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EXPANSIVITY ON ORBITAL INVERSE LIMIT SYSTEMS

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Abstract. In this article, we study expansiveness of the shift maps on orbital inverse limit spaces which consist of two cross bonding mappings. On orbital inverse limit systems, horizontal directions express inverse limit systems and vertical directions mean orbits based on horizontal axes. We characterize the c-expansiveness of functions on orbital spaces. We also prove that the c-expansiveness of the functions is equivalent to the expansiveness of the shift maps on orbital inverse limit spaces.

1. Introduction

Inverse limit systems are important to study dynamical systems. For a continuous surjective map on a topological space, the systems consist of all orbital sequences of the map. The systems appear in various works in dynamical systems. See [2, 5, 8, 9]. We generalize the concept to orbital inverse limit systems induced by a homeomorphism on the space with satisfying commutative condition. For studying continuous dynamical systems induced from given two flows, the condition plays an important role to study the spaces [3, 4]. Roy [10] got several results for dynamics on fiberwise systems. One considers that a continuous orbital inverse limit system has a kind of orbit fibers on the whole phase space. Recently, Chu and Lee [6, 7] investigate several topological dynamic properties on the orbital inverse limit systems.

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In this paper, we deal with discrete orbital inverse limit systems. We introduce shift maps on orbital spaces and investigate several expansiveness between the map on the spaces and its shift map.

Let X be a compact metric space with metric d and f a continuous surjection from X to itself. Let $X^{\mathbb{Z}} := \{(x_i)_i \mid x_i \in X,\ i \in \mathbb{Z}\}$ be the product space of all sequences in X. So $X^{\mathbb{Z}}$ is also a compact metric space equipped with the product topology. For points $(x_i)_i$, $(y_i)_i \in X^{\mathbb{Z}}$, we define a compatible metric d_∞ on $X^{\mathbb{Z}}$ given by

$$d_\infty((x_i)_i, (y_i)_i) := \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}}.$$

Define an orbital space $\mathbf{X}_f$ of f on $X^{\mathbb{Z}}$ given by

$$\mathbf{X}_f := \{(x_i)_i \in X^{\mathbb{Z}} \mid f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$

It is clear that $\mathbf{X}_f$ is closed in $X^{\mathbb{Z}}$. We call the space $\mathbf{X}_f$ an *inverse limit space* induced by f. We say that the function f is c-expansive if there is a constant $e > 0$ such that for $(x_i)_i$, $(y_i)_i \in \mathbf{X}_f$ if $d(x_i, y_i) \le e$ for all $i \in \mathbb{Z}$ then $(x_i)_i = (y_i)_i$. Such a constant e is called a *c-expansive constant* for f. For $k \in \mathbb{Z}$ let $p^k : X^{\mathbb{Z}} \to X$ be the cannonical projection. This means that $p^k((x_i)_i) = x_k$ for every $k \in \mathbb{Z}$. To get much more information about the inverse limit systems, see [1].

Now, we introduce the notion of orbital inverse limit systems. Let f be a continuous surjection from X to itself. We denote a space $Hom_f(X)$ the set of all homeomorphisms on X with the commutative property with respect to f given by

$$Hom_f(X) := \{g \mid g : X \to X \text{ is a homeomorphism },\ f \circ g = g \circ f\}.$$

For $g \in Hom_f(X)$, we define a subspace $\mathbf{X}^g$ of $X^{\mathbb{Z}}$ by

$$\mathbf{X}^g := \{{}^t(x_i)_i \in X^{\mathbb{Z}} \mid g(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$

