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GLOBAL GENERALIZED CROSS VALIDATION IN THE PRECONDITIONED GL-LSQR

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Abstract. This paper present the global generalized cross validation as the appropriate choice of the regularization parameter in the preconditioned Gl-LSQR method in solving image deblurring problems. The regularization parameter, chosen from the global generalized cross validation, with preconditioned Gl-LSQR method can give better reconstructions of the true image than other parameters considered in this study.

1. Introduction

Regularized deblurring problems in the imaging system are often modeled as a linear least squares problem:

$$\min_x(\|Hx-b\|_2^2+\lambda^2\|x\|_2^2), \tag{1.1}$$

where $H_{M\times N}(M\geq N)$ is a blurring ill-conditioned matrix with some block structures, b and x represent the observed and the original image respectively. An appropriate choice of the regularization parameter λ is important thing to do in the regularization process. There are various techniques to choose the approximate regularization parameters such as Morozov's discrepancy principle, L-curve criterion, and generalized cross valication(GCV)[2, 3, 4]. Especially the GCV method is prominent for the selection of the crucial regularization parameters since GCV has good asymptotic properties for large number of noisy data.

Since the implementations of image restoration problems typically require the need of formidable data, we generalized the problem (1.1) to

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the following minimization problem with respect to the Frobenius norm

$$\min_{X}\{\|HX - B\|_F^2 + \lambda^2 \|X\|_F^2\}, \tag{1.2}$$

where $B_{N\times s}(N \gg s)$ is a collection of the column stacking of each small blocks obtained from partitioning the blurred and noisy image ([1, 6]).

In [1], it was shown that the global least squares(Gl-LSQR) method can be applied to image restoration problems (1.2). In [7], the global generalized cross validation(GCV) method for the problem (1.2) was suggested and it was proved that the preconditioned Gl-CGLS method with the global GCV for large image restoration problems is stable since the choice of regularization parameter from the global GCV has results in the better performances.

To solve image deblurring problems this pager suggests Gl-LSQR method with the global GCV for the appropriate choice of the regularization parameter in (1.2).

The outline of this paper follows. The brief description of the Gl-LSQR method for the problem (1.2) is summarized in Section 2. In Section 3, we present an appropriate generalized cross validation to (1.2) with Gl-LSQR method for solving image deblurring problems. Numerical experiments and final remarks are described in Section 4.

2. Global least squares procedure

The global least squares (Gl-LSQR) method is a method for solving linear system with multiple right hand sides,

$$HX = B, \tag{2.1}$$

where B and X are $n \times s$ matrices([8]). This section summarizes the concept of the Gl-LSQR algorithm.

Frobenius norm is defined by $\|X\|_F = \sqrt{\langle X, X\rangle_F}$, where $\langle X, Y\rangle_F$ denotes the trace of the square matrix X^TY for two $n \times s$ matrices X and Y. If some $n \times s$ block vectors $V_1, V_2, \ldots$ are orthonormal with respect to $< \cdot\, , \,\cdot >_F$, then $V_1, V_2, \ldots$ becomes F-orthonormal basis.

The two sets of k number of of the $n \times s$ block vectors $V_1, V_2, \ldots, V_k$ and $U_1, U_2, \ldots, U_k$ which are two F-orthonormal basis of $R^{n\times ks}$ can be constructed as the following global bidiagonalization with starting matrix B:

$$\begin{aligned} &\beta_1 U_1 = B,\ \alpha_1 V_1 = H^T U_1 \\ &\beta_{i+1} U_{i+1} = H V_i - \alpha_i U_i, (i = 1, 2, \ldots, k) \\ &\alpha_{i+1} V_{i+1} = H^T U_{i+1} - \beta_{i+1} V_i, (i = 1, 2, \ldots, k) \end{aligned} \tag{2.2}$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are chosen so that $\|U_i\|_F = \|V_i\|_F = 1$.

