JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 1, February 2019
http://dx.doi.org/10.14403/jcms.2019.32.1.127

ON THE STABILITY OF AN ADDITIVE-QUARTIC SET-VALUED FUNCTIONAL EQUATION

Yang-Hi Lee

Abstract. In this paper, I investigate a stability of the following set-valued functional equation

$$f(x+3y)\oplus 10f(x+y)\oplus 7f(-x)\oplus 5f(x-y)$$
$$= 5f(x+2y)\oplus f(x)\oplus f(2x)\oplus f(x-2y)$$

in the sense of P. Găvruta.

1. Introduction

In 1940, Ulam [14] first posed a question about the stability of group homomorphisms. In 1941, Hyers [4] provided a positive answer to this question about additive mappings between Banach spaces. Since then many mathematicians have dealt with this problem [3, 8, 9, 13].

A solution of the functional equation

$$f(x+y)-f(x)-f(y)=0$$

is called an additive mapping and a solution of the functional equation

$$f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-24f(y)=0$$

is called a quartic mapping. Now I consider the following functional equation

$$\begin{aligned} &f(x+3y)-5f(x+2y)+10f(x+y)-f(x)+7f(-x)-f(2x)\\ &\quad+5f(x-y)-f(x-2y)=0. \end{aligned} \tag{1.1}$$

The mapping $f(x)=ax^4+bx$ is a solution of this functional equation, where a,b are real constants. A mapping f is called an additive-quartic mapping if f is represented by the sum of an additive mapping and a quartic mapping.

Received December 15, 2018; Accepted February 01, 2019.
2010 Mathematics Subject Classification: Primary 39B82, 39B52.
Key words and phrases: stability, additive-quartic set-valued functional equation.

Lu and Park [7] initiated the study of the Ulam stability of set-valued functional equations as a generalization of the stability of single-valued functional equations. Along the way, many authors have studied the stability problem of set-value functional equations of various types [2, 5, 6, 10, 11].

The following terminologies used in this paper will be adopted from the article of Kenary et al. [6].

Throughout this paper, unless otherwise specified, let X be a real vector space and Y be a Banach space with the norm $\|\cdot\|_Y$. We denote by $C_b(Y)$, $C_c(Y)$ and $C_{cb}(Y)$ the set of all closed bounded subsets of Y, the set of all closed convex subsets of Y and the set of all closed convex bounded subsets of Y, respectively. For the nonempty sets $A, B \subset Y$ and $\lambda \in \mathbb{R}$, let $A + B, \lambda A$ be the sets defined as follows

$$A + B = \{a + b | a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a | a \in A\}.$$

Furthermore, for the subsets $A, B \in C_c(Y)$, we write $A \oplus B = \overline{A + B}$, where $\overline{A + B}$ denotes the closure of $A + B$. Generally, we can obtain that

$$\lambda A + \lambda B = \lambda(A + B) \text{ and } (\lambda + \mu)A \subset \lambda A + \mu A$$

for arbitrary $\lambda, \mu \in \mathbb{R}^+$. In particular, if A is convex, then we have $(\lambda+\mu)A = \lambda A+\mu A$. For $A, B \in C_b(Y)$, the Hausdorff distance between A and B is defined by

$$h(A, B) := \inf\{\varepsilon > 0 | A \subset B + \varepsilon\overline{S_1}, B \subset A + \varepsilon\overline{S_1}\},$$

where S_1 denotes the closed unit ball in Y, i.e. $S_1 = \{y \in Y | \|y\|_Y \leq 1\}$. I easily know that

$$h(A, B) = h(-A, -B)$$

for all $A, B \in C_c(Y)$ from the definition of h. Since Y is a Banach space, it is proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup [1]. Rädström [12] proved that $(C_{cb}(Y), \oplus, h)$ can be isometrically embedded in a Banach space. The following are some properties of the Hausdorff distance.

Lemma 1.1. (Castaing and Valadier [1]). *For any $A_1, A_2, B_1, B_2, C \in C_{cb}(Y)$ and $\lambda \in \mathbb{R}^+$, the following expressions hold*
(i) $h(A_1 \oplus A_2, B_1 \oplus B_2) \leq h(A_1, B_1) + h(A_2, B_2)$;
(ii) $h(\lambda A_1, \lambda B_1) = \lambda h(A_1, B_1)$;
(iii) $h(A_1 \oplus C, B_1 \oplus C) = h(A_1, B_1)$.

