instead of splitting the given mapping $f : V \to Y$ into two parts, we will prove the stability of the functional equations $D_1f(x,y) = 0$,

$D_2f(x,y)=0$, $D_3f(x,y)=0$ at once by using the fixed point theory.

2. Main theorems

We recall the following Margolis and Diaz's fixed point theorem to prove the main theorem.

THEOREM 2.1. ([5]) *Suppose that a complete generalized metric space (X,d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:
(1) $d(J^n x, J^{n+1}x) < +\infty$ for all $n \geq k$;
(2) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;
(4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.

Lee [12] proved the following lemma from Baker's theorem [2].

LEMMA 2.2. (Corollary 2.2 in [12]) *Let V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $r \in \mathbb{Q} - \{0, \pm 1\}$. Suppose that $n_1, \ldots, n_m$ are natural numbers, and $c_{l_i}, d_{l_i}, \alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If a mapping $f : V \to W$ satisfies the equality $f(rx) = r^k f(x)$ for all $x \in V$ and the inequality*

$$f(\alpha_0 x + \beta_0 y) + \sum_{l=1}^{m}\sum_{i=1}^{n_l} c_{l_i} f(d_{l_i}(\alpha_l x + \beta_l y)) = 0$$

for all $x, y \in V$, then f is a monomial mapping of degree k.

The monomial mappings of degree 2 and 4 are also called a quadric mapping and a quartic mapping, respectively.

LEMMA 2.3. *Let $m \in \{1,2,3\}$ and $f : V \to W$ with $f(0) = 0$. Then the equality*

$$f(4x) - 20f(2x) + 64f(x) = E_m f(x) \tag{2.1}$$

holds for all $x \in V$, where $E_m f : V \to W$ is given by

$$\begin{aligned}
E_1 f(x) :=& \frac{1}{k^4 - k^2}(-D_1 f_e(x, (k+2)x) - D_1 f_e(x, (k-2)x) \\
& - 4D_1 f_e(x, (k+1)x) - 4D_1 f_e(x, (k-1)x) + 10 D_1 f_e(x, kx) \\
& + D_1 f_e(2x, 2x) + 4D_1 f_e(2x, x) - k^2 D_1 f_e(x, 3x) \\
& - 2(k^2+1) D_1 f_e(x, 2x) + (17k^2 - 8) D_1 f_e(x, x)) \\
& + \frac{D_1 f(0, 4x) - 20 D_1 f(0, 2x) + 64 D_1 f(0, x)}{2(k^2 - 1)} \\
E_2 f(x) :=& D_2 f(-x, x) + 5 D_2 f(-2x, x) + 25 D_2 f(x, 0) \\
E_3 f(x) :=& \frac{6}{k^4 - k^2}(2D_3 f(x, kx) - D_3 f(2x, 0) + 2k^2 D_3 f(x, x)) \\
& - \frac{12 D_3 f(x, 0)}{k^2} + \frac{12 D_3 f(0, (1-k)x)}{(k^2-1)^2}
\end{aligned}$$

for all $x \in V$.

THEOREM 2.4. *Let m be a fixed integer such that $m \in \{1, 2, 3\}$. If a mapping $f : V \to W$ satisfies $D_m f(x, y) = 0$ for all $x, y \in V$, then f is a quadratic-quartic mapping.*

Proof. Note that $f(0) = 0$ is easily obtained from $D_m f(x, y) = 0$. Define the mappings f_1 and f_2 by

$$f_1(x) := \frac{f(2x) - 16f(x)}{12} \quad \text{and} \quad f_2(x) := \frac{-f(2x) + 4f(x)}{12}.$$

If a mapping $f : V \to W$ satisfies $D_m f(x, y) = 0$ for all $x, y \in V$, then f_1 and f_2 satisfy the equalities

$$D_m f_1(x, y) = 0 \quad \text{and} \quad D_m f_2(x, y) = 0$$

for all $x, y \in V$. We can obtain the equalities $f_1(2x) = 2^2 f_1(x)$ and $f_2(2x) = 2^4 f_2(x)$ from the equalities

$$f(4x) - 20f(2x) + 64f(x) = E_m f(x)$$

for all $x \in V$, where $E_m f : V \to W$ is given in Lemma 2.3. According to Lemma 2.2, f_1 is a quadratic mapping and f_2 is a quartic mapping. Since the equality $f = f_1 + f_2$ holds, f is a quadratic-quartic mapping. □

Now we can prove some stability results of the functional equation $D_m f(x, y) = 0$ $(m = 1, 2, 3)$ by using the fixed point theory.

