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ON AP-HENSTOCK-STIELTJES INTEGRAL FOR FUZZY-NUMBER-VALUED FUNCTIONS ON INFINITE INTERVAL

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ABSTRACT. In this paper we introduce the AP-Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval and investigate some properties.

1. Introduction and Preliminaries

It is well-known that the Henstock integral for real valued function was first defined by Henstock [2,3] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Feynman integrals.

In 2012, Zengtai Gong and L. Wang introduced the concept of the Henstock-Stieltjes integrals of fuzzy-number-valued functions and obtained some properties([9]).

In this paper we introduce the concept of the AP-Henstock-Stieltjes integral of fuzzy-number-valued functions on infinite interval and investigate some properties.

A Henstock partition of $[a,b]$ is a finite collection $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ such that $\{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a,b]$ covering $[a,b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $1 \leq i \leq n$. A gauge on $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$. A Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is said to be δ-fine on $[a,b]$ if $[x_{i-i}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for each $1 \leq i \leq n$.

Let α be an increasing function on $[a,b]$. A function $f : [a,b] \to R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a,b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such

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that $|\sum_{i=1}^n f(\xi_i)(\alpha(v_i)-\alpha(u_i))-L|<\epsilon$ whenever $P=\{([u_i,v_i],\xi_i):1\le i\le n\}$ is a δ-fine Henstock partition of $[a,b]$. We denote this fact as $(HS)\int_a^b f(x)d\alpha = L$ and $f\in HS[a,b]$. The function f is Henstock-Stieltjes integrable with respect to α on a set $E\subset[a,b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a,b]$, where χ_E denotes the characteristic function of E.

Fuzzy set $u:R\to[0,1]$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and supp $u=\overline{\{x\in R|u(x)>0\}}$ is compact. Here $\overline{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space ([9]).

Let $u,v\in E^1, k\in R$, the addition and scalar multiplication are defined by

$$[u+v]_\lambda=[u]_\lambda+[v]_\lambda,\ [ku]_\lambda=k[u]_\lambda,$$

where $[u]_\lambda=\{x:u(x)\ge\lambda\}=[u_\lambda^-,\,u_\lambda^+]$ for any $\lambda\in[0,1]$.

We use the Hausdorff distance between fuzzy numbers given by $D: E^1\times E^1\to[0,\,\infty)$ as follows

$$D(u,v)=\sup_{\lambda\in[0,1]} d([u]_\lambda,\,[v]_\lambda)=\sup_{\lambda\in[0,1]}\max\{|u_\lambda^- - v_\lambda^-|,|u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(u,v)$ is called the distance between u and v.

A function $F:[a,b]\to E^1$ is said to be bounded if there exists $M\in R$ such that $\|F(x)\|=D(F(x),0)\le M$ for any $x\in[a,b]$.

Lemma 1.1. (**[9]**) *If $u\in E^1$, then*

(1) $[u]_\lambda$ *is non-empty bounded closed interval for all* $\lambda\in[0,1]$.
(2) $[u]_{\lambda_1}\supset[u]_{\lambda_2}$ *for any* $0\le\lambda_1\le\lambda_2\le1$.
(3) *for any* $\{\lambda_n\}$ *converging increasingly to* $\lambda\in(0,1]$,

$$\bigcap_{n=1}^{\infty}[u]_{\lambda_n}=[u]_\lambda.$$

Conversely, if for all $\lambda\in[0,1]$, *there exists* $A_\lambda\subset R$ *satisfying* (1) $\sim$ (3), *then there exists a unique* $u\in E^1$ *such that* $[u]_\lambda=A_\lambda$, $\lambda\in(0,1]$, *and* $[u]_0=\overline{\cup_{\lambda\in(0,1]}[u]_\lambda}\subset A_0$.

Definition 1.2. ([9]) Let α be an increasing function on $[a,b]$. A fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if there exists a fuzzy number $K\in E^1$ such that for every $\epsilon>0$ there exists a positive function $\delta(x)$ such that

$$D(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a δ-fine Henstock partition of $[a, b]$. We denote this fact as $(FHS) \int_a^b F(x) d\alpha = K$ and $(F, \alpha) \in FHS[a, b]$.

In this paper, $\overline{R}$ denoted the extended real number and for the fuzzy number valued function F defined on $[a, \infty]$, we define $F(\infty) = 0$ and $0 \cdot \infty = 0$.