In particular, from (i) and (iii) in Lemma 1.1, I have

$$h(A, C) = h(A \oplus B, B \oplus C) \leq h(A, B) + h(B, C)$$

for any $A, B, C \in C_{cb}(Y)$.

The main purpose of this paper is to establish the stability of the following additive-quartic set-valued functional equation

$$\begin{aligned} f(x+3y)\oplus 10f(x+y) \oplus 7f(-x) \oplus 5f(x-y) \\ =5f(x+2y) \oplus f(x) \oplus f(2x) \oplus f(x-2y) \end{aligned} \tag{1.2}$$

in the sense of P. Găvruta [3]. Namely, starting from the given mapping f that approximately satisfies the functional equation (1.2), a solution F of the functional equation (1.2) is explicitly constructed by using the formula either

$$F(x) = \lim_{n\to\infty}\left(\frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x)\right)$$

or

$$F(x) = \lim_{n\to\infty}\left(\frac{16^n+2^n}{2} f(2^{-n} x) \oplus \frac{16^n-2^n}{2} f(-2^{-n} x)\right)$$

which approximates the mapping f.

2. Stability of the additive-quartic set-valued functional equation (1.2)

Now I will establish the stability of the set-valued functional equation (1.2) in the sense of P. Găvruta by employing the direct method.

THEOREM 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) = \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} < \infty \tag{2.1}$$

for all $x, y \in X$. Suppose that $f : X \to C_{cb}(Y)$ is a mapping satisfying

$$\begin{aligned} h(f(x+3y)\oplus 10f(x+y) \oplus 7f(-x) \oplus 5f(x-y), \\ 5f(x+2y) \oplus f(x) \oplus f(2x) \oplus f(x-2y)) \le \varphi(x,y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique mapping $F : X \to C_{cb}(Y)$ such that F is a solution of the functional equation (1.2) *and*

(2.3)

$$h(f(x), F(x)) \le \sum_{n=0}^{\infty}\left(\frac{8^{n+1}+1}{2\cdot 16^{n+1}}\varphi(2^n x, 0) + \frac{8^{n+1}-1}{2\cdot 16^{n+1}}\varphi(-2^n x, 0)\right)$$

for all $x \in X$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x)\right) \tag{2.4}$$

for all $x \in X$.

Proof. Setting $y = 0$ in (2.2), I have

$$\begin{aligned} h(9f(x) \oplus 7f(-x), f(2x)) &= h(16f(x) \oplus 7f(-x), f(2x) \oplus 7f(x)) \\ &\le \varphi(x, 0) \end{aligned} \tag{2.5}$$

for all $x \in X$. Replacing x by $2^n x$ in (2.5) and dividing both sides by $2 \cdot 16^{n+1}$, from the equality (ii) in Lemma 1.1, we get

$$h\left(\frac{9f(2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7f(-2^n x)}{2 \cdot 16^{n+1}}, \frac{f(2^{n+1}x)}{2 \cdot 16^{n+1}}\right) \le \frac{\varphi(2^n x, 0)}{2 \cdot 16^{n+1}}$$

for all $x \in X$ and $n \in \mathbb{N}$. By the above inequality and the inequality (i) in Lemma 1.1, I have

$$\begin{aligned} &h\bigg(\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x), \\ &\quad \frac{1+8^{n+1}}{2 \cdot 16^{n+1}} f(2^{n+1}x) \oplus \frac{1-8^{n+1}}{2 \cdot 16^{n+1}} f(-2^{n+1}x)\bigg) \\ &\le h\left(\frac{9(1+8^{n+1})f(2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7(1+8^{n+1})f(-2^n x)}{2 \cdot 16^{n+1}}, \frac{1+8^{n+1}}{2 \cdot 16^{n+1}} f(2^{n+1}x)\right) \\ &\quad + h\left(\frac{9(1-8^{n+1})f(-2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7(1-8^{n+1})f(2^n x)}{2 \cdot 16^{n+1}}, \frac{1-8^{n+1}}{2 \cdot 16^{n+1}} f(-2^{n+1}x)\right) \\ &\le \frac{8^{n+1}+1}{2 \cdot 16^{n+1}} \varphi(2^n x, 0) + \frac{8^{n+1}-1}{2 \cdot 16^{n+1}} \varphi(-2^n x, 0) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. From the above inequality and the property $h(A, C) \le h(A, B) + h(B, C)$, we obtain