We say $\mathbf{X}^g$ an *orbital space* of g on X and an element ${}^t(x_i)_i$ in $\mathbf{X}^g$ a *g-orbit* of x where $x = x_0$. We define a function $\mathbf{F}_g$ from $\mathbf{X}^g$ to itself given by for ${}^t(x_i)_i \in \mathbf{X}^g$

$$\mathbf{F}_g({}^t(x_i)_i) := {}^t(f(x_i))_i.$$

Now we define the product topological space $(\mathbf{X}^g)^{\mathbb{Z}}$ as

$$(\mathbf{X}^g)^{\mathbb{Z}} := \{(x_{ij})_{ij} \mid [(x_{ij})_{ij}]^j \in \mathbf{X}^g,\ i, j \in \mathbb{Z}\}$$

where $[(x_{ij})_{ij}]^j$ is the j-th column of the matrix. The element $(x_{ij})_{ij}$ of $(\mathbf{X}^g)^{\mathbb{Z}}$ is expressed as the following form

$$\left(\cdots, \begin{pmatrix} \vdots \\ x_{(-i)(-j)} \\ \vdots \\ x_{0(-j)} \\ \vdots \\ x_{i(-j)} \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} \vdots \\ x_{(-i)0} \\ \vdots \\ x_{00} \\ \vdots \\ x_{i0} \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} \vdots \\ x_{(-i)j} \\ \vdots \\ x_{0j} \\ \vdots \\ x_{ij} \\ \vdots \end{pmatrix}, \cdots \right)$$

where $g(x_{ij}) = x_{(i+1)j}$ for all $i, j \in \mathbb{Z}$. We define a metric $\tilde{d}$ on $(\mathbf{X}^g)^{\mathbb{Z}}$ by

$$\tilde{d}((x_{ij})_{ij}, (y_{ij})_{ij}) := \sum_{i,j=-\infty}^{\infty} \frac{d(x_{ij}, y_{ij})}{2^{|i|} \cdot 3^{|j|}}$$

where $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{Z}}$. It is clear that $\tilde{d}$ is a metric on $(\mathbf{X}^g)^{\mathbb{Z}}$. For $l \in \mathbb{Z}$, we denote the projection map $\mathbf{p}^l : (\mathbf{X}^g)^{\mathbb{Z}} \to \mathbf{X}^g$ given by $\mathbf{p}^l((x_{ij})_{ij}) = [(x_{ij})_{ij}]^l$. So actually $\mathbf{p}^l((x_{ij})_{ij})$ is the g-orbit of an element x_{0l} of X.

A homeomorphism $\tilde{\sigma} : (\mathbf{X}^g)^{\mathbb{Z}} \to (\mathbf{X}^g)^{\mathbb{Z}}$ is called the (*left*) *shift map* on $(\mathbf{X}^g)^{\mathbb{Z}}$ if for $(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{Z}}$, $\tilde{\sigma}((x_{ij})_{ij}) = (y_{ij})_{ij}$ and $y_{ij} = x_{i(j+1)}$ for all $i, j \in \mathbb{Z}$. For a continuous surjection $f : X \to X$ and $g \in Hom_f(X)$, we define a subspace $\mathbf{X}_{f,g}$ of $(\mathbf{X}^g)^{\mathbb{Z}}$ given by

$$\mathbf{X}_{f,g} := \{(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{Z}} \mid [(x_{ij})_{ij}]^j \in \mathbf{X}^g \text{ and } f(x_{ij}) = x_{i(j+1)},\ \ i, j \in \mathbb{Z}\}.$$

The space $\mathbf{X}_{f,g}$ is called the *orbital inverse limit space* induced by f with respect to g and $\mathbf{F}_g$ is called the *orbital function* on $\mathbf{X}_{f,g}$. We say $(\mathbf{X}_{f,g}, \mathbf{F}_g)$ an *orbital inverse limit system.*

Remark 1.1. Let f and g be functions as above. Then the space $\mathbf{X}_{f,g}$ is a nonempty closed set.

