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STABILITY OF CLOSED SETS IN FLOWS ON TVS-CONE METRIC SPACES

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Abstract. The concept of the stability is very important in dynamical systems. This paper is devoted to the study some properties of stability on a TVS-cone metric space.

1. Introduction and Preliminaries

Stability has been studied in the continuous flow (X, f) on an arbitrary metric space X by N.P. Bhatia and G.P. Szego [2], and in the compact closed relation dynamical systems by G.S. Kim and K.B. Lee [5]. Recently Long-Guang and Xian [1] generalized the notion of metric space by replacing the set of real numbers by an ordered Banach space, defined a cone metric space. I. Beg, A. Abbas, and M. Arshad [3] introuduced a topological vector space valued cone metric space(or shortly TVS-cone metric space). The purpose of this paper is to study some properties of stability in TVS-cone metric space.

We first mention some definitions and theorems.

Definition 1.1. [1] Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if

1) P is closed, nonempty, and $P \neq \{0_E\}$ where 0_E is the zero vector in E,

2) if $x, y \in P$, then $ax + by \in P$ for $a, b \geq 0$ $a, b \in \mathbb{R}$,

3) If $x \in P$ and $-x \in P$, then $x = 0_E$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}P$, $\text{Int}P$ dentoes the interior of P.

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Definition 1.2. [1] Let X be a nonempty set and let E be a Banach space with a cone P. We say (X, d) is a cone metric space if the mapping $d : X \times X \to E$ satisfies

(1) $0_E \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In the case of general metric spaces, the negation of $d(a, b) \geq c$ is $d(a, b) < c$, but it does not hold in the cone metric space, which is shown in the following example.

Example 1.3. *Let* $E = \mathbb{R}^2$ *and* $P = \{(x, y) : x \geq 0,\ y \geq 0\}$. *Then* P *is the cone of* $\mathbb{R}^2$ *under the partial ordering:* $(x_1, y_1) \preceq (x_2, y_2)$ *iff* $x_1 \leq x_2$ *and* $y_1 \leq y_2$. *Also it is clear that* $\mathrm{Int}(P) = \{(x, y) : x > 0, y > 0\} \neq \emptyset$. *Let* $X = \mathbb{R}^2$ *and define* $d : X \times X \to E$ *by* $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|)$. *Then* d *is a cone metric on* X. *Let* $a = (0, 2)$, $b = (1, 0)$ *and* $c = (2, 1)$. *Then* $d(a, b) = (1, 2) \in \mathrm{Int}P$ *and* $c \in \mathrm{Int}P$ *but* $d(a, b)) \npreceq c$ *and* $c \nprec d(a, b)$.

Definition 1.4. [1] Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x if for every $c \in E$ with $0_E \ll c$ there is an $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$. We denote this by $x_n \underset{X}{\longrightarrow} x$.

Lemma 1.5. [4] *Let* P *be a TVS-cone of a topological vector space* E *and* $x, y \in E$. *Then the following statements hold:*

1) If $0_E \ll x$, *then* $0_E \ll ax$ *for each* $a \in \mathbb{R}^+$.
2) If $x \ll y$ *and* $p \preceq q$, *then* $x + p \ll y + q$.
3) If $0_E \ll x$ *and* $0_E \ll y$, *then there is* $z \in E$ *such that* $0_E \ll z$, $z \ll x$ *and* $z \ll y$.

Theorem 1.6. [4] *Let* (X, d) *be a TVS-cone metric space. Put* $\mathfrak{B} = \{B(x, \epsilon) : x \in X$ *and* $0_E \ll \epsilon\}$, *where* $B(x, \epsilon) = \{y \in X : d(x, y) \ll \epsilon\}$. *Then* $\mathfrak{B}$ *is a base for some topology on* X.

In this paper, we always suppose that a cone P is a TVS-cone of a topological vector space E and a TVS-cone metric space (X, d) is a topological space with the topology $\Im$, which is generated by $\mathfrak{B}$.

