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THE WEAK LAWS OF LARGE NUMBERS FOR SUMS OF ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VECTORS IN HILBERT SPACES

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Abstract. In this paper, the weak laws of large numbers for sums of asymptotically almost negatively associated random vectors in Hilbert spaces are investigated. Some results in Hien and Thanh ([3]) are generalized to asymptotically almost negatively random vectors in Hilbert space.

1. Introduction

Ko et al. ([6]) introduced the concept of negative association (NA) for $\mathbb{R}^d$-valued random vectors as follows: A finite sequence $\{X_1, \cdots, X_m\}$ of $\mathbb{R}^d$-valued random vectors is said to be negatively associated (NA) if for any disjoint nonempty subsets $A, B \subset \{1, \cdots, m\}$ and any nondecreasing functions f on $\mathbb{R}^{|A|d}$ and g on $\mathbb{R}^{|B|d}$

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0 \tag{1.1}$$

whenever the covariance exists. Here and in the sequel, $|A|$ denotes the cardinality of A. An infinite sequence $\{X_i, i \geq 1\}$ of $\mathbb{R}^d$-valued random vectors is negatively associated (NA) if every finite subsequence is negatively associated.

In the case $d = 1$ the concept of negative association had already been introduced by Joag-Dev and Proschan ([5]). A number of well-known multivariate distributions possess the negatively associated property,

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such as multinomial distribution, multivariate hypergeometric distribution, negatively correlated normal distribution and joint distribution of ranks.

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $< \cdot, \cdot >$, let $\{e_j, j \geq 1\}$ be an orthonormal basis in H let $< X, e_j >$ be denoted by $X^{(j)}$, where X is an H-valued random vector. Ko et al. ([6]) extended the concept of negative association for $\mathbb{R}^d$-valued random vectors to the random vectors with values in real separable Hilbert space as follows : A sequence $\{X_i, i \geq 1\}$ of random vectors taking values in a real separable Hilbert space $(H, < \cdot, \cdot >)$ is called negatively associated if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $\{(< X_i, e_1 >, \cdots, < X_i, e_d >); i \geq 1\}$ is negatively associated.

As the above definition of negatively associated random vectors in $\mathbb{R}^d$ we can define asymptotically almost negative association (AANA) in $\mathbb{R}^d$ as follows:

Definition 1.1. A sequence $\{X_1, \cdots, X_m\}$ of $\mathbb{R}^d$-valued random vectors is said to be asymptotically almost negatively associated (AANA) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$(1.2) \qquad \begin{aligned} &Cov(f(X_n), g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})) \\ &\leq q(n)(Var(f(X_n))Var(g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})))^{\frac{1}{2}} \end{aligned}$$

for all, $n, k \geq 1$ and for all coordinatewise nondecreasing continuous functions f and g whenever the variances exist.

In the case $d = 1$ the concept of asymptotically almost negative association has been introduced by Chandra and Ghosal ([1], [2]). Obviously, asymptotically almost negatively associated random variables contain independent random variables (with $q(n) = 0$ for $n \geq 1$) and negatively associated random variables. Chandra and Ghosal ([1]) pointed out that negatively associated implies asymptotically almost negatively associated, but asymptotically almost negatively associated does not imply negatively associated. Because negatively associated has been applied to the reliability theory, multivariate statistical analysis and percolation theory, extending the limit properties of negatively associated random variables to asymptotically almost negatively associated random variables is of interest in the theory and application.

Since the concept of asymptotically almost negative association was introduced, many applications have been established by many authors. For more details, we can refer to Chandra and Ghosal ([1], [2]), Ko et al.

([7]), Yuan and An ([11], [12]), Yuan and Wu ([13]), Wang et al. ([10]), Tang ([9]), Shen and Wu ([8]), and so forth.

As the definition of negatively associated random vectors in Hilbert space we can define asymptotically almost negatively associated random vectors in Hilbert space as follows.