Proof. For $n \in \mathbb{Z}$, we define a set $\overrightarrow{G}_n$ given by

$$\overrightarrow{G}_n = \{(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{Z}} \mid f(x_{0j}) = x_{0(j+1)},\ j \geq n\}.$$

We first show that the set is closed. Let $(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{Z}} - \overrightarrow{G}_n$. Then there exists $m \in \mathbb{Z}$ with $m \geq n$ such that $f(x_{0m}) \neq x_{0(m+1)}$. Since X is Hausdorff, we can choose disjoint open sets U, V in X containing $f(x_{0m})$, $x_{0(m+1)}$, respectively. Thus we get that

$$(\mathbf{p}^m)^{-1}((p^0)^{-1}(f^{-1}(U)) \cap \mathbf{X}^g) \cap (\mathbf{p}^{m+1})^{-1}((p^0)^{-1}(V) \cap \mathbf{X}^g)$$

is an open neighborhood of $(x_{ij})_{ij}$ in $(\mathbf{X}^g)^{\mathbb{Z}} - \overrightarrow{G}_n$. Since $\{\overrightarrow{G}_n\}_{n\in\mathbb{Z}}$ is a descending sequence of nonempty closed subsets in the compact metric space $(X^g)^{\mathbb{Z}}$, $\bigcap_{n=-\infty}^{\infty} \overrightarrow{G}_n$ is a nonempty closed subset of $(\mathbf{X}^g)^{\mathbb{Z}}$. Then we immediately get the conclusion from the fact that

$$\mathbf{X}_{f,g} = \bigcap_{n=-\infty}^{\infty} \overrightarrow{G}_n.$$

□

By the remark 1.1, $\mathbf{X}_{f,g}$ is compact in $(\mathbf{X}^g)^{\mathbb{Z}}$. Moreover we have $\tilde{\sigma}((x_{ij})_{ij}) = (f(x_{ij}))_{ij}$ for all $(x_{ij})_{ij} \in \mathbf{X}_{f,g}$, and thus $\tilde{\sigma}(\mathbf{X}_{f,g}) = \mathbf{X}_{f,g}$. We express the restriction of $\tilde{\sigma}$ to $\mathbf{X}_{f,g}$ as $\tilde{\sigma}_{f,g}$.

From now on, let g be a homeomorphism on a compact metric space X and f a continuous surjection on X.

2. Expansivity on orbital inverse limit systems

In this section, we deal with the notion of the c-expansiveness for orbital functions on orbital spaces. For orbital inverse limit systems, the notion of the c-expansiveness of the functions conicides with the notion of the expansiveness of the corresponding shift maps. The orbital inverse limit space $\mathbf{X}_{f,g}$ is also a compact space. The orbital function $\mathbf{F}_g$ on $\mathbf{X}^g$ is *c-expansive* if there is a constant $e > 0$ such that for $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}_{f,g}$ if

$$d_\infty([(x_{ij})_{ij}]^j, [(y_{ij})_{ij}]^j) \le e$$

for all $j \in \mathbb{Z}$ then $(x_{ij})_{ij} = (y_{ij})_{ij}$. Such a constant e is called a *c-expansive constant* for $\mathbf{F}_g$.

Proposition 2.1. *Let $g \in Hom_f(X)$. Then f is c-expansive on X if and only if $\mathbf{F}_g$ is c-expansive on $\mathbf{X}^g$.*

Proof. Let $e > 0$ be a c-expansive constant for f. Suppose that for $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}_{f,g}$, $d_\infty([(x_{ij})_{ij}]^l, [(y_{ij})_{ij}]^l) \le e$ for all $l \in \mathbb{Z}$. Then we have that for all $l \in \mathbb{Z}$,

$$d(x_{0l}, y_{0l}) \le \sum_{i=-\infty}^{\infty} \frac{d(x_{il}, y_{il})}{2^{|i|}} = d_\infty([(x_{ij})_{ij}]^l, [(y_{ij})_{ij}]^l) \le e.$$

For each $l \in \mathbb{Z}$, put $x_l = x_{0l}$ and $y_l = y_{0l}$. Then $(x_l)_l$, $(y_l)_l \in \mathbf{X}_f$ and $(x_l)_l = (y_l)_l$ by the expansivity of f. Hence $(x_{ij})_{ij} = (y_{ij})_{ij}$ because g is a homeomorphism.