Theorem 1.7. *A TVS-cone metric space* X *is first countable.*

Proof. Let $0_E \ll \epsilon$ be given. We show that $\{B(x, \frac{1}{n}\epsilon) : n = 1, 2, 3, \cdots\}$ is a conuntable basis at x for any $x \in X$. For any $\delta \gg 0_E$ define a map $\theta : E \to E$ by $\theta(v) = v + \delta$. Since $\theta(0_E) = \delta \in \mathrm{Int}P$ and θ is continuous, there exists a symmetric neighborhood U of 0_E such that $\theta(U) \subset \text{Int}P$. Since $\frac{1}{n}\epsilon \underset{X}{\longrightarrow} 0_E$, there is a natural number m such that $\frac{1}{m}\epsilon \in U$. Since $-\frac{1}{m}\epsilon \in -U = U$, $\delta - \frac{1}{m}\epsilon = \theta(-\frac{1}{m}\epsilon) \in \theta(U) \subset \text{Int}P$. Therefore $\frac{1}{m}\epsilon \ll \delta$. Hence $B(x, \frac{1}{m}\epsilon) \subset B(x, \delta)$. □

Now we define a flow and also define the positive semi-trajectory, positive limit set, and positive prolongational limit set and study some properties of them on a TVS-cone metric space.

Definition 1.8. Let (X, d) be a TVS-cone metric space. A flow on X is the triplet $(X, \mathbb{R}, f)$, where f is a map from the product space $X \times \mathbb{R}$ into the space X satisfying the following axioms;

(1) (Identity axiom) $f(x, 0) = x$ for every $x \in X$;

(2) (Group axiom) $f(f(x, t_1), t_2) = f(x, t_1 + t_2)$ for every $x \in X$ and $t_1, t_2 \in \mathbb{R}$;

(3) (Continuous axiom) f is continuous:

In the sequel we shall generally delete the symbol f. Thus the image $f(x, t)$ will be written simply as xt.

Definition 1.9. [2] Define maps γ^+, Λ^+, and J^+ from X into 2^X by defining for any $x \in X$,

$\gamma^+(x) = \{xt : t \in \mathbb{R}\}$, $\Lambda^+(x) = \{y \in X$: there is a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to +\infty$ and $xt_n \underset{X}{\longrightarrow} y\}$, $J^+(x) = \{y \in X$: there is a sequence$\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \underset{X}{\longrightarrow} x$, $t_n \to +\infty$, and $x_n t_n \underset{X}{\longrightarrow} y\}$.

For any $x \in X$, the sets $\gamma^+(x)$, $\Lambda^+(x)$ and $J^+(x)$ are called the positive semi-trajectory, positive (or omega) limit set, and positive prolongational limit set of x, respectively.

Theorem 1.10. *Let $x \in X$.*

(1) $\Lambda^+(x)$, $J^+(x)$ are closed invariant sets.

(2) $\overline{\gamma^+(x)} = \gamma^+(x) \cup \Lambda^+(x)$.

(3) If $\overline{\gamma^+(x)}$ is compact, then $\Lambda^+(x) \neq \emptyset$.

Proof. (1) Let $\{y_n\}$ be a sequence in $\Lambda^+(x)$ with $y_n \underset{X}{\longrightarrow} y$. For each k since $y_k \in \Lambda^+(x)$, there is a sequence $\{t_n^k\}$ in $\mathbb{R}^+$ with $t_n^k \to +\infty$ and $xt_n^k \underset{X}{\longrightarrow} y_k$. For any $\epsilon \gg 0_E$ we may assume without loss of generality that $d(y_k, xt_n^k) \ll \frac{1}{k}\epsilon$ and $t_n^k \geq k$ for $n \geq k$. Consider now the sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n = t_n^n$. Then $t_n \to +\infty$ and we claim that $xt_n \underset{X}{\longrightarrow} y$. To see that, observe that

$$d(y, xt_n) \preceq d(y, y_n) + d(y_n, xt_n) \ll d(y, y_n) + \tfrac{1}{n}\epsilon.$$

Since $\frac{1}{n}\epsilon$ and $d(y, y_n)$ tend to the zero vector we conclude that $d(y, xt_n)$ is converges to 0_E. Consequently, $xt_n \underset{X}{\longrightarrow} y$ and $y \in \Lambda^+(x)$. Therefore $\Lambda^+(x)$ is closed.

Let $y \in \Lambda^+(x)$ and $t \in \mathbb{R}$. Then there is a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to +\infty$ and $xt_n \underset{X}{\longrightarrow} y$. Then by the continuity axiom $(xt_n)t \underset{X}{\longrightarrow} yt$. Since $(xt_n)t = x(t_n + t)$ and $t_n + t \to +\infty$ we have $yt \in \Lambda^+(x)$ and $\Lambda^+(x)$ is invariant.