Definition 1.2. A sequence $\{X_i, i \geq 1\}$ of random vectors taking values in a real separable Hilbert space $(H, <\cdot,\cdot>)$ is said to be asymptotically almost negatively associated if for some orthonormal basis $\{e_k : k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $\{(<X_i, e_1>, \cdots, <X_i, e_d>), i \geq 1\}$ is asymptotically almost negatively associated.

The concept of coordinatewise negative association was introduced by Hien and Thanh ([3]) as follows. A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be coordinatewise negatively associated (CNA) if for each $\{e_j, j \geq 1\}$, the sequence $\{X_n^{(j)}, n \geq 1\}$ of random variable is negatively associated, where $X_n^{(j)} = <X_n, e_j>$ for $n \geq 1$.

As above definition of coordinatewise negative association we can define coordinatewise asymptotically almost negative association for random vectors with values in Hilbert space as follows.

Definition 1.3. A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be coordinatewise asymptotically almost negatively associated (CAANA) if for each $\{e_j, j \geq 1\}$ the sequence $\{X_n^{(j)}, n \geq 1\}$ of random variables is asymptotically almost negatively associated, where $X_n^{(j)} = <X_n, e_j>$ for $n \geq 1$.

It is clear that if a sequence of H-valued random vectors is asymptotically almost negatively associated, then it is coordinatewise asymptotically almost negatively associated. However, the reverse is not true in general.

Let $\{X, X_n, n \geq 1\}$ be a sequence of H-valued random vectors. We consider the following inequalities

$$C_1 P(|X^{(j)}| > t) \leq \frac{1}{n}\sum_{k=1}^{n} P(|X_k^{(j)}| > t) \leq C_2 P(|X^{(j)}| > t). \tag{1.3}$$

(1) If there exists a positive constant C_1 (C_2) such that the left-hand side (right-hand side) of (1.3) is satisfied for all $j \geq 1, n \geq 1$ and $t \geq 0$, then the sequence $\{X_n, n \geq 1\}$ is said to be coordinatewise weakly lower (upper) bounded by X.

(2) The sequence $\{X_n, n \geq 1\}$ is said to be coordinatewise weakly bounded by X, if it is both coordinatewise lower and upper bounded by X (see [4]).

In this paper we generalize weak laws of large numbers for H-valued negatively associated random vectors obtained by [3] to asymptotically almost negatively associated random vectors in Hilbert space.

Throughout the paper, let $I(A)$ be the indicator function of the set A. The symbol C denotes a positive constant, which may be different in different places and $a_n = O(b_n)$ stands for $a_n \leq C b_n$.

2. Some lemmas

From the definition of a sequence of asymptotically almost negatively associated random variables, we have

LEMMA 2.1 ([11]). *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables with mixing coefficients $\{q(n), n \geq 1\}$ and let $\{f_n, n \geq 1\}$ be a sequence of nondecreasing continuous functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of asymptotically almost negatively associated random variables with mixing coefficients $\{q(n), n \geq 1\}$.*

LEMMA 2.2 ([11]). *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $EX_n = 0$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C such that for all $n \geq 1$,*

$$E(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i|^2) \leq C(\sum_{i=1}^{n} E|X_i|^2). \tag{2.1}$$

Inspired by the proof of Lemma 2.3 in [14] from Lemma 2.2 we obtain the Rosenthal-type maximal inequality for the H-valued asymptotically almost negatively associated random vectors as follows.

LEMMA 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued asymptotically almost negatively associated random vectors with mixing coefficients $\{q(n), n \geq 1\}$. Let $EX_i = 0$ and $E\|X_i\|^2 < \infty$ for every $i \geq 1$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C such that for all $n \geq 1$,*

$$E \max_{1 \leq k \leq n} \| \sum_{i=1}^{k} X_i\|^2 \leq C(\sum_{i=1}^{n} E\|X_i\|^2). \tag{2.2}$$

REMARK 2.4. If $\{X_n, n \geq 1\}$ is a sequence of H-valued negatively associated random vectors with $EX_i = 0$ and $E\|X_i\|^2 < \infty$, then (2.2) holds when $C = 2$.