$$\begin{aligned} &h\left(f(x), \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\right) \\ &\quad \le \sum_{k=0}^{n-1} h\bigg(\frac{1+8^k}{2 \cdot 16^k} f(2^n x) \oplus \frac{1-8^k}{2 \cdot 16^k} f(-2^k x), \\ &\qquad\qquad \frac{1+8^{k+1}}{2 \cdot 16^{k+1}} f(2^{k+1}x) \oplus \frac{1-8^{k+1}}{2 \cdot 16^{k+1}} f(-2^{k+1}x)\bigg) \\ &\quad \le \sum_{k=0}^{n-1} \left(\frac{8^{k+1}+1}{2 \cdot 16^{k+1}} \varphi\left(2^k x, 0\right) + \frac{8^{k+1}-1}{2 \cdot 16^{k+1}} \varphi\left(-2^k x, 0\right)\right) \end{aligned} \tag{2.6}$$

for all $x \in X$ and $n \in \mathbb{N}$. In particular, if $\varphi(x,y) := 0$ i.e., f is a solution of the functional equation od (1.2), then I know that

$$f(x) = \frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x) \tag{2.7}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now I claim that the sequence $\{\frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y),\ h)$. Indeed, for all $m, n \in \mathbb{N}$, by (2.6), one can shows that

$$\begin{aligned}
&h\Big(\frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x), \frac{1+8^{m+n}}{2\cdot 16^{m+n}} f(2^n x) \oplus \frac{1-8^{m+n}}{2\cdot 16^{m+n}} f(-2^n x)\Big) \\
&\le \sum_{k=n}^{m+n-1} \left(\frac{8^{k+1}+1}{2\cdot 16^{k+1}} \varphi\left(2^k x, 0\right) + \frac{8^{k+1}-1}{2\cdot 16^{k+1}} \varphi\left(-2^k x, 0\right)\right) \\
&\le \sum_{k=n}^{m+n-1} \frac{\varphi\left(2^k x, 0\right) + \varphi\left(-2^k x, 0\right)}{2^k}
\end{aligned}$$

for all $x \in X$. From the condition (2.1), it follows that the right side of the above inequality tends to zero as $n \to \infty$. So the sequence $\{\frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x)\}$ is a Cauchy sequence. Therefore, from the completeness of $C_{cb}(Y)$, I can define a set-valued mapping $F : X \to (C_{cb}(Y)$ represented by (2.4) for all $x \in X$.

Next, I will show that F satisfies the set-valued functional equation (1.1). From the properties $h(A,D) \le h(A,B) + h(B,C) + h(C,D)$ and $h(A_1 \oplus A_2 \oplus \cdots \oplus A_n, B_1 \oplus B_2 \oplus \cdots \oplus B_n) \le h(A_1,B_1) + h(A_2,B_2) + \cdots + h(A_n,B_n)$, I have