Let $\{y_n\}$ be a sequence in $J^+(x)$ with $y_n \underset{X}{\longrightarrow} y \in X$. For each k since $y_k \in J^+(x)$, there are sequences $\{x_n^k\}$ in X and $\{t_n^k\}$ in $\mathbb{R}^+$ such that $x_n^k \underset{X}{\longrightarrow} x \in X$, $t_n^k \to +\infty$ and $x_n^k t_n^k \underset{X}{\longrightarrow} y_k$. For any $\epsilon \gg 0_E$ we may assume without loss of generality that $d(x, x_n^k) \ll \frac{1}{k}\epsilon$, $t_n^k \geq k$ and $d(y_k, x_n^k t_n^k) \ll \frac{1}{k}\epsilon$ for all $n \geq k$. Consider now the sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ with $x_n = x_n^n$ and $t_n = t_n^n$. Then $t_n \to +\infty$, $x_n \underset{X}{\longrightarrow} x$ and we claim that $x_n t_n \underset{X}{\longrightarrow} y$.

To see that, observe that

$$d(y, x_n t_n) \preceq d(y, y_n) + d(y_n, x_n t_n) \ll d(y, y_n) + \tfrac{1}{n}\epsilon.$$

Since $d(y, y_n)$ and $\frac{1}{n}\epsilon$ tend to 0_E we conclude that $d(y, x_n t_n) \underset{X}{\longrightarrow} 0_E$. Consequently, $x_n t_n \underset{X}{\longrightarrow} y$ and $y \in J^+(x)$. Therefore $J^+(x)$ is closed.

Let $y \in J^+(x)$ and $t \in \mathbb{R}$. Then there are sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \underset{X}{\longrightarrow} x \in X$, $t_n \to +\infty$ and $x_n t_n \underset{X}{\longrightarrow} y$. Then by the continuity axiom $(x_n t_n)t \underset{X}{\longrightarrow} yt$. Since $(x_n t_n)t = x_n(t_n + t)$ and $t_n + t \to +\infty$ we have $yt \in J^+(x)$ and $J^+(x)$ is invariant.

(2) For this recall that $\gamma^+(x) = x\mathbb{R}^+$. By the definition of $\Lambda^+(x)$ we have $\gamma^+(x) \cup \Lambda^+(x) \subset \overline{\gamma^+(x)}$. To see that $\overline{\gamma^+(x)} \subset \gamma^+(x) \cup \Lambda^+(x)$, let $y \in \overline{\gamma^+(x)}$. Then there is a sequence $\{y_n\}$ in $\gamma^+(x)$ such that $y_n \underset{X}{\longrightarrow} y$. Now $y_n = xt_n$ for a $t_n \in \mathbb{R}^+$. Either the sequence $\{t_n\}$ has the property that $t_n \to +\infty$, in which case $y \in \Lambda^+(x)$, or there is a subsequence $t_{n_k} \to t \in \mathbb{R}^+$ (as $\mathbb{R}^+$ is closed). But $xt_{n_k} \underset{X}{\longrightarrow} xt \in \gamma^+(x)$, and since $xt_{n_k} \underset{X}{\longrightarrow} y$ we have $y = xt \in \gamma^+(x)$. Thus $\overline{\gamma^+(x)} \subset \gamma^+(x) \cup \Lambda^+(x)$.

(3) Let $x_n = xn$. Then $\{x_n\}$ is a sequence in $\overline{\gamma^+(x)}$. Since $\overline{\gamma^+(x)}$ is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_n \underset{X}{\longrightarrow} y$. Then $y \in \Lambda^+(x) \neq \emptyset$. □

PROPOSITION 1.11. *A TVS-cone metric space X is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$ and let $\epsilon \gg 0_E$ be given. If $B(x, \frac{1}{n}\epsilon) \cap B(y, \frac{1}{n}\epsilon) \neq \emptyset$ for any natural number n, then we can choose $x_n \in B(x, \frac{1}{n}\epsilon) \cap B(y, \frac{1}{n}\epsilon)$. Since $0_E \preceq d(x,y) \preceq d(x,x_n) + d(x_n,y) \ll \frac{1}{n}\epsilon + \frac{1}{n}\epsilon = \frac{2}{n}\epsilon$ and $\frac{2}{n}\epsilon \underset{X}{\longrightarrow} 0_E$, $d(x,y) = 0_E$. This is a contradiction. Thus there is a natrual number n such that $B(x, \frac{1}{n}\epsilon) \cap B(y, \frac{1}{n}\epsilon) = \emptyset$. Therefore X is Hausdorff. □