2. The fuzzy number-valued AP-Henstock-Stieltjes Integral

In this section, we will define the fuzzy number-valued AP-Henstock-Stieltjes integral on $[a, \infty]$, which is an extension of the fuzzy number-valued Riemann-Stieltjes integrals on infinite interval and investigate some of their properties.

Let E be a measurable set in $\overline{R}$ and let $x \in \overline{R}$. The density of E at x is defined by

$$d_x E = \lim_{h \to 0+} \frac{\mu(E \cap (x - h, x + h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E.

An approximate neighborhood(or ap-nbd) of $x \in [a, b] \subset \overline{R}$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x and S_x is finite measurable set if $x \in R$. then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If all the members of $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is fine to a choice S, then we say that P is S-fine. Let $E \subset [a, \infty]$. If P is S-fine and each $\xi_i \in E$, then P is called S-fine on E. If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is S-fine and $[a, \infty] = \cup_{i=1}^{n} [x_{i-1}, x_i]$, then we say that P is S-fine partition of $[a, \infty]$. Note that if $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is S-fine partition of $[a, \infty]$, then $\xi_n = \infty$ and $[x_{n-1}, x_n]$ is infinite interval in $[a, \infty]$.

Definition 2.1. ([8]) Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a choice S on $[a,b]$ such that

$$D(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a S-fine Henstock partition of $[a,b]$.

Definition 2.2. Let α be an increasing function on $[a,\infty]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a,\infty]$ if there exists a fuzzy number $K \in E^1$ and $T \in R^+$ such that for every $\epsilon > 0$ there exists a choice S on $[a,\infty]$ such that

$$D(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a S-fine Henstock partition on $[a,b]$ of a finite interval $[a,b] \supset [a,\infty] \cap [-T,T]$. We write $(APFHS) \int_a^\infty F(x)d\alpha = K$ and $(F,\alpha) \in APFHS[a,\infty]$.

The definition of $(F,\alpha) \in APFHS[-\infty,a]$ is similar. Naturally $(F,\alpha) \in APFHS[-\infty,\infty]$ if and only if $(F,\alpha) \in APFHS[-\infty,a]$ and $(F,\alpha) \in APFHS[a,\infty]$, and then

$$\begin{aligned}(APFHS) &\int_{-\infty}^{\infty} F(x)d\alpha \\ &= (APFHS) \int_{-\infty}^{a} F(x)d\alpha + (APFHS) \int_{a}^{\infty} F(x)d\alpha.\end{aligned}$$

The fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a,\infty]$ if F_{χ_E} is AP-Henstock-Stieltjes integrable with respect to α on $[a,\infty]$, where χ_E denotes the characteristic function of E.

Remark 2.3. If $F \in APFHS[a,\infty]$, then the integral value is unique.

From the definition of the fuzzy number-valued AP-Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following theorem.

Theorem 2.4. *Let α be an increasing function on $[a,\infty]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a,\infty]$ if and only if for every $\epsilon > 0$ there exists a choice*

S on $[a,\infty]$ such that for any S-fine partitions $P=\{([u,v],\xi)\}$ and $P^{'}=\{([u^{'},v^{'}],\xi^{'})\}$, we have

$$D(\sum_{P}F(\xi)(\alpha(v)-\alpha(u)),\sum_{P'}F(\xi^{'})(\alpha(v^{'})-\alpha(u^{'})))<\epsilon.$$

THEOREM 2.5. *Let α be an increasing function on $[a,\infty]$ and let F be a fuzzy number-valued function on $[a,\infty]$. Then the following statements are equivalent:*

(1) $(F,\alpha)\in APFHS[a,\infty]$ and $(APFHS)\int_a^\infty F(x)d\alpha=K$.
(2) *For any $b>a$, $(F,\alpha)\in APFHS[a,b]$ and $\lim_{b\to\infty}(APFHS)\int_a^b F(x)d\alpha$ exists and then $\lim_{b\to\infty}(APFHS)\int_a^b F(x)d\alpha=(APFHS)\int_a^\infty F(x)d\alpha$.*

Proof. (1) implies (2) : Since $(APFHS)\int_a^\infty F(x)d\alpha=K$, for any $\epsilon>0$, there exists a choice S_1 on $[a,\infty]$ and $T\in R^+$ such that for any S_1-fine partition $P=\{([x_{i-1},x_i],\xi_i):1\le i\le n\}$ on $[a,b]$ of a finite interval $[a,b]\supset[a,\infty]\cap[-T,T]$, we have

$$D(\sum_{i=1}^{n}F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\,K)<\epsilon \tag{2.1}$$

