JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 3, August 2019
http://dx.doi.org/10.14403/jcms.2019.32.3.281

# QUASI-CONCIRCULAR CURVATURE TENSOR ON A LORENTZIAN $\beta$-KENMOTSU MANIFOLD

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Abstract. In the present paper, we study quasi-concircular curvature tensor satisfying certain curvature conditions on a Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection.

## 1. Introduction

Let $(M^n, g)$ be an odd dimensional $(n = 2m + 1 > 1)$ smooth manifold. It is well known that an almost contact metric structure $(\phi, \xi, \eta, g)$ can be defined on $M$ by a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$. If $M$ has a Sasakian (resp. Kenmotsu) structure, then $M$ is called a Sasakian (resp. Kenmotsu) manifold. Sasakian manifolds and Kenmotsu manifolds have been studied by various authors.

In the Gray-Hervella classification of almost Hermitian manifolds [10], there appears a class $W_4$ of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure $(\phi, \xi, \eta, g)$ on a manifold $M$ is called a trans-Sasakian structure [16], if the product manifold $(M\times R, J, G)$ belongs to the class $W_4$, where $J$ is the almost complex structure on $M\times R$ defined by

$$J(X, ad/dt) = (\phi X - a\xi, \eta(X)ad/dt)$$

for all vector fields $X$ on $M$ and smooth function $a$ on $M\times R$ and $G$ is the product metric on $M\times R$. This may be expressed by the condition

Received February 27, 2019; Accepted June 13, 2019.
2010 Mathematics Subject Classification: 53C50, 53C25, 53C40, 52B25.
Key words and phrases: Lorentzian $\beta$-Kenmotsu manifold, quasi-concircular curvature tensor, symmetric and Ricci pseudo-symmetric Lorentzian $\beta$-Kenmotsu manifold, semi-symmetric semi-metric connection.
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[2]

$$(1.1)\qquad (\nabla_X\phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X]$$

for some smooth functions $\alpha$ and $\beta$ on $M$ and we say that the trans-Sasakian structure is of type $(\alpha,\beta)$.
From the condition (1.1) it follows that

$$(1.2)\qquad \nabla_X\xi = -\alpha\phi X + \beta[X - \eta(X)\xi],$$

$$(1.3)\qquad (\nabla_X\eta)Y = -\alpha g(\phi X,Y)\xi + \beta g(\phi X,\phi Y).$$

In 1981, Janssens and Vanhecke introduced the notion of $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds where $\alpha$ and $\beta$ are non zero real numbers. It is known that [12] trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic ([1], [2]), $\beta$-Kenmotsu [12] and $\alpha$-Sasakian [12] respectively. The local structure of trans-Sasakian manifolds of dimension $n\geq 5$ has been completely characterized by Marrero [13]. He proved that a trans-Sasakian manifold of dimension $n\geq 5$ is either cosymplectic or $\alpha$-Sasakian or $\beta$-Kenmotsu manifold. Trans-Sasakian manifolds have been studied by many authors in several ways to a different extent such as ([3], [5], [6], [18]).

Let $M$ be a differentiable manifold with a Lorentzian metric $g$, that is, a symmetric non-degenerate $(0,2)$-tensor field of index 1, then $M$ is called a Lorentzian manifold. A Lorentzian manifold $M$ has not only spacelike vector fields but also timelike and lightlike vector fields due to the Lorentzian metric of index 1. Hence odd dimensional manifold is able to have a Lorentzian metric.

In the present paper, we study quasi-concircular curvature tensor satisfying certain curvature conditions on a Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection.

The paper is organized as follows : In Section 2, we give a brief introduction of a Lorentzian $\beta$-Kenmotsu manifold and define semi-symmetric semi-metric connection. In Section 3, we deduce the relation between the curvature tensor of Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection and the Levi-Civita connection. Section 4 deals with the study of quasi-concircularly flat and $\xi$-quasi-concircularly flat Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection. In Section 5, we study $\phi$-quasi-concircularly semi-symmetric Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection. Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection satisfying the condition $\bar{C}\cdot\bar{S} = 0$

have discussed in Section 6. Sections 7 and 8 are devoted to investigate symmetric and Ricci-pseudo-symmetric Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection and it is shown that in both cases the manifolds are an $\eta$-Einstein manifold.