PROPOSITION 1.12. *Let X be a locally compact TVS-cone metric space and M be a compact subset of X. Then there exists an $\epsilon \gg 0_E$ such that $\overline{B(M,\epsilon)} \subset U$ for any neighborhood U of M and $\overline{B(M,\epsilon)}$ is compact.*

Proof. Since X is Hausdorff locally compact, there exists a neighborhood V of M such that $\overline{V} \subset U$ and $\overline{V}$ is compact. For every $x \in M$ we can find $\epsilon(x) \gg 0_E$ so that $B(x, \epsilon(x)) \subset V$. Since $\{B(x, \frac{1}{2}\epsilon(x)) : x \in M\}$ is an open cover of M and M is compact, we can find $x_1, x_2, \cdots, x_n \in M$ so that $M \subset \cup_{k=1}^{n} B(x_k, \frac{1}{2}\epsilon(x_k))$. By Lemma 1.5, there exists an $\epsilon \gg 0_E$ such that $\epsilon \ll \frac{1}{2}\epsilon(x_1), \cdots, \epsilon \ll \frac{1}{2}\epsilon(x_n)$. For any $y \in B(M,\epsilon)$ we can find $x \in M$ such that $d(x,y) \ll \epsilon$. Choose k with $d(x_k, x) \ll \frac{1}{2}\epsilon(x_k)$, and $d(x_k, y) \preceq d(x_k, x) + d(x, y) \ll \frac{1}{2}\epsilon(x_k) + \epsilon \ll \frac{1}{2}\epsilon(x_k) + \frac{1}{2}\epsilon(x_k) = \epsilon(x_k)$. It means that $y \in B(x_k, \epsilon(x_k))$. Therefore $B(M,\epsilon) \subset \cup_{k=1}^{n} B(x_k, \epsilon(x_k)) \subset V$. Hence $\overline{B(M,\epsilon)} \subset \overline{V} \subset U$ and $\overline{B(M,\epsilon)}$ is compact. □

PROPOSITION 1.13. *Let X be locally compact TVS-cone metric space. Then $\Lambda^+(x) \neq \emptyset$ whenever $J^+(x)$ is nonempty and compact.*

Proof. Suppose that $\Lambda^+(x) = \emptyset$. Since $\overline{\gamma^+(x)} = \gamma^+(x) \cup \Lambda^+(x) = \gamma^+(x)$, $\gamma^+(x)$ is a closed set. If $\gamma^+(x) \cap J^+(x) \neq \emptyset$, then $\gamma^+(x) \subset J^+(x)$ because of $J^+(x)$ is invariant. Since $J^+(x)$ is compact, $\gamma^+(x)$ is also compact. By Theorem 1.10, $\Lambda^+(x) \neq \emptyset$. This is a contradiction. Therefore $\gamma^+(x) \cap J^+(x) = \emptyset$. By Proposition 1.11, there exists an $\epsilon \gg 0_E$ such that $\overline{B(J^+(x),\epsilon)} \cap \gamma^+(x) = \emptyset$ and $\overline{B(J^+(x),\epsilon)}$ is compact. If $y \in B(J^+(x), \frac{1}{2}\epsilon) \cap B(x, \frac{1}{2}\epsilon)$, then there exists $z \in J^+(x)$ such that $d(z,y) \ll \frac{1}{2}\epsilon$ and $d(z,x) \preceq d(z,y) + d(y,x) \ll \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$. It means that $x \in B(J^+(x),\epsilon) \cap \gamma^+(x)$. This is a contradiction. Hence $B(J^+(x), \frac{1}{2}\epsilon) \cap B(x, \frac{1}{2}\epsilon) = \emptyset$.

Let $y \in J^+(x)$. There exist sequence $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \underset{X}{\longrightarrow} x$, $t_n \to +\infty$ and $x_n t_n \underset{X}{\longrightarrow} y$. We can suppose that $x_n \in B(x, \frac{1}{2}\epsilon)$ and $x_n t_n \in B(J^+(x), \frac{1}{2}\epsilon)$ for all n. Since $x_n[0, t_n]$ is connected, there is a $0 < \tau_n < t_n$ such that $x_n \tau_n \in \partial B(J^+(x), \frac{1}{2}\epsilon)$. Since

$\{x_n\tau_n\}$ is a sequence in $\partial B(J^+(x), \frac{1}{2}\epsilon)$ and $\partial B(J^+(x), \frac{1}{2}\epsilon)$ is compact, $\{x_n\tau_n\}$ has a convergent subsequence. Let $x_n\tau_n \underset{X}{\longrightarrow} z$.