For $b>T$, we find a choice S_2 on $[a,b]$ such that for any S_2-fine partition $P=\{([x_{i-1},x_i],\xi_i):1\le i\le n\}$ on $[a,b]$

$$D(\sum_{i=1}^{n}F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\int_a^b F(x)d\alpha)<\epsilon \tag{2.2}$$

Define $S=S_1\cap S_2$ on $[a,b]$ and let P be S-fine partition on $[a,b]$. then by (2.1) and (2.2)

$$D\big((APFHS)\int_a^b F(x)d\alpha,(APFHS)\int_a^\infty F(x)d\alpha\big)<2\epsilon.$$

Hence, $\lim_{b\to\infty}(APFHS)\int_a^b F(x)d\alpha=(APFHS)\int_a^\infty F(x)d\alpha$.

(2) implies (1) : Let $\lim_{b\to\infty}(APFHS)\int_a^b F(x)d\alpha=L$. Then for any $\epsilon>0$, there exists $T\in R^+$ such that

$$D(\sum_{i=1}^{n}F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\,L)<\frac{\epsilon}{2} \tag{2.3}$$

for $b>T$. Let $\{b_n\}$ be a strictly increasing sequence with $b_1=a$ and $\lim_{n\to\infty}b_n=\infty$.

For $\epsilon > 0$ there exists a choice S^n on $[b_n, b_{n+1}]$ such that for any S^n-fine partition P_n on $[b_n, b_{n+1}]$

$$(2.4)\quad D\big(\sum_{P_n} F(\xi)(\alpha(v)-\alpha(u)),\,(APFHS)\int_{b_n}^{b_{n+1}} F(x)d\alpha\big) < \frac{\epsilon}{2^{n+2}}.$$

Define $S = \{\, S_x : x \in [a,\infty]\}$ by $S_\xi \in S^n$ if $\xi \in [b_n, b_{n+1}]$, $S_\xi \in (b_n, b_{n+1})$ if $\xi \in (b_n, b_{n+1})$ and $S_\infty = (T,\infty]$ if $\xi = \infty$.

Let $b > T$. Since $\lim_{n\to\infty} b_n = \infty$, there exists N such that $b_N < b \le b_{N+1}$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ be a S-fine partition on $[a,b]$. Note that the partition P anchors on all b_n with $n \le N$ and (2.5)

$$\sum_P F\chi_{[a,b]}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}) = \sum_1^{N-1}\sum_P F\chi_{[b_n,b_{n+1}]}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}))$$

$$+\sum_P F\chi_{[b_N,b]}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})).$$

By (2.4),

(2.6)
$$D\Big(\sum_P F\chi_{[b_n,b_{n+1}]}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}),(APFHS)\int_{b_n}^{b_{n+1}} F(x)d\alpha\Big) < \frac{\epsilon}{2^{n+2}}.$$

By (2.3), (2.5), (2.6) and Saks-Henstock lemma (lemma 2.8.1 in [7])

$$D\big(\sum_{i=1}^n F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\, L)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{4}\sum_1^{N-1}\frac{1}{2^n}$$

$$+D\big(\sum_P F\chi_{[b_N,b]}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),(APFHS)\int_{b_N}^{b} F(x)d\alpha\big) < \epsilon.$$

Hence, $(F,\alpha) \in APFHS[a,\infty]$. □

THEOREM 2.6. *Let α be an increasing function on $[a,\infty]$ and let F be a fuzzy number-valued function on $[a,\infty]$. Then the following statements are equivalent:*

(1) $(F,\alpha) \in APFHS[a,\infty]$ and $(APFHS)\int_a^\infty F(x)d\alpha = K$.
(2) *For any $\lambda \in [0,1]$, F_λ^- and F_λ^+ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a,\infty]$ for $\lambda \in [0,1]$ uniformly (Choice S is independent of $\lambda \in [0,1]$) and*

$$\big[(APFHS)\int_a^\infty F(x)d\alpha\big]_\lambda$$

$$= \big[(APHS)\int_a^\infty F_\lambda^-(x)d\alpha,\, (APHS)\int_a^\infty F_\lambda^+(x)d\alpha\big].$$