$$\begin{aligned}
&h\big(F(x+3y) \oplus 10F(x+y) \oplus 7F(-x) \oplus 5F(x-y), \\
&\quad 5F(x+2y) \oplus F(x) \oplus F(2x) \oplus F(x-2y)\big) \\
&\le h\Big(\frac{5(1+8^n)}{2\cdot 16^n} f(2^n(x+2y)) \oplus \frac{5(1-8^n)}{2\cdot 16^n} f(-2^n(x+2y)) \oplus \frac{1+8^n}{2\cdot 16^n} f(2^n x) \\
&\quad \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x) \oplus \frac{1+8^n}{2\cdot 16^n} f(2^{n+1} x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^{n+1} x) \\
&\quad \oplus \frac{1+8^n}{2\cdot 16^n} f(2^n(x-2y)) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n(x-2y)), \\
&\quad 5F(x+2y) \oplus F(x) \oplus F(2x) \oplus F(x-2y)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+h\bigg(F(x+3y)\oplus 10F(x+y)\oplus 7F(-x)\oplus 5F(x-y),\\
&\quad \frac{1+8^n}{2\cdot 16^n}f(2^n(x+3y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x+3y))\oplus\frac{10(1+8^n)}{2\cdot 16^n}f(2^n(x+y))\\
&\quad \oplus\frac{10(1-8^n)}{2\cdot 16^n}f(-2^n(x+y))\oplus\frac{7(1+8^n)}{2\cdot 16^n}f(-2^nx)\oplus\frac{7(1-8^n)}{2\cdot 16^n}f(2^nx))\\
&\quad \oplus\frac{5(1+8^n)}{2\cdot 16^n}f(2^n(x-y))\oplus\frac{5(1-8^n)}{2\cdot 16^n}f(-2^n(x-y))\bigg)\\
&+h\bigg(\frac{1+8^n}{2\cdot 16^n}f(2^n(x+3y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x+3y))\oplus\frac{10(1+8^n)}{2\cdot 16^n}f(2^n(x+y))\\
&\quad \oplus\frac{10(1-8^n)}{2\cdot 16^n}f(-2^n(x+y))\oplus\frac{7(1+8^n)}{2\cdot 16^n}f(-2^nx)\oplus\frac{7(1-8^n)}{2\cdot 16^n}f(2^nx))\\
&\quad \oplus\frac{5(1+8^n)}{2\cdot 16^n}f(2^n(x-y))\oplus\frac{5(1-8^n)}{2\cdot 16^n}f(-2^n(x-y)),\\
&\quad \frac{5(1+8^n)}{2\cdot 16^n}f(2^n(x+2y))\oplus\frac{5(1-8^n)}{2\cdot 16^n}f(-2^n(x+2y))\\
&\quad \oplus\frac{1+8^n}{2\cdot 16^n}f(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^nx)\oplus\frac{1+8^n}{2\cdot 16^n}f(2^{n+1}x)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^{n+1}x)\\
&\quad \oplus\frac{1+8^n}{2\cdot 16^n}f(2^n(x-2y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x-2y))\bigg)\\
&\le 5h\bigg(\frac{1+8^n}{2\cdot 16^n}f(2^n(x+2y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x+2y)),F(x+2y)\bigg)\\
&\quad +h\bigg(\frac{1+8^n}{2\cdot 16^n}f(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^nx),F(x)\bigg)\\
&\quad +h\bigg(\frac{1+8^n}{2\cdot 16^n}f(2^{n+1}x)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^{n+1}x),F(2x)\bigg)\\
&\quad +h\bigg(\frac{1+8^n}{2\cdot 16^n}f(2^n(x-2y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x-2y)),F(x-2y)\bigg)\\
&\quad +h\bigg(F(x+3y),\frac{1+8^n}{2\cdot 16^n}f(2^n(x+3y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x+3y))\bigg)\\
&\quad +10h\bigg(F(x+y),\frac{1+8^n}{2\cdot 16^n}f(2^n(x+y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x+y))\bigg)\\
&\quad +7h\bigg(F(-x),\frac{1+8^n}{2\cdot 16^n}f(-2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(2^nx))\bigg)\\
&\quad +5h\bigg(F(x-y),\frac{1+8^n}{2\cdot 16^n}f(2^n(x-y))\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^n(x-y))\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2^n}h\big(f(2^n(x+3y))\oplus 10f(2^n(x+y))\oplus 7f(-2^nx)\oplus 5f(2^n(x-y)),\\
&\quad 5f(2^n(x+2y))\oplus f(2^nx)\oplus f(2^{n+1}x)\oplus f(2^n(x-2y))\big)\\
&+\frac{1}{2^n}h\big(f(-2^n(x+3y))\oplus 10f(-2^n(x+y))\oplus 7f(2^nx)\oplus 5f(-2^n(x-y)),\\
&\quad 5f(-2^n(x+2y))\oplus f(-2^nx)\oplus f(-2^{n+1}x)\oplus f(-2^n(x-2y))\big)
\end{aligned}
\tag{2.8}
$$

for all $x, y \in X$. Since $\lim_{n\to\infty} h\left(F(x), \frac{1+8^n}{2\cdot 16^n}f(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^nx)\right) = 0$ and