If $\tau_n \to \tau$, then $x_n\tau_n \underset{X}{\longrightarrow} x\tau$. It means that $\partial B(J^+(x), \frac{1}{2}\epsilon) \cap \gamma^+(x) \neq \emptyset$. This is a contradiction. If $\tau_n \to +\infty$, then $z \in J^+(x)$, i.e., $\partial B(J^+(x), \frac{1}{2}\epsilon) \cap J^+(x) \neq \emptyset$. This is a contradiction. Hence $\Lambda^+(x) \neq \emptyset$. □

2. Main theorem

Stability has been studied in a flow $f : X \times \mathbb{R} \to X$ on an arbitrary metric space X by N.P. Bhatia and G.P. Szego [2]. We look into the stability in a flow $f : X \times \mathbb{R} \to X$ on a TVS-cone metric space X.

Let X be a TVS-cone metric space and $f : X \times \mathbb{R} \to X$ be a flow on X. For $A \subset X$, we denote $\gamma^+(A) = \cup_{a\in A}\gamma^+(a)$.

Definition 2.1. Let M be a closed subset of X. M is said to be stable if for every $x \in M$ and $\epsilon \gg 0_E$ there exists a $\delta \gg 0_E$ such that $\gamma^+(B(x,\delta)) \subset B(M,\epsilon)$. M is said to be uniformly stable if for every $x \notin M$ there exists an $\epsilon \gg 0_E$ such that $x \notin \overline{\gamma^+(B(M,\epsilon))}$. M is said to be Lyapunov stable if for every $\epsilon \gg 0_E$ there exists a $\delta \gg 0_E$ such that $\gamma^+(B(M,\delta)) \subset B(M,\epsilon)$.

Theorem 2.2. *Let M be a closed subset of X. If M is Lyapunov stable, then M is stable and uniformly stable.*

Proof. Since M is Lyapunov stable, for any $\epsilon \gg 0_E$, there exists a $\delta \gg 0_E$ such that $\gamma^+(B(M,\delta)) \subset B(M,\epsilon)$. $\gamma^+(B(x,\delta)) \subset \gamma^+(B(M,\delta)) \subset B(M,\epsilon)$ for every $x \in M$. Hence M is stable.

Let $x \notin M$. Since $X - M$ is an open set, there exists an $\epsilon \gg 0_E$ such that $B(x,\epsilon) \subset X - M$. Since M is Lyapunov stable, there exists a $\delta \gg 0_E$ such that $\gamma^+(B(M,\delta)) \subset B(M, \frac{1}{2}\epsilon)$. Suppose that $B(x, \frac{1}{2}\epsilon) \cap \gamma^+(B(M,\delta)) \neq \emptyset$. We can find $y \in B(x, \frac{1}{2}\epsilon) \cap \gamma^+(B(M,\delta))$. Since $y \in \gamma^+(B(M,\delta)) \subset B(M, \frac{1}{2}\epsilon)$, there exists $z \in M$ such that $d(z,y) \ll \frac{1}{2}\epsilon$. Since $d(x,z) \preceq d(x,y) + d(y,z) \ll \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$, $z \in B(x,\epsilon) \subset X - M$. This is a contradiction. Hence $B(x, \frac{1}{2}\epsilon) \cap \gamma^+(B(M,\delta)) = \emptyset$. Therefore $x \notin \overline{\gamma^+(B(M,\delta))}$. It means that M is uniformly stable. □

Theorem 2.3. *If a compact subset M of X is stable, then M is Lyapunov stable.*

Proof. Let $\epsilon \gg 0_E$ be given. Since M is stable, for every $x \in M$ there exists a $\delta_x \gg 0_E$ such that $\gamma^+(B(x,\delta(x))) \subset B(M,\epsilon)$. Since

$\{B(x, \frac{1}{2}\delta(x)) : x \in M\}$ is an open cover of M and M is compact, there exist finitely many $x_1, x_2, \cdots, x_n \in M$ such that $M \subset \cup_{k=1}^n B(x_k, \frac{1}{2}\delta(x_k))$. By Lemma 1.5, there exsits an $\alpha \gg 0_E$ such that $\alpha \ll \frac{1}{2}\delta(x_1), \cdots, \alpha \ll \frac{1}{2}\delta(x_n))$. For any $y \in B(M, \alpha)$, there exist $x \in M$ and k such that $d(x, y) \ll \alpha$ and $x \in B(x_k, \frac{1}{2}\delta(x_k))$. Since $d(x_k, y) \preceq d(x_k, x) + d(x, y) \ll \frac{1}{2}\delta(x_k) + \frac{1}{2}\delta(x_k) = \delta(x_k)$, we have $y \in B(x_k, \delta(x_k))$. Therefore $B(M, \alpha) \subset \cup_{k=1}^n B(x_k, \delta(x_k))$.