Set $\mathcal{V}_k \equiv \left[\begin{array}{cccc} V_1 & V_2 & \ldots & V_k \end{array} \right], \mathcal{U}_k \equiv \left[\begin{array}{cccc} U_1 & U_2 & \ldots & U_k \end{array} \right]$ and a lower bidiagonal matrix

$$T_k \equiv \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix}. \tag{2.3}$$

Define $\mathcal{V}_k * t = \sum_{j=1}^{k} V_j t_j, (t \in \mathbb{R}^k)$. Then the recurrence formula (2.2) of the global bidiagonalization can be rewritten as

$$\begin{aligned} &\mathcal{U}_{k+1} * (\beta_1 e_1) = B, H\mathcal{V}_k = \mathcal{U}_{k+1} * T_k, \\ &H^T \mathcal{U}_{k+1} = \mathcal{V}_k * T_k^T + \alpha_{k+1} V_{k+1} * e_{k+1}^T, \end{aligned} \tag{2.4}$$

where e_i is the ith column of identity matrix.

The form of an approximate solution X_k at iteration k is $X_k = \mathcal{V}_k * y_k, (y_k \in \mathbb{R}^k)$ and the corresponding residual matrix of the equation (2.1) is

$$\mathcal{R}_k = \beta_1 U_1 - (\mathcal{U}_{k+1} * T_k) * y_k = \mathcal{U}_{k+1} * (\beta_1 e_1 - T_k y_k).$$

The global LSQR algorithm chooses the vector y_k which minimizes $\|\mathcal{R}_k\|_F$,

$$\min \|\mathcal{R}_k\|_F = \min_{y_k \in \mathbb{R}^k} \|\beta_1 e_1 - T_k y_k\|_2 . \tag{2.5}$$

The QR factorization of T_k induces

$$Q \left[\begin{array}{cc} T_k & \beta_1 e_1 \end{array} \right] = \left[\begin{array}{cc} R_k & f_k \\ 0 & \bar{\phi}_{k+1} \end{array} \right],$$

where the matrix Q is a product $G_{k,k+1} G_{k-1,k} \cdots G_{1,2}$ of $G_{i,i+1}$, $i = k, k-1, \cdots, 1$ chosen to eliminate the subdiagonal element $\beta_2, \ldots \beta_{k+1}$

of T_k and

$$R_k = \begin{bmatrix} \rho_1 & \theta_2 & & & \\ & \rho_2 & \theta_3 & & \\ & & \ddots & \ddots & \\ & & & \rho_{k-1} & \theta_k \\ & & & & \rho_k \end{bmatrix} \text{ and } f_k = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{k-1} \\ \phi_k \end{bmatrix}.$$

From $R_k y_k = f_k$, the solution y_k of (2.5) can be obtained. Then an approximate solution is formed as

$$X_k = \mathcal{V}_k * y_k = \mathcal{V}_k * (R_k^{-1} * f_k) = (\mathcal{V}_k * R_k^{-1}) * f_k.$$

Setting $\mathcal{P}_k \equiv \mathcal{V}_k * R_k^{-1} \equiv \begin{bmatrix} P_1 & P_2 & \ldots & P_k \end{bmatrix}$, the approximate solution $X_k = \mathcal{P}_k * f_k = X_{k-1} + P_k\phi_k$ with the initial guess $P_0 = X_0 = O$. The last block column P_k of $\mathcal{P}_k$ can be updated by

$$P_k = (V_k - P_{k-1}\theta_k)\rho_k^{-1}$$

where $f_k = \begin{bmatrix} f_{k-1} \\ \phi_k \end{bmatrix}$ and $\phi_k = c_k\bar{\phi}_k$. Thus the matrix residual norm $\|R_k\|_F$ is equal to $|\bar{\phi}_{k+1}|$.

3. Gl-LSQR with the global GCV

This section presents to use Gl-LSQR method with the global GCV for the problem (1.2). The best way to solve (1.2) numerically is to treat it as a minimization problem

$$X_\lambda = \arg\min_X \left\| \begin{pmatrix} H \\ \lambda I \end{pmatrix} X - \begin{pmatrix} B \\ O \end{pmatrix} \right\|_F. \tag{3.1}$$

The parameter λ acts on the smoothness of the solution. A generalization of the cross validation for the problem (1.2) can define the following global GCV funtion.