Conversely, let $D = \mathrm{diam}X$ and let $e > 0$ be a c-expansive constant for $\mathbf{F}_g$. Choose $N \in \mathbb{Z}_+$ such that $\frac{D}{2^{N-2}} < e$. Since g is a homeomorphism on a compact space X, we can choose $\delta > 0$ such that for $x, y \in X$

$$d(g^k(x), g^k(y)) < \frac{e}{8} \quad \text{for } |k| \leq N$$

if $d(x,y) < \delta$. Let $(x_j)_j, (y_j)_j \in \mathbf{X}_f$. Suppose that $d(x_j, y_j) \leq \delta$ for all $j \in \mathbb{Z}$. Take $(x_{ij})_{ij}, (y_{ij})_{ij} \in \mathbf{X}_{f,g}$ satisfying $x_{0j} = x_j$ and $y_{0j} = y_j$ for all $j \in \mathbb{Z}$. Then $d(x_{0j}, y_{0j}) \leq \delta$ for all $j \in \mathbb{Z}$. Since $g \in Hom_f(X)$, we get that for $l \in \mathbb{Z}$,

$$\begin{aligned}
& d_\infty([(x_{ij})_{ij}]^l, [(y_{ij})_{ij}]^l) \\
= & \sum_{i=-\infty}^{\infty} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} \\
= & \sum_{i=-N}^{N} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} + \sum_{i=N+1}^{\infty} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{i}} \\
& + \sum_{i=-\infty}^{-N-1} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} \\
< & \frac{e}{8} \sum_{i=-N}^{N} \frac{1}{2^{|i|}} + 2D \sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{7}{8} e < e
\end{aligned}$$

Therefore $(x_{ij})_{ij} = (y_{ij})_{ij}$, and hence $(x_j)_j = (y_j)_j$. This completes the proof. □

THEOREM 2.2. *Let $g \in Hom_f(X)$ and k be a positive integer. Then $\mathbf{F}_g$ is c-expansive if and only if $\mathbf{F}_g^k$ is c-expansive.*

Proof. Let $e > 0$ be a c-expansive constant for $\mathbf{F}_g$ and let $D = \mathrm{diam}X$. Choose $N \in \mathbb{Z}_+$ such that $\frac{D}{2^{N-2}} < e$. By uniformly continuity of g, we can choose $\lambda > 0$ such that for x, $y \in X$,

$$d(g^n(x), g^n(y)) < \frac{e}{8} \quad \text{for } |n| \leq N$$

if $d(x,y) < \lambda$. Since f is uniformly continuous, we can choose $\delta > 0$ such that for x, $y \in X$,

$$d(f^n(x), f^n(y)) < \lambda, \quad 1 \leq n \leq k$$

if $d(x,y) < \delta$.

We embark that δ is a c-expansive constant for $\mathbf{F}_g^k$. For $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}_{f,g}$, suppose that $d_\infty([(x_{ij})_{ij}]^{kl}, [(y_{ij})_{ij}]^{kl}) \leq \delta$ for all $l \in \mathbb{Z}$.

Then

$$d(x_{0(kl)}, y_{0(kl)}) \le \delta$$

for all $l \in \mathbb{Z}$. By choosing δ, for $l \in \mathbb{Z}$ and $1 \le n \le k$,

$$d(f^n(x_{0(kl)}), f^n(y_{0(kl)})) < \lambda.$$

Therefore we have that

$$\begin{aligned} & d_\infty([(x_{ij})_{ij}]^l, [(y_{ij})_{ij}]^l) \\ = & \sum_{i=-\infty}^{\infty} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} \\ = & \sum_{i=-N}^{N} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} + \sum_{i=N+1}^{\infty} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{i}} \\ & + \sum_{i=-\infty}^{-N-1} \frac{d(g^i(x_{0l}), g^i(y_{0l}))}{2^{|i|}} < \frac{7}{8}e < e \end{aligned}$$

By the expansivity of $\mathbf{F}_g$, $(x_{ij})_{ij} = (y_{ij})_{ij}$, so $(x_{i(kj)})_{ij} = (y_{i(kj)})_{ij}$. Conversely, the proof directly follows from the definitions. □