Since $\gamma^+(B(M, \alpha)) \subset \gamma^+(\cup_{k=1}^n B(x_k, \delta(x_k))) = \cup_{k=1}^n \gamma^+(B(x_k, \delta(x_k))) \subset B(M, \alpha)$, we see that M is Lyapunov stable. □

THEOREM 2.4. *Let X be sequentially compact. If a closed subset M of X is uniformly stable, then M is Lyapunov stable.*

Proof. Suppose that M is not Lyapunov stable. Then there exsits an $\epsilon \gg 0_E$ such that for any $\delta \gg 0_E$, $\gamma^+(B(M, \delta)) \not\subset B(M, \epsilon)$. For every positive integer n since $\gamma^+(B(M, \frac{1}{n}\epsilon)) \not\subset B(M, \epsilon)$, there exists $y_n \in \gamma^+(B(M, \frac{1}{n}\epsilon)) - B(M, \epsilon)$. We can find an $x_n \in B(M, \frac{1}{n}\epsilon)$ so that $y_n \in \gamma^+(x_n)$. Since X is sequentially compact, $\{y_n\}$ has a convergent subsequence. Let $y_n \underset{X}{\longrightarrow} y \in X$. Since $y_n \in X - B(M, \epsilon)$, $y \in \overline{X - B(M, \epsilon)} = X - B(M, \epsilon)$, we have $y \notin M$. Since M is Lyapunov stable there exists a $\delta \gg 0_E$ such that $y \notin \overline{\gamma^+(B(M, \delta))}$ and $\delta \in \mathrm{Int}P$. Define a map $\theta : E \to E$ by $\theta(v) = v + \delta$. Since $\theta(0_E) = \delta \in \mathrm{Int}P$ and θ is continuous there exists a symmetric neighborhood U of 0_E such that $\theta(U) \subset \mathrm{Int}P$. Since $\frac{1}{n}\epsilon \underset{X}{\longrightarrow} 0_E$ and $y_n \underset{X}{\longrightarrow} y$ there exists a n such that $\frac{1}{n}\epsilon \in U$ and $y_n \in X - \overline{\gamma^+(B(M, \delta))}$. Since $-\frac{1}{n}\epsilon \in -U = U$, $\delta - \frac{1}{n}\epsilon = \theta(-\frac{1}{n}\epsilon) \in \theta(U) \subset \mathrm{Int}P$. Thus $\frac{1}{n}\epsilon \ll \delta$. Hence $y_n \in \gamma^+(x_n) \subset \gamma^+(B(M, \frac{1}{n}\epsilon)) \subset \gamma^+(B(M, \delta)) \subset \overline{\gamma^+(B(M, \delta))}$. This is a contradiction. Thus M is Lyapunov stable. □

THEOREM 2.5. *Let M be a closed subset of X. If M is stable or uniformly stable, M is positively invariant.*

Proof. Suppose that there exists $x \in M$ and $t > 0$ such that $xt \notin M$. Since $X - M$ is open there exsits an $\epsilon \gg 0_E$ such that $B(xt, \epsilon) \subset X - M$.

Suppose M is stable. There exists a $\delta \gg 0_E$ such that $\gamma^+(B(x, \delta)) \subset B(M, \epsilon)$. Since $xt \in \gamma^+(x) \subset \gamma^+(B(x, \delta)) \subset B(M, \epsilon)$ there is $y \in M$ such that $d(y, xt) \ll \epsilon$, i.e., $B(xt, \epsilon) \subset X - M$. This is a contradiction.

Suppose M is uniformly stable. There exists an $\epsilon \gg 0_E$ such that $xt \notin \overline{\gamma^+(B(M, \epsilon))}$. But $xt \in \gamma^+(x) \subset \gamma^+(B(M, \epsilon)) \subset \gamma^+(B(M, \epsilon))$. This is a contradiction. So M is positively invariant. □

Corollary 2.6. *If a closed subset M of X is Lyapunov stable, then M is positively invariant.*

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