Proof. (1) implies (2) : Since $(APFHS)\int_a^\infty F(x)d\alpha = K$, for any $\epsilon > 0$, there exists a choice S on $[a,\infty]$ and $T \in R^+$ such that for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ on $[a,b]$ of a finite interval $[a,b] \supset [a,\infty] \cap [-T,T]$, we have

$$D(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\, K) < \epsilon$$

By definition of D, for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ on $[a,b]$

$$\sup_{\lambda\in[0,1]} \max\big\{|\big[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))\big]_\lambda^- - K_\lambda^-|,$$
$$|\big[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))\big]_\lambda^+ - K_\lambda^+|\big\} < \epsilon.$$

Hence, we have

$$\big|\sum_{i=1}^n F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^-\big| < \epsilon,$$
$$\big|\sum_{i=1}^n F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+\big| < \epsilon.$$

Therefore, $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes with respect to α on $[a,\infty]$ for any $\lambda \in [0,1]$, and

$$(APHS)\int_a^\infty F_\lambda^-(x)d\alpha = K_\lambda^-, \quad (APHS)\int_a^\infty F_\lambda^+(x)d\alpha = K_\lambda^+.$$

(2) implies (1) : Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a,\infty]$ for any $\lambda \in [0,1]$, for $\epsilon > 0$, there exists a choice S on $[a,\infty]$ and $T \in R^+$ such that for any S-fine partition $P = \{([x_{i-1}, x_i], xi_i) : 1 \le i \le n\}$ on $[a,b]$ of a finite interval $[a,b] \supset [a,\infty] \cap [-T,T]$ and for any $\lambda \in [0,1]$, we have

$$\big|\sum_{i=1}^n F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^-\big| < \epsilon,$$
$$\big|\sum_{i=1}^n F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+\big| < \epsilon.$$

Then we can prove that $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0,1]\}$ satisfies the conditions of Lemma 1.1. By the Lemma 1.1, $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0,1]\}$ determines a fuzzy number K. By the definition of D, for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ on $[a,b]$, we obtain

$$D(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K) < 2\epsilon$$

Thus, $(F, \alpha) \in APFHS[a, \infty]$ and $(APFHS) \int_a^\infty F(x) d\alpha = K$. □

From theorem 2.6, the following theorem is obvious.

THEOREM 2.7. *Let $F, G \in APFHS[a, \infty]$ and $\beta, \gamma \in R$. Then $\beta F + \gamma G \in\in APFHS[a, \infty]$ and*

$$(APFHS) \int_a^\infty (\beta F + \gamma G) d\alpha = \beta (APFHS) \int_a^\infty F d\alpha + \gamma (APFHS) \int_a^\infty G d\alpha.$$

THEOREM 2.8. *Let $F, G \in APFHS[a, \infty]$ and $D(F, G)$ is Lebesgue-Stieltjes integrable on $[a, \infty]$. Then*

$$D((APFHS) \int_a^\infty F\ d\alpha, (APFHS) \int_a^\infty G\ d\alpha) \le (L) \int_a^\infty D(F, G)\ d\alpha.$$

Proof. By definition of distance,

$$\begin{aligned}
&D((APFHS) \int_a^\infty F d\alpha,\ (APFHS) \int_a^\infty G d\alpha) \\
&= \sup_{\lambda \in [0,1]} \max \Big(|[((APHS) \int_a^\infty F d\alpha)]_\lambda^- - [((APHS) \int_a^\infty G d\alpha)]_\lambda^-|, \\
&\qquad\qquad |[((APHS) \int_a^\infty F d\alpha)]_\lambda^+ - [((APHS) \int_a^\infty G d\alpha)]_\lambda^+| \Big) \\
&= \sup_{\lambda \in [0,1]} \max \Big(|(APHS) \int_a^\infty ([F]_\lambda^- - [G]_\lambda^-) d\alpha|, \\
&\qquad\qquad |(APHS) \int_a^\infty ([F]_\lambda^+ - [G]_\lambda^+) d\alpha| \Big) \\
&\le \sup_{\lambda \in [0,1]} \max((L) \int_a^\infty |[F]_\lambda^- - [G]_\lambda^-| d\alpha,\ (L) \int_a^\infty |[F]_\lambda^+ - [G]_\lambda^+| d\alpha) \\
&\le (LS) \int_a^\infty D(F, G) d\alpha.
\end{aligned}$$

□

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