THEOREM 2.5. *Let m be a fixed integer such that $m \in \{1,2,3\}$ and let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality*

$$\|D_m f(x,y)\| \le \varphi(x,y) \tag{2.2}$$

holds for all $x, y \in V$ with $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(2x,2y) \le 2(\sqrt{41}-5)L\varphi(x,y) \tag{2.3}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying the functional equation $D_m F(x,y) = 0$ and the inequality

$$\|f(x) - F(x)\| \le \frac{\Phi_m(x)}{64(1-L)} \tag{2.4}$$

for all $x \in V$, where $\varphi_e : V^2 \to [0,\infty)$ and Φ_m are defined by

$$\begin{aligned}
\varphi_e(x,y) :=& \frac{\varphi(x,y)+\varphi(-x,-y)}{2},\\
\Phi_1(x) :=& \frac{1}{|k^4-k^2|}\big(\varphi_e(x,(k+2)x) + \varphi_e(x,(k-2)x) + 4\varphi_e(x,(k+1)x)\\
&+ k^2\varphi_e(x,3x) + 4\varphi_e(x,(k-1)x) + 10\varphi_e(x,kx) + \varphi_e(2x,2x)\\
&+ 4\varphi_e(2x,x) + 2(k^2+1)\varphi_e(x,2x) + |17k^2-8|\varphi_e(x,x)\big)\\
&+ \frac{\varphi_e(0,4x) + 20\varphi_e(0,2x) + 64\varphi_e(0,x)}{|k^2-1|},\\
\Phi_2(x) :=& 2\varphi_e(-x,x) + 10\varphi_e(-2x,x) + 50\varphi_e(x,0),\\
\Phi_3(x) :=& \frac{12(\varphi_e(x,kx) + \varphi_e(2x,0) + 2k^2\varphi_e(x,x))}{|k^4-k^2|} + \frac{24\varphi_e(x,0)}{k^2}\\
&+ \frac{24\varphi_e(0,(1-k)x)}{|k^2-1|^2}.
\end{aligned}$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(20)^i}{64^n} f(2^{2n-i}x) \tag{2.5}$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h) = \inf\big\{K \in \mathbb{R}_+ \big|\, \|g(x) - h(x)\| \le K\Phi_m(x) \text{ for all } x \in V\big\}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{20g(2x)}{64} - \frac{g(4x)}{64}$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(20)^i}{64^n} g(2^{2n-i}x)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| \leq & K(\frac{1}{64}\Phi_m(4x) + \frac{20}{64}\Phi_m(2x)) \\
\leq & K(\frac{\sqrt{41}-5}{32}L\Phi_m(2x) + \frac{10}{32}\Phi_m(2x)) \\
\leq & K(\frac{(\sqrt{41}-5)^2}{16}L^2\Phi_m(x) + \frac{10(\sqrt{41}-5)}{16}L\Phi_m(x)) \\
\leq & K\frac{(\sqrt{41}-5)^2 + 10(\sqrt{41}-5)}{16}L\Phi_m(x) \\
\leq & LK\Phi_m(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Using (2.1) we obtain that

$$\|f(x) - Jf(x)\| = \left\| \frac{f(4x) - 20f(2x) + 64f(x)}{64} \right\| = \left\| \frac{E_m f(x)}{64} \right\| \leq \frac{\Phi_m(x)}{64}$$

for all $x \in V$. It means that $d(f, Jf) \leq \frac{1}{64} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (2.5) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1-L}d(f, Jf) \leq \frac{1}{64(1-L)},$$

which implies (2.4). By the definition of F, together with (2.2) and (2.3), we have

$$\begin{aligned}
\|D_mF(x,y)\| &= \lim_{n\to\infty} \|D_mJ^nf(x,y)\| \\
&= \lim_{n\to\infty} \Big\| \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(20)^i}{64^n} D_mf(2^{2n-i}x, 2^{2n-i}y) \Big\| \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{20^i}{64^n} \varphi(2^{2n-i}x, 2^{2n-i}y) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{20^i}{64^n} 2^{n-i}(\sqrt{41}-5)^{n-i}L^{n-i}\varphi(2^nx, 2^ny) \\
&\le \lim_{n\to\infty} \frac{(2(\sqrt{41}-5)+20)^n}{64^n}\varphi(2^nx, 2^ny) \\
&\le \lim_{n\to\infty} \frac{(\sqrt{41}+5)^n(\sqrt{41}-5)^n}{16^n} L^n\varphi(x,y) \\
&\le \lim_{n\to\infty} L^n\varphi(x,y) \\
&= 0
\end{aligned}$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $D_mF(x,y) = 0$. Notice that if F is a solution of the functional equation $D_mF(x,y) = 0$, then the equality $F(x) - JF(x) = \frac{E_mF(x)}{64}$ implies that F is a fixed point of J. □

We continue our investigation with the next result.

THEOREM 2.6. *Let m be a fixed integer such that $m \in \{1,2,3\}$ and let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality (2.2) holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property*

$$L\varphi(2x,2y) \ge \frac{32}{\sqrt{41}-5}\varphi(x,y) \tag{2.6}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying the functional equation $D_mF(x,y) = 0$ and the inequality

$$\|f(x) - F(x)\| \le \frac{(66-10\sqrt{41})L^2}{1024(1-L)}\Phi_m(x) \tag{2.7}$$

for all $x \in V$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 20^i(-64)^{n-i} f\left(\frac{x}{2^{2n-i}}\right) \tag{2.8}$$

for all $x \in V$.