$$
\begin{aligned}
&\frac{1}{2^n}h\big(f(2^n(x+3y))\oplus 10f(2^n(x+y))\oplus 7f(-2^nx)\oplus 5f(2^n(x-y)),\\
&\quad 5f(2^n(x+2y))\oplus f(2^nx)\oplus f(2^{n+1}x)\oplus f(2^n(x-2y))\big)\\
&+\frac{1}{2^n}h\big(f(-2^n(x+3y))\oplus 10f(-2^n(x+y))\oplus 7f(2^nx)\oplus 5f(-2^n(x-y)),\\
&\quad 5f(-2^n(x+2y))\oplus f(-2^nx)\oplus f(-2^{n+1}x)\oplus f(-2^n(x-2y))\big)\\
&\le\frac{1}{2^n}\varphi\left(2^nx, 2^ny\right)+\frac{1}{2^n}\varphi\left(-2^nx, -2^ny\right)\\
&\quad\to 0 \quad\text{as}\quad n\to\infty,
\end{aligned}
$$

all terms on the right hand side of the inequality (2.8) tend to zero as $n \to \infty$. This implies that F is an additive-quartic set-valued mapping. Moreover, letting $n \to \infty$ in (2.6), I get the desired inequality (2.3). To prove the uniqueness of F, assume that a set-valued mapping F' is another solution of the functional equation of (1.2) satisfying the inequality (2.3). Then the equality $F'(x) = \frac{1+8^n}{2\cdot 16^n}F'(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}F'(-2^nx)$ follows from (2.7). Thus I have

$$
\begin{aligned}
&h\left(F'(x), \frac{1+8^n}{2\cdot 16^n}f(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^nx)\right)\\
&\qquad\le h\left(\frac{1+8^n}{2\cdot 16^n}F'(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}F'(-2^nx),\right.\\
&\qquad\quad\left.\frac{1+8^n}{2\cdot 16^n}f(2^nx)\oplus\frac{1-8^n}{2\cdot 16^n}f(-2^nx)\right)\\
&\qquad\le\frac{1+8^n}{2\cdot 16^n}h\left(F'(2^nx), f(2^nx)\right)+\frac{8^n-1}{2\cdot 16^n}h\left(F'(-2^nx), f(-2^nx)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1+8^n}{2\cdot 16^n}\sum_{k=0}^{\infty}\left(\frac{8^{k+1}+1}{16^{k+1}}\varphi\left(2^{k+n}x,0\right)+\frac{8^{k+1}-1}{16^{k+1}}\varphi\left(-2^{k+n}x,0\right)\right)\\
&\leq \frac{1}{2^n}\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\varphi\left(2^{k+n}x,0\right)+\frac{1}{2^k}\varphi\left(-2^{k+n}x,0\right)\right)\\
&\leq \sum_{k=n}^{\infty}\left(\frac{1}{2^k}\varphi\left(2^{k}x,0\right)+\frac{1}{2^k}\varphi\left(-2^{k}x,0\right)\right).
\end{aligned}
$$

for all $x \in X$. It is easy to see from the condition (2.1) that all terms on the right hand side of the above inequality tend to zero as $n \to \infty$, i.e. $F'(x) = \lim_{n\to\infty} \frac{1+8^n}{2\cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2\cdot 16^n} f(-2^n x) = F(x)$ for all $x \in X$. This completes the proof of this theorem. □

THEOREM 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$
\Phi(x,y) = \sum_{k=0}^{\infty} 16^k \varphi(2^{-k}x, 2^{-k}y) < \infty \tag{2.9}
$$

for all $x, y \in X$. Suppose that $f : X \to C_{cb}(Y)$ is the mapping satisfying (2.2) *for all $x, y \in X$. Then the mapping $F : X \to C_{cb}(Y)$ defined by*

$$
F(x) = \lim_{n\to\infty}\left(\frac{16^n+2^n}{2} f(2^{-n}x) \oplus \frac{16^n-2^n}{2} f(-2^{-n}x)\right)
$$

for all $x \in X$, is a unique solution of the functional equation (1.2) *such that*

$$
h(f(x),F(x)) \leq \sum_{k=0}^{\infty}\left(\frac{16^k+2^k}{2}\varphi\left(\frac{x}{2^{k+1}},0\right)+\frac{16^k-2^k}{2}\varphi\left(\frac{-x}{2^{k+1}},0\right)\right) \tag{2.10}
$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2^{n+1}}$ in (2.5), I get