3. Main results

H will denote a real separable Hilbert space with orthonormal basis $\{e_j, j \geq 1\}$ and inner product $< \cdot, \cdot >$. We assume that $\{X_n, n \geq 1\}$ is a sequence of asymptotically almost negatively associated random vectors in Hilbert space. For $n, i, j \geq 1$, we set

$$X_i^{(j)} =< X_i, e_j > \text{ and } Y_{ni} = \sum_{j=1}^{\infty} Y_{ni}^{(j)} e_j,$$

where $Y_{ni}^{(j)} = -nI(X_i^{(j)} < -n) + X_i^{(j)} I(|X_i^{(j)}| \leq n) + nI(X_i^{(j)} > n)$.

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued asymptotically almost negatively associated random vectors with coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$. Assume $EX_i = 0$ and $E\|X_i\|^2 < \infty$ for all $i \geq 1$. If*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} P(|X_i^{(j)}| > n) = 0 \tag{3.1}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sum_{j=1}^{\infty} E(|X_i^{(j)}|^2 I(|X_i^{(j)}| \leq n))}{n^2} = 0, \tag{3.2}$$

then we obtain the weak law of large numbers

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - EY_{ni}) \to^p 0 \text{ as } n \to \infty. \tag{3.3}$$

Proof. By the first part of the proof of Theorem 2.1 in [3] we have, for any $\epsilon > 0$

$$P(\frac{1}{n} \| \sum_{i=1}^{n} (X_i - Y_{ni}) \| > \epsilon) \leq P(\cup_{i=1}^{n} (X_i \neq Y_{ni}))$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{\infty} P(|X_i^{(j)}| > n) \to 0 \text{ as } n \to \infty \text{ (by (3.1))}. \tag{3.4}$$

Thus, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{ni}) \to^p 0 \ as \ n \to \infty. \tag{3.5}$$

It is enough to show that

$$\frac{1}{n}\sum_{i=1}^{n}(Y_{ni} - EY_{ni}) \to^p 0 \ as \ n \to \infty. \tag{3.6}$$

By Lemma 2.1 $\{Y_{ni} - EY_{ni}, i \geq 1\}$ is a sequence of H-valued asymptotically almost negatively associated random vectors with coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$. Hence, by Markov inequality, Lemma 2.3 and Hölder's inequality we have

$$\begin{aligned}
& P(\frac{1}{n}\|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})\| > \epsilon) \\
\leq & \frac{1}{\epsilon^2 n^2} E\|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})\|^2 \\
\leq & \frac{C}{n^2}\sum_{i=1}^{n} E\|Y_{ni} - EY_{ni}\|^2) \text{ (by (2.2))} \\
= & \frac{C}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\infty} E|Y_{ni}^{(j)} - EY_{ni}^{(j)}|^2 \\
\leq & \frac{C}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\infty} E|Y_{ni}^{(j)}|^2 \\
\leq & \frac{C}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\infty} n^2 P(|X_i^{(j)}| > n) + \frac{C}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\infty} E(|X_i^{(j)}|^2 I(|X_i^{(j)}| \leq n)) \\
\leq & C\sum_{i=1}^{n}\sum_{j=1}^{\infty} P(|X_i^{(j)}| > n) + C\sum_{i=1}^{n}\sum_{j=1}^{\infty} \frac{E(|X_i^{(j)}|^2 I(|X_i^{(j)}| \leq n))}{n^2} \\
& \to 0 \ as \ n \to \infty \text{ (by (3.1) and (3.2))},
\end{aligned}$$

Thus (3.6) is proved and by combining (3.5) and (3.6) the result (3.3) is obtained. □

REMARK 3.2. Note that Theorem 3.1 for H-valued negatively associated random vectors under assumptions (3.1) and (3.2) holds (see Theorem 2.1 in [3]).