Definition 3.1. Using the normal equations for (3.1)

$$(H^T H + \lambda^2 I)X = H^T B,$$

the regularization solution is written by $X_\lambda = (H^T H + \lambda^2 I)^{-1} H^T B$. Then the global GCV function is defined by

$$\mathcal{G}_{\text{global}}(\lambda) = \frac{\|HX_\lambda - B\|_F^2}{[\text{trace}(I - H(H^T H + \lambda^2 I)^{-1} H^T)]^2}. \tag{3.2}$$

In particular, when the reflective boundary conditions are used, H can be diagonalized by the orthogonal two-dimensional discrete cosine transform matrix and thus the follows can be obtained.

LEMMA 3.2. *If* $\{\eta_i, i = 1, ..., N\}$ *represents the spectrum of* H*, we can rewrite* $\mathcal{G}_{global}(\lambda)$ *as*

$$\mathcal{G}_{\text{global}}(\lambda) = \frac{\sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{1}{\eta_i^2+\lambda^2}[\mathcal{C}B_j]_i\right)^2}{\left(\sum_{i=1}^{N}\frac{1}{\eta_i^2+\lambda^2}\right)^2}, \tag{3.3}$$

where B_j *is the* j*-th column of* B.

Proof. From a unitary spectral decomposition $H = \mathcal{C}^T\Lambda_H\mathcal{C}$ ($\Lambda_H = diag(\eta_1, \eta_2, \ldots, \eta_N)$),

$$HX_\lambda - B = \mathcal{C}(\Lambda_H(\Lambda_H^2 + \lambda^2 I)^{-1}\Lambda_H - I)\mathcal{C}^T B.$$

Then direct caculation yields $\|Hx_\lambda - B\|_F^2 = \sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{\lambda^2[\mathcal{C}B_j]_i}{\eta_i^2+\lambda^2}\right)^2$ and $\text{trace}(I - H(H^TH + \lambda^2 I)^{-1}H^T) = \left(\sum_{i=1}^{N}\frac{\lambda^2}{\eta_i^2+\lambda^2}\right)$. Substitute the two expresions into (3.2) to obtain (3.3). □

Now we want to find a regularization parameter λ_{gGCV}. In other words, the following constrained optimization problem can be solved

$$\min_{\lambda} \quad \mathcal{G}_{\text{global}}(\lambda) \text{ subject to } \eta_1 \le \lambda \le \eta_N, \tag{3.4}$$

where η_1 is the smallest eigenvalue of H and η_N is the largest eigenvalue of H to get an appropriate parameter λ that gives the least squares solution X_λ. The problem (3.1) is restated by

$$\min_{X}\left\|\begin{pmatrix} H \\ \lambda_{gGCV} I\end{pmatrix} X - \begin{pmatrix} B \\ O\end{pmatrix}\right\|_F. \tag{3.5}$$

The preconditioned Gl-LSQR algorithm solves (3.5) by transforming the problem with a preconditioner Ω,

$$\min_{Y}\left\|\hat{H}\Omega^{-1}Y - \hat{B}\right\|_F \tag{3.6}$$

with $Y = \Omega X$, where $\hat{H} = \begin{pmatrix} H \\ \lambda_{gGCV} I\end{pmatrix}$ and $\hat{B} = \begin{pmatrix} B \\ O\end{pmatrix}$.

Summarizing the above process so far, the algorithm for preconditioned Gl-CGLS with the extended global GCV can be presented as follows.

Algorithm 1. Preconditioned Gl-LSQR with the global GCV.