$$
h\left(f\left(\frac{x}{2^n}\right), 9f\left(\frac{x}{2^{n+1}}\right) \oplus 7f\left(\frac{-x}{2^{n+1}}\right)\right) \leq \varphi\left(\frac{x}{2^{n+1}},0\right)
$$

for all $x \in X$ and $n \in \mathbb{N}$. By the above inequality and the inequality (i) in Lemma 1.1, I have

$$
\begin{aligned}
h\Bigg(&\frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right), \\
&\quad \frac{16^{n+1}+2^{n+1}}{2} f\left(\frac{x}{2^{n+1}}\right) \oplus \frac{16^{n+1}-2^{n+1}}{2} f\left(\frac{-x}{2^{n+1}}\right)\Bigg) \\
\leq & h\left(\frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right), \frac{9(16^n+2^n)}{2} f\left(\frac{x}{2^{n+1}}\right) \oplus \frac{7(16^n+2^n)}{2} f\left(\frac{-x}{2^{n+1}}\right)\right) \\
&+ h\left(\frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right), \frac{9(16^n-2^n)}{2} f\left(\frac{-x}{2^{n+1}}\right) \oplus \frac{7(16^n-2^n)}{2} f\left(\frac{x}{2^{n+1}}\right)\right) \\
\leq & \frac{16^n+2^n}{\varphi}\left(\frac{x}{2^{n+1}}, 0\right) + \frac{16^n-2^n}{2}\varphi\left(\frac{-x}{2^{n+1}}, 0\right)
\end{aligned}
\tag{2.11}
$$

for all $x \in X$ and $n \in \mathbb{N}$. From the above inequality and the property $h(A, C) \leq h(A, B) + h(B, C)$,

$$
\begin{aligned}
h\Bigg(f(x), &\frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right)\Bigg) \\
&\leq \sum_{k=0}^{n-1} h\Bigg(\frac{16^k+2^k}{2} f\left(\frac{x}{2^k}\right) \oplus \frac{16^k-2^k}{2} f\left(\frac{-x}{2^k}\right), \\
&\qquad \frac{16^{k+1}+8^{k+1}}{2} f\left(\frac{x}{2^{k+1}}\right) \oplus \frac{16^{k+1}-8^{k+1}}{f}\left(\frac{-x}{2^{k+1}}\right)\Bigg) \\
&\leq \sum_{k=0}^{n-1}\left(\frac{16^k+2^k}{2}\varphi\left(\frac{x}{2^{k+1}}, 0\right) + \frac{16^k-2^k}{2}\varphi\left(\frac{-x}{2^{k+1}}, 0\right)\right)
\end{aligned}
\tag{2.12}
$$

for all $x \in X$ and $n \in \mathbb{N}$. In particular, if f is a solution of the functional equation od (1.2), then I know that

$$
f(x) = \frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right) \tag{2.13}
$$

for all $x \in X$ and $n \in \mathbb{N}$. Now I claim that the sequence $\{\frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right)\}$ is a Cauchy sequence in $(C_{cb}(Y),\ h)$. Indeed, for all

$m, n \in \mathbb{N}$, by (2.12), I know

$$h\bigg(\frac{16^m+2^m}{2}f\left(\frac{x}{2^m}\right) \oplus \frac{16^m-2^m}{2}f\left(\frac{-x}{2^m}\right), \\ \frac{16^{m+n}+2^{m+n}}{2}f\left(\frac{x}{2^{m+n}}\right) \oplus \frac{16^{m+n}-2^{m+n}}{f}\left(\frac{-x}{2^{m+n}}\right)\bigg) \\ \leq \sum_{k=m}^{m+n-1}\left(\frac{16^k+2^k}{2}\varphi\left(\frac{x}{2^{k+1}},0\right)+\frac{16^k-2^k}{2}\varphi\left(\frac{-x}{2^{k+1}},0\right)\right)$$