THEOREM 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued asymptotically almost negatively associated random vectors with mixing coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$, $EX_i = 0$ and $E\|X_i\|^2 < \infty$ for all $i \geq 1$. If $\{X_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector X with*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} nP(|X^{(j)}| > n) = 0, \tag{3.7}$$

then we obtain the weak law of large numbers (3.3).

Proof. We first show that (3.5) holds. By (1.3), (3.4) and (3.7) we have

$$\begin{aligned} P(\frac{1}{n}\|\sum_{i=1}^{n}(X_i - Y_{ni})\|) &\leq \sum_{j=1}^{\infty}\sum_{i=1}^{n} P(|X_i^{(j)}| > n) \text{ (by (3.4))} \\ &\leq C\sum_{j=1}^{\infty} nP(|X^{(j)}| > n) \text{ (by (1.3))} \\ &= o(1) \text{ (by (3.7))}, \end{aligned}$$

which yields (3.5). Next we will show that (3.6) holds.

Note that $\{Y_{ni} - EY_{ni}, i \geq 1\}$ is a sequence of H-valued asymptotically almost negatively associated random vectors with coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$ by Lemma 2.1. Hence, it follows from the Markov inequality, Hölder's inequality, Lemma 2.3 and the last part of the proof of Theorem 2.3 in [3] that

$$\begin{aligned} &P(\frac{1}{n}\|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})\| > \epsilon) \\ &\leq Cn^{-1}\sum_{i=1}^{n}(\sum_{j=1}^{\infty} iP(|X^{(j)}| > i)) = o(1) \text{ (by (3.7))}. \end{aligned}$$

which yields (3.6). Combining (3.5) and (3.6) we obtain the weak law of large numbers (3.3). □

COROLLARY 3.4. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued asymptotically almost negatively associated random vectors with mixing coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$. If $\{X_n, n \geq 1\}$ is identically distributed random vectors with $EX_1 = 0$, $E\|X_1\|^2 < \infty$ and*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} nP(|X_1^{(j)}| > n) = 0, \tag{3.8}$$

then we obtain the weak law of large numbers (3.3).

REMARK 3.5. Hien and Thanh ([3]) had already showed that Corollary 3.4 for H-valued negatively associated random vectors holds (see Theorem 2.3 in [3]).

THEOREM 3.6. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued asymptotically almost negatively associated random vectors with mixing coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$, $EX_i = 0$, $E\|X_i\|^2 < \infty$. If $\{X_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector X with*

$$\sum_{j=1}^{\infty} E(|X^{(j)}|) < \infty, \tag{3.9}$$

then we obtain

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to^p 0 \text{ as } n \to \infty. \tag{3.10}$$

Proof. It follows from (3.9) that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} E(|X^{(j)}|I(|X^{(j)}| > n)) = 0. \tag{3.11}$$

Then, by applying the proof of Corollary 2.5 in [3] we obtain

$$\|\frac{1}{n}\sum_{i=1}^{n} EY_{ni}\| \leq C\sum_{j=1}^{\infty} E(|X^{(j)}|)I(|X^{(j)}| > n) \to^p 0 \text{ as } n \to \infty \text{ (by (3.11))}. \tag{3.12}$$

Therefore, the theorem is proved if we show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - EY_{ni}) \to^p 0 \text{ as } n \to \infty. \tag{3.13}$$

Since (3.11) ensure (3.7), it follows from Theorem 3.3 that (3.13) hold. □

COROLLARY 3.7. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued coordinatewise asymptotically almost negatively associated random vectors with coefficients $\{q(n), n \geq 1\}$ such that $\sum_{n=1}^{\infty} q^2(n) < \infty$. If $\{X_n, n \geq 1\}$ is a sequence of identically distributed random vectors with $EX_1 = 0$, $E\|X_1\|^2 < \infty$ and*

$$\sum_{j=1}^{\infty} E(|X_1^{(j)}|) < \infty, \tag{3.14}$$

then we obtain the weak law of large numbers (3.10).

REMARK 3.8. Corollary 2.5 of Hien and Thanh ([3]) shows that Corollary 3.7 for H-valued negatively associated random vectors holds.

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