## 2. Preliminaries

A differentiable manifold $M$ of dimension $n(= 2m + 1 > 1)$ is called *Lorentzian $\beta$-Kenmotsu manifold* if it admits a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy

(2.1)
$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ g(X,\xi) = \eta(X),\ \phi\xi = 0,\ \eta(\phi X) = 0,$$

(2.2) $$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y)$$

for all $X, Y \in \chi(M)$. Then such a structure $(\phi, \eta, \xi, g)$ is termed as Lorentzian para-contact structure and the manifold $M$ with a Lorentzian para-contact structure is called a *Lorentzian para-contact manifold* [14]. On a Lorentzian para-contact manifold, we also have

(2.3) $$(\nabla_X \phi)(Y) = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X]$$

for any $X, Y \in \chi(M)$, where $\nabla$ is the Levi-Civita connection with respect to the Lorentzian metric $g$. Thus a Lorentzian para-contact manifold satisfying (2.3) is called a Lorentzian $\beta$-Kenmotsu manifold [20]. From (2.3), it is easy to obtain that

(2.4) $$\nabla_X \xi = -\beta\phi^2 X = -\beta[X + \eta(X)\xi],$$

(2.5) $$(\nabla_X \eta)Y = -\beta g(\phi X, \phi Y) = -\beta[g(X,Y) + \eta(X)\eta(Y)].$$

Further, on a Lorentzian $\beta$-Kenmotsu manifold the following relations hold [20]:

(2.6) $$R(X,Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y],$$

(2.7) $$R(\xi, X)Y = \beta^2[g(X,Y)\xi - \eta(Y)X],$$

(2.8) $$R(\xi, X)\xi = \beta^2[\eta(X)\xi + X],$$

(2.9) $$S(X,\xi) = (n-1)\beta^2\eta(X),\ S(\xi,\xi) = -(n-1)\beta^2,$$

(2.10) $$Q\xi = (n-1)\beta^2\xi,$$

where $X, Y \in \chi(M)$ and $S(X,Y) = g(QX, Y)$.

Definition 2.1. The *quasi-concircular curvature tensor* $C$ on an $n$-dimensional Lorentzian $\beta$-Kenmotsu manifold $M$ with respect to the connection $\nabla$ is given by ([15], [17])

$$C(X,Y)Z = aR(X,Y)Z - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{2.11}$$

where $a$ and $b$ are constants such that $a, b \neq 0$ and $R$ is the curvature tensor, $r$ is the scalar curvature with respect to the connection $\nabla$ on $M$. If $a = 1$ and $b = -\frac{1}{n-1}$, then (2.11) takes the form

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] = C^*(X,Y)Z,$$

where $C^*$ is the concircular curvature tensor.

Definition 2.2. A Lorentzian $\beta$-Kenmotsu manifold is said to be an *$\eta$-Einstein manifold* if its Ricci tensor $S$ of type $(0,2)$ satisfies

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y), \tag{2.12}$$

where $\lambda_1$ and $\lambda_2$ are smooth functions on $M$. In particular, if $\lambda_2 = 0$, then an $\eta$-Einstein manifold is an Einstein manifold.