1. Find the minimizer λ_{gGCV} for the problem:

$$\min_{\lambda} \mathcal{G}_{\text{global}}(\lambda) \text{ subject to } \eta_1 \leq \lambda \leq \eta_N.$$

2. Solve (3.6) using preconditioned Gl-LSQR :
   i. Set $X_0 = O_{n\times s}$.
   ii. $\beta_1 = \left\|\widehat{B}\right\|_F$, $U_1 = B/\beta_1$, $\alpha_1 = \left\|\widehat{H}^T U_1\right\|_F$, $V_1 = \widehat{H}^T U_1/\alpha_1$.
   iii. Set $W_1 = V_1$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$.
   iv. For k=1, 2, ...
      (i) $\varpi_k = \widehat{H}V_k - \alpha_k U_k$, $\beta_{k+1} = \|\varpi_k\|_F$, $U_{k+1} = \varpi_k/\beta_{k+1}$
      (ii) $\tau_k = \widehat{H}^T U_{k+1} - \beta_{k+1}V_k$, $\alpha_{k+1} = \|\tau_k\|_F$, $V_{k+1} = \tau_k/\alpha_{k+1}$
      (iii) $\rho_k = (\bar{\rho}^2 + \beta_{k+1}^2)^{1/2}$, $c_k = \bar{\rho}_k/\rho_k$, $s_k = \beta_{k+1}/\rho_k$
      (iv) $\theta_{k+1} = s_k\alpha_{k+1}$, $\bar{\rho}_{k+1} = c_k\alpha_{k+1}$
      (v) $\phi_k = c_k\bar{\phi}_k$, $\bar{\phi}_{k+1} = -s_k\bar{\phi}_k$
      (vi) $X_k = X_{k-1} + (\phi_k/\rho_k)W_k$
      (vii) $W_{k+1} = V_{k-1} - (\theta_{k+1}/\rho_k)W_k$
      (viii) If $|\bar{\phi}_{k+1}|$ is small enough, then stop.

4. Numerical experiments

Employing the global GCV in Gl-LSQR method for solving image restoration problems with two test images, we investigated numerical results to illustrate the effectiveness of the regularization parameters chosen from the minimization of the global GCV function. For comparison purposes, various possible regularization parameters chosen experimentally by attempting to minimize the relative accuracy are used in each test.

Our two test images (128-by-128) are degraded by Gaussian blur with adding Gaussian noises. Using reflective boundary condition, these images are divided into the collection of 64 small block images with the sizes of 16-by-16 respectively.

Table 1 presents the performance results of the preconditioned Gl-LSQR method with regularization parameters chosen from both of the global GCV and the numerical experiments having the relative accuracy as small as possible. The relative error $\frac{\|X^* - \widehat{X}\|}{\|X^*\|}$ shows how well the true image has been approximated. The peak-to-signal ratio(PSNR) is defined as $10\log_{10}\left(\frac{255^2}{\frac{1}{mn}\sum_{i,j}(x^*_{i,j} - \widehat{x}_{i,j})^2}\right)$, where $x^*_{i,j}$ and $\widehat{x}_{i,j}$ denote the pixel value of the original image and restored image respectively.

TABLE 1

Regularization parameter	Test image	Relative error	PSNR
global GCV	I	0.002336	57.8466
$\lambda = 0.05$	I	0.002946	55.8319
$\lambda = 0.1$	I	0.003386	54.6220
$\lambda = 0.8$	I	0.003788	53.6467
global GCV	II	0.018308	38.3272
$\lambda = 0.05$	II	0.037062	32.2017
$\lambda = 0.1$	II	0.040806	31.3658
$\lambda = 0.8$	II	0.043520	30.8065

In the preconditioned Gl-LSQR, the determination of regularization parameter by using the global GCV is more efficient than by the other choices of λ from the perspective of smaller relative error and a larger PSNR. The reconstructed image for test image (II) by preconditioned Gl-LSQR with the global GCV is given in Figure 1(c).

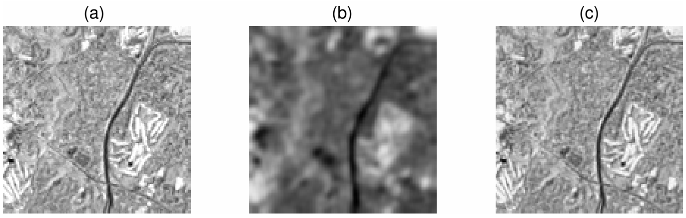

FIGURE 1. (a) Original (b) Gaussian blur and noisy (c) Reconstructed images.

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