Proof. Let the set (S, d) be as in the proof of Theorem 2.5. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := 20g\left(\frac{x}{2}\right) - 64g\left(\frac{x}{4}\right)$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i 20^i(-64)^{n-i} g\left(\frac{x}{2^{2n-i}}\right)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le& 20\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + 64\left\|g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right)\right\||\\ \le& 64K\Phi_m\left(\frac{x}{4}\right) + 20K\Phi_m\left(\frac{x}{2}\right)\\ \le& L^2\frac{(\sqrt{41}-5)^2}{16}K\Phi_m(x) + 10\frac{\sqrt{41}-5}{16}LK\Phi_m(x)\\ \le& LK\Phi_m(x)\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) and (2.2), we see that

$$\|f(x) - Jf(x)\| = \left\|E_m\left(\frac{x}{4}\right)\right\| \le \Phi_m\left(\frac{x}{4}\right) \le \frac{(\sqrt{41}-5)^2L^2}{32^2}\Phi_m(x)$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{(66-10\sqrt{41})L^2}{1024} < \infty$ by the definition of d. Therefore according to Theorem 2.5, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (2.8) for all $x \in V$. Notice that

$$d(f, F) \le \frac{1}{1-L}d(f, Jf) \le \frac{(66-10\sqrt{41})L^2}{1024(1-L)},$$

which implies (2.7). By the definition of F, together with (2.2) and (2.8), we have

$$
\begin{aligned}
\|D_mF(x,y)\| &= \lim_{n\to\infty} \|D_mJ^nf(x,y)\| \\
&= \lim_{n\to\infty} \left\| \sum_{i=0}^{n} {}_nC_i 20^i(-64)^{n-i} D_mf\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \right\| \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 20^i 64^{n-i} \varphi\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 20^i 64^{n-i} \frac{(\sqrt{41}-5)^{n-i}L^{n-i}}{32^{n-i}} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 20^i (2(\sqrt{41}-5))^{n-i} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&\le \lim_{n\to\infty} (20+2(\sqrt{41}-5))^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&\le \lim_{n\to\infty} 2^n(\sqrt{41}+5)^n \frac{(\sqrt{41}-5)^nL^n}{32^n} \varphi(x,y) \\
&\le \lim_{n\to\infty} L^n\varphi(x,y) \\
&= 0
\end{aligned}
$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $D_mF(x,y) = 0$. Notice that if F is a solution of the functional equation $D_mF(x,y) = 0$, then the equality $F(x) - JF(x) = E_mF\left(\frac{x}{4}\right)$ implies that F is a fixed point of J. □

Since f is a quadratic-quartic mapping if $D_mf(x,y) = 0$, and $f - Jf = 0$ if f is a quadratic-quartic mapping, we obtain the following corollaries from Theorem 2.5 and Theorem 2.6.

Corollary 2.7. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality* (2.2) *holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property* (2.2) *for all $x, y \in V$, then there exists a unique quadratic-quartic mapping $F : V \to Y$ satisfying the inequality* (2.4) *for all $x \in V$.*

Corollary 2.8. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality* (2.2) *holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such* *that φ has the property* (2.6) *for all $x, y \in V$, then there exists a unique quadratic-quartic mapping $F : V \to Y$ satisfying the inequality* (2.7) *for all $x \in V$.*

Corollary 2.9. *Let X be a normed space and $p \in (0, 1+\log_2(\sqrt{41}-5) \cup (6-\log_2(\sqrt{41}-5), \infty)$. If a function $f : X \to Y$ satisfies the inequality*

$$\|D_m f(x,y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and for some $\theta \geq 0$, then there exists a quadratic-quartic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{5(2^p+9)\theta\|x\|^p}{64(1-2^{p-\log_2(\sqrt{41}-5)})} & \text{if } p < 1 + \log_2(\sqrt{41}-5), \\ \frac{(\sqrt{41}-5)^2\theta(2^p+9)2^{2(5-\log_2(\sqrt{41}-5)-p)}\|x\|^p}{32^2(1-2^{4-\log_2(\sqrt{41}-5)-p})} & \\ & \text{if } p > 6 - \log_2(\sqrt{41}-5) \end{cases}$$

for all $x \in X\backslash\{0\}$.

Proof. If we put

$$\varphi(x,y) := \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X\backslash\{0\}$ and

$$L := \begin{cases} 2^{p-1-\log_2(\sqrt{41}-5)} & \text{when } p < 1 + \log_2(\sqrt{41}-5), \\ 2^{6-\log_2(\sqrt{41}-5)-p} & \text{when } p > 6 - \log_2(\sqrt{41}-5), \end{cases}$$

then our assertions follow from Theorems 2.5 and 2.6. □

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