THEOREM 2.3. *Let f be a continuous surjection on X and f_i a continuous surjection on a compact space X_i and let $g \in Hom_f(X)$, $g_i \in Hom_{f_i}(X_i)$ for $i = 1, 2$. Then the following properties hold.*

(1) *If $\mathbf{F}_g$ is c-expansive and Y is a closed subset of X with $f(Y) = Y$ and $g(Y) = Y$, then $\mathbf{F}_g|_{\mathbf{Y}^g} : \mathbf{Y}^g \to \mathbf{Y}^g$ is also c-expansive.*
(2) *If $\mathbf{F}^1_{g_1}$ and $\mathbf{F}^2_{g_2}$ are c-expansive, then the product mapping $\mathbf{F}^1_{g_1} \times \mathbf{F}^2_{g_2} : \mathbf{X}_1^{g_1} \times \mathbf{X}_2^{g_2} \to \mathbf{X}_1^{g_1} \times \mathbf{X}_2^{g_2}$ is also c-expansive where*

$$(\mathbf{F}^1_{g_1} \times \mathbf{F}^2_{g_2})(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{F}^1_{g_1}(\mathbf{x}_1), \mathbf{F}^2_{g_2}(\mathbf{x}_2)),\ (\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{X}_1^{g_1} \times \mathbf{X}_2^{g_2}.$$

Proof. The proofs are obvious.

□

The shift map $\tilde{\sigma}_{f,g}$ on $\mathbf{X}_{f,g}$ is *expansive* if there is a constant $e > 0$ such that for $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}_{f,g}$ if $\tilde{d}(\tilde{\sigma}^n_{f,g}((x_{ij})_{ij}), \tilde{\sigma}^n_{f,g}((y_{ij})_{ij})) \le e$ for all $n \in \mathbb{Z}$ then $(x_{ij})_{ij} = (y_{ij})_{ij}$. Such a constant e is called an *expansive constant* for $\tilde{\sigma}_{f,g}$.

The last theorem mentions that the c-expansivity of orbital maps in orbital inverse limit systems is equivalent to the expansivity of the corresponding shift maps.

THEOREM 2.4. *Let $g \in Hom_f(X)$. Then $\mathbf{F}_g$ is c-expansive if and only if $\tilde{\sigma}_{f,g} : \mathbf{X}_{f,g} \to \mathbf{X}_{f,g}$ is expansive.*

Proof. Let e be a c-expansive constant for $\mathbf{F}_g$. If $(x_{ij})_{ij} \neq (y_{ij})_{ij}$ then there exists $m \in \mathbb{Z}$ such that

$$d_\infty([(x_{ij})_{ij}]^m, [(y_{ij})_{ij}]^m) > e.$$

Then

$$\tilde{d}(\tilde{\sigma}^m_{f,g}((x_{ij})_{ij}), \tilde{\sigma}^m_{f,g}((y_{ij})_{ij})) \geq d_\infty([(x_{ij})_{ij}]^m, [(y_{ij})_{ij}]^m) > e.$$

Hence e is an expansive constant for $\tilde{\sigma}_{f,g}$.

Conversely, let e be an expansive constant for $\tilde{\sigma}_{f,g}$ and $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}_{f,g}$. Then $\frac{e}{2}$ is a c-expansive constant for $\mathbf{F}_g$. Indeed, suppose that

$$d_\infty([(x_{ij})_{ij}]^l, [(y_{ij})_{ij}]^l) \leq \frac{e}{2}$$

for all $l \in \mathbb{Z}$. Then we have that for $n \in \mathbb{Z}$,

$$\tilde{d}(\tilde{\sigma}^n_{f,g}((x_{ij})_{ij}), \tilde{\sigma}^n_{f,g}((y_{ij})_{ij})) = \sum_{i,j=-\infty}^{\infty} \frac{d(x_{i(j+n)}, y_{i(j+n)})}{2^{|i|} \cdot 3^{|j|}} < e.$$

This completes the proof. □

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