A linear connection $\bar{\nabla}$ on $M$ is said to be a *semi-symmetric connection* [9, 19] if its torsion tensor $T$ of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{2.13}$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.14}$$

where $\eta$ is a 1-form. If moreover, a semi-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{2.15}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold $M$, then $\bar{\nabla}$ is said to be a *semi-symmetric semi-metric connection.*

A relation between the semi-symmetric semi-metric connection $\bar{\nabla}$ and the Levi-Civita connection $\nabla$ on $M$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi. \tag{2.16}$$

## 3. Curvature tensor of Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection

Let $\bar{R}$, $\bar{S}$, $\bar{Q}$ and $\bar{r}$ be the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature of the connection $\bar{\nabla}$ respectively on $M$. Then we have the following relations [11]

(3.1)
$$\bar{R}(X,Y)Z = R(X,Y)Z - (\beta-1)\eta(X)g(Y,Z)\xi + (\beta-1)\eta(Y)g(X,Z)\xi - \beta g(Y,Z)X + \beta g(X,Z)Y,$$

$$\bar{R}(X,Y)\xi = \beta(\beta-1)[\eta(Y)X - \eta(X)Y], \tag{3.2}$$

(3.3)
$$\bar{R}(\xi,X)Y = (\beta^2-1)g(X,Y)\xi - \beta(\beta-1)\eta(Y)X + (\beta-1)\eta(X)\eta(Y)\xi,$$

$$\bar{R}(\xi,X)\xi = \beta(\beta-1)[\eta(X)\xi + X], \tag{3.4}$$

$$\bar{S}(Y,Z) = S(Y,Z) + (2\beta - n\beta - 1)g(Y,Z) + (\beta-1)\eta(Y)\eta(Z), \tag{3.5}$$

$$\bar{S}(Y,\xi) = \beta(\beta-1)(n-1)\eta(Y), \quad \bar{S}(\xi,\xi) = -\beta(\beta-1)(n-1), \tag{3.6}$$

$$\bar{Q}Y = QY + (2\beta - n\beta - 1)Y + (\beta-1)\eta(Y)\xi, \tag{3.7}$$

$$\bar{r} = r - (n-1)[(n-1)\beta + 1] \tag{3.8}$$

for all $X, Y, Z \in \chi(M)$.

Lemma 3.1. *Let $M$ be an $n$-dimensional Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection $\bar{\nabla}$. Then we have*

$$\bar{\nabla}_X\xi = -\beta[X + \eta(X)\xi], \tag{3.9}$$

$$(\bar{\nabla}_X\eta)Y = (1-\beta)[g(X,Y) + \eta(X)\eta(Y)] \tag{3.10}$$

*for all $X, Y \in \chi(M)$.*

## 4. Quasi-concircularly flat and $\xi$-quasi-concircularly flat Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection

Analogous to the Definition 2.1, the quasi-concircular curvature tensor $\bar{C}$ on an $n$-dimensional Lorentzian $\beta$-Kenmotsu manifold $M$ with respect to the semi-symmetric semi-metric connection $\bar{\nabla}$ is given by

$$\bar{C}(X,Y)Z = a\bar{R}(X,Y)Z - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)X - g(X,Z)Y], \tag{4.1}$$

where $a$ and $b$ are constants such that $a, b \neq 0$ and $\bar{R}$, $\bar{S}$ and $\bar{r}$ are the curvature tensor, the Ricci tensor and the scalar curvature with respect to the semi-symmetric semi-metric connection $\bar{\nabla}$ respectively on $M$.

First we assume that the manifold $M$ with respect to the semi-symmetric semi-metric connection is quasi-concircularly flat, that is, $\bar{C}(X,Y)Z = 0$. Then from (4.1), we have

$$a\bar{R}(X,Y)Z - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)X - g(X,Z)Y] = 0. \tag{4.2}$$

Taking inner product of (4.2) with $\xi$ and using (2.1) and (3.1), we have

$$[a(\beta^2 - 1) - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0. \tag{4.3}$$

Thus we have either

$$\bar{r} = \frac{an(n-1)(\beta^2 - 1)}{a + 2b(n-1)}, \quad a + 2b(n-1) \neq 0 \tag{4.4}$$

or

$$g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = 0. \tag{4.5}$$

Putting $Y = \xi$ in (4.5) and using (2.1), we find

$$g(X,Z) + \eta(X)\eta(Z) = 0. \tag{4.6}$$

Replacing $X$ by $\bar{Q}X$ in (4.6) and using (2.1) and (3.7), we obtain

$$\bar{S}(X,Z) = -\beta(\beta - 1)(n-1)\eta(X)\eta(Z). \tag{4.7}$$

Thus we can state the following theorem:

THEOREM 4.1. *If a Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection is quasi-concircularly flat, then either the scalar curvature $\bar{r}$ is $\frac{an(n-1)(\beta^2-1)}{a+2b(n-1)}$, $a + 2b(n-1) \neq 0$ or the manifold is a special type of $\eta$-Einstein manifold.*

Next we assume that the manifold $M$ with respect to the semi-symmetric semi-metric connection is $\xi$-quasi-concircularly flat, that is, $\bar{C}(X,Y)\xi = 0$. Then from (4.1), we have

$$a\bar{R}(X,Y)\xi - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,\xi)X - g(X,\xi)Y] = 0, \tag{4.8}$$

which by using (2.1) and (3.2) yields

$$[a\beta(\beta-1) - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)][\eta(Y)X - \eta(X)Y] = 0. \tag{4.9}$$

Since $\eta(Y)X - \eta(X)Y \neq 0$, therefore we get

$$\bar{r} = \frac{an\beta(\beta-1)(n-1)}{a+2b(n-1)}, \quad a + 2b(n-1) \neq 0. \tag{4.10}$$

Thus we can state the following theorem:

Theorem 4.2. *If a Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection is $\xi$-quasi-concircularly flat, then the scalar curvature $\bar{r}$ is given by* (4.10).

## 5. $\phi$-quasi-concircularly semi-symmetric Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection

Definition 5.1. ([4]) A Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection $(M^n, g)$, $n > 1$, is said to be *$\phi$-quasi-concircularly semi-symmetric* if $\bar{C}(X,Y)\cdot\phi = 0$ on $M$ for all $X$, $Y \in \chi(M)$.

Let $M$ be an $n$-dimensional $(n > 1)$ $\phi$-quasi-concircularly semi-symmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection. Therefore $\bar{C}(X,Y)\cdot\phi = 0$ turns into

$$(\bar{C}(X,Y)\cdot\phi)Z = \bar{C}(X,Y)\phi Z - \phi\bar{C}(X,Y)Z = 0 \tag{5.1}$$

for any vector fields $X$, $Y$ and $Z \in \chi(M)$. In view of (4.1), we have

$$\bar{C}(X,Y)\phi Z = a\bar{R}(X,Y)\phi Z - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,\phi Z)X - g(X,\phi Z)Y] \tag{5.2}$$

and

$$\phi\bar{C}(X,Y)Z = a\phi\bar{R}(X,Y)Z - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)\phi X - g(X,Z)\phi Y]. \tag{5.3}$$

From the equations (5.1), (5.2) and (5.3), we have
(5.4)

$$a\bar{R}(X,Y)\phi Z - a\phi\bar{R}(X,Y)Z + \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)\phi X - g(X,Z)\phi Y$$
$$-g(Y,\phi Z)X + g(X,\phi Z)Y] = 0.$$

By taking $Y = \xi$ in (5.4) and using (2.1), we get

$$a\bar{R}(X,\xi)\phi Z - a\phi\bar{R}(X,\xi)Z + \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[\eta(Z)\phi X + g(X,\phi Z)\xi] = 0$$

which in view of (3.3) takes the form

$$\text{(5.5)} \qquad -a(\beta^2-1)g(X,\phi Z)\xi - a\beta(\beta-1)\eta(Z)\phi X$$
$$+\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)[\eta(Z)\phi X + g(X,\phi Z)\xi] = 0.$$

Now considering $Z$ to be orthogonal to $\xi$, then $\eta(Z) = 0$ and $g(X,\phi Z)\xi \neq 0$. Then we have

$$\text{(5.6)} \quad -a(\beta^2-1) + \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) = 0 \implies \bar{r} = \frac{an(n-1)(\beta^2-1)}{a+2(n-1)b},$$

where $a + 2(n-1)b \neq 0$. Thus we can state the following theorem:

THEOREM 5.2. *For an $n$-dimensional $\phi$-quasi-concircularly semisymmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection, the scalar curvature $\bar{r}$ is*

$$\bar{r} = \frac{an(n-1)(\beta^2-1)}{a+2(n-1)b}, \qquad a + 2(n-1)b \neq 0.$$

## 6. Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection satisfying the condition $\bar{C}\cdot\bar{S} = 0$

Let us consider that the manifold $M$ with respect to the semi-symmetric semi-metric connection $\bar{\nabla}$ satisfying the condition

$$(\bar{C}(X,Y)\cdot\bar{S})(Z,U) = 0.$$

Then we have

$$\text{(6.1)} \qquad \bar{S}(\bar{C}(X,Y)Z,U) + \bar{S}(Z,\bar{C}(X,Y)U) = 0$$

for all $X, Y, Z, U \in \chi(M)$.
Taking $X = U = \xi$ in (6.1), we have

$$\text{(6.2)} \qquad \bar{S}(\bar{C}(\xi,Y)Z,\xi) + \bar{S}(Z,\bar{C}(\xi,Y)\xi) = 0.$$

In view of (4.1), we find

(6.3)
$$\bar{C}(\xi, Y)Z = a(\beta^2 - 1)g(Y,Z)\xi - a\beta(\beta-1)\eta(Z)Y + a(\beta-1)\eta(Y)\eta(Z)\xi$$
$$-\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)(g(Y,Z)\xi - \eta(Z)Y),$$

(6.4) $$\bar{C}(\xi, Y)\xi = a\beta(\beta-1)(\eta(Y)\xi + Y) - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)(\eta(Y)\xi + Y).$$

From the equations (6.2), (6.3) and (6.4), we have
$$[a\beta(\beta-1) - \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)][\eta(Y)\bar{S}(\xi, Z) + \bar{S}(Y,Z)]$$
$$+a(\beta^2-1)g(Y,Z)\bar{S}(\xi,\xi) - a\beta(\beta-1)\bar{S}(\xi,Y)\eta(Z) + a(\beta-1)\bar{S}(\xi,\xi)\eta(Y)\eta(Z)$$
$$-\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b)][g(Y,Z)\bar{S}(\xi,\xi) - \eta(Z)\bar{S}(\xi,Y)] = 0,$$
which by using (3.6) takes the form
$$\bar{S}(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z),$$
where

(6.5)
$$A = \frac{[\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) - a(\beta^2-1)]\beta(\beta-1)(n-1)}{\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) - a\beta(\beta-1)}, \quad \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) \neq a\beta(\beta-1)$$

and

(6.6) $$B = -\frac{a\beta(\beta-1)^2(n-1)}{\frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) - a\beta(\beta-1)}, \quad \frac{\bar{r}}{n}(\frac{a}{n-1} + 2b) \neq a\beta(\beta-1).$$

Thus we can state the following theorem:

THEOREM 6.1. *An $n$-dimensional Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection satisfying the condition $\bar{C} \cdot \bar{S} = 0$ is an $\eta$-Einstein manifold.*

## 7. Symmetric Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection

DEFINITION 7.1. A Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection is said to be *symmetric* if

(7.1) $$(\bar{\nabla}_X \bar{R})(Y,Z)W = 0$$

for all vector fields $X, Y, Z$ and $W$ on $M$, where $\bar{R}$ is the curvature tensor with respect to the connection $\bar{\nabla}$.

Let $M$ be a symmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the connection $\bar{\nabla}$. Then $(\bar{\nabla}_X \bar{R})(Y, Z)W = 0$. By suitable contraction of this equation, we have

$$(\bar{\nabla}_X \bar{S})(Z, W) = \bar{\nabla}_X \bar{S}(Z, W) - \bar{S}(\bar{\nabla}_X Z, W) - \bar{S}(Z, \bar{\nabla}_X W) = 0.$$

Taking $W = \xi$ in the above equation, we have

$$\bar{\nabla}_X \bar{S}(Z, \xi) - \bar{S}(\bar{\nabla}_X Z, \xi) - \bar{S}(Z, \bar{\nabla}_X \xi) = 0. \tag{7.2}$$

By using (3.6) and (3.9) in (7.2), we find

$$(\beta-1)(n-1)(\bar{\nabla}_X \eta)Z + \bar{S}(X, Z) + \beta(\beta-1)(n-1)\eta(X)\eta(Z) = 0, \quad \beta \neq 0, \tag{7.3}$$

which in view of (3.10) yields

$$\bar{S}(X, Z) = (\beta - 1)^2(n - 1)g(X, Z) - (\beta - 1)(n - 1)\eta(X)\eta(Z). \tag{7.4}$$

Contracting (7.4) over $X$ and $Z$, we obtain

$$\bar{r} = (n - 1)(\beta - 1)(n\beta - n + 1).$$

Thus we have the following theorem:

THEOREM 7.2. *Let $M$ be an $n$-dimensional symmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection. Then the manifold is an $\eta$-Einstein manifold with the scalar curvature $(n - 1)(\beta - 1)(n\beta - n + 1)$.*

## 8. Ricci pseudo-symmetric Lorentzian $\beta$-Kenmotsu manifolds with respect to the semi-symmetric semi-metric connection

DEFINITION 8.1. A Lorentzian $\beta$-Kenmotsu manifold is said to be *Ricci pseudo-symmetric* if and only if the relation [7, 8]

$$R \cdot S = fQ(g, S) \tag{8.1}$$

holds on the set $U_S = [x \in M : S \neq 0 \text{ at } x]$, where $f$ is some function on $U_S$, $R \cdot S$ and $Q(g, S)$ are respectively defined by

$$(R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V) \tag{8.2}$$

and

$$Q(g, S) = ((X \wedge_g Y) \cdot S)(U, V), \tag{8.3}$$

where $(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$ for all $X, Y, U$ and $V \in \chi(M)$.

Assume that the manifold $M$ is a Ricci pseudo-symmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric connection. Then we have

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = f\bar{Q}(g,\bar{S})(X,Y;U,V) \tag{8.4}$$

for all $X,Y,U,V \in \chi(M)$. It is equivalent to

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = f((X\wedge_g Y)\cdot\bar{S})(U,V), \tag{8.5}$$

which in view of (8.2) and (8.3) becomes

$$\begin{aligned}-\bar{S}(\bar{R}(X,Y)U,V)-\bar{S}(U,\bar{R}(X,Y)V) &= f[-g(Y,U)\bar{S}(X,V)+g(X,U)\bar{S}(Y,V)\\ &\quad -g(Y,V)\bar{S}(U,X)+g(X,V)\bar{S}(U,Y)].\end{aligned} \tag{8.6}$$

Putting $X=U=\xi$ in (8.6) and using (2.1), (3.4) and (3.6), we get

$$\begin{aligned}-\beta(\beta-1)[\bar{S}(Y,V)+\eta(Y)\bar{S}(\xi,V)]+\beta(\beta-1)(\beta^2-1)(n-1)[g(Y,V)+\eta(Y)\eta(V)]\\ = f[-\bar{S}(Y,V)+\beta(\beta-1)(n-1)g(Y,V)],\end{aligned} \tag{8.7}$$

which by using (3.6) and simplifying takes the form

$$\bar{S}(Y,V) = Ag(Y,V)+B\eta(Y)\eta(V), \tag{8.8}$$

where

$$A = \frac{\beta(\beta-1)(n-1)(f-\beta^2+1)}{f-\beta(\beta-1)}, \quad f\neq\beta(\beta-1) \tag{8.9}$$

and

$$B = -\frac{\beta(\beta-1)^2(n-1)}{f-\beta(\beta-1)}, \quad f\neq\beta(\beta-1). \tag{8.10}$$

Thus we have the following theorem:

THEOREM 8.2. *An $n$-dimensional Ricci pseudo-symmetric Lorentzian $\beta$-Kenmotsu manifold with respect to the semi-symmetric semi-metric is an $\eta$-Einstein manifold.*

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