for all $x \in X$. It follows from the condition (2.9) that the right hand side of the above inequality tends to zero as $m \to \infty$. So the sequence $\{\frac{16^n+2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2}f\left(\frac{-x}{2^n}\right)\}$ is a Cauchy sequence. Since $C_{cb}(Y)$ is complete, I can define a set valued mapping $F : X \to C_{cb}(Y)$ represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{16^n+2^n}{2}f(2^{-n}x) \oplus \frac{16^n-2^n}{2}f(-2^{-n}x)\right)$$

for all $x \in X$. Next, I will show that F satisfies the the functional equation (1.2). Similar to Theorem 2.1, the inequality

$$h\bigg(\frac{16^n+2^n}{2}f\left(\frac{x+3y}{2^n}\right) \oplus \frac{16^n-2^n}{2}f\left(\frac{-x-3y}{2^n}\right) \\ \oplus \frac{10(16^n+2^n)}{2}f\left(\frac{x+y}{2^n}\right) \oplus \frac{10(16^n-2^n)}{2}f\left(\frac{-x-y}{2^n}\right) \\ \oplus \frac{7(16^n+2^n)}{2}f\left(\frac{-x}{2^n}\right) \oplus \frac{7(16^n-2^n)}{2}f\left(\frac{x}{2^n}\right) \\ \oplus \frac{5(16^n+2^n)}{2}f\left(\frac{x-y}{2^n}\right) \oplus \frac{5(16^n-2^n)}{2}f\left(\frac{-x+y}{2^n}\right), \\ \frac{5(16^n+2^n)}{2}f\left(\frac{x+2y}{2^n}\right) \oplus \frac{5(16^n-2^n)}{2}f\left(\frac{-x-2y}{2^n}\right) \\ \oplus \frac{16^n+2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2}f\left(\frac{-x}{2^n}\right) \\ \oplus \frac{16^n+2^n}{2}f\left(\frac{2x}{2^n}\right) \oplus \frac{16^n-2^n}{2}f\left(\frac{-2x}{2^n}\right) \\ \oplus \frac{16^n+2^n}{2}f\left(\frac{x-2y}{2^n}\right) \oplus \frac{16^n-2^n}{2}f\left(\frac{-x+2y}{2^n}\right)\bigg)$$

$$
\begin{aligned}
\leq & \frac{16^n+2^n}{2} h\Bigg(f\left(\frac{x+3y}{2^n}\right) \oplus 10f\left(\frac{x+y}{2^n}\right) \oplus 7f\left(\frac{-x}{2^n}\right) \oplus 5f\left(\frac{x-y}{2^n}\right), \\
& \qquad 5f\left(\frac{x+2y}{2^n}\right) \oplus f\left(\frac{x}{2^n}\right) \oplus f\left(\frac{2x}{2^n}\right) \oplus f\left(\frac{x-2y}{2^n}\right)\Bigg) \\
& + \frac{16^n-2^n}{2} h\Bigg(f\left(\frac{-x-3y}{2^n}\right) \oplus 10f\left(\frac{-x-y}{2^n}\right) \oplus 7f\left(\frac{x}{2^n}\right) \\
& \qquad \oplus 5f\left(\frac{-x+y}{2^n}\right), f\left(\frac{-x-2y}{2^n}\right) 5 \oplus f\left(\frac{-x}{2^n}\right) \\
& \qquad \oplus f\left(\frac{-2x}{2^n}\right) \oplus f\left(\frac{-x+2y}{2^n}\right)\Bigg) \\
\leq & \frac{16^n+2^n}{2} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \frac{16^n-2^n}{2} \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)
\end{aligned}
$$

for all $x, y \in X$, leads us that F satisfies the the functional equation (1.2). Moreover, letting $n \to \infty$ in (2.12), I get the desired inequality (2.10). To prove the uniqueness of F, assume that a set-valued mapping F' is another solution of (1.2) satisfying the inequality (2.10). Then the equality $F'(x) = \frac{16^n+2^n}{2} F'\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} F'\left(\frac{-x}{2^n}\right)$ follows from the equality (2.13). Thus I have

$$
\begin{aligned}
& h\Bigg(F'(x), \frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right)\Bigg) \\
= & h\Bigg(\frac{16^n+2^n}{2} F'\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} F'\left(\frac{-x}{2^n}\right), \\
& \qquad \frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right)\Bigg) \\
\leq & \frac{16^n+2^n}{2} h\left(F'\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right) + \frac{16^n-2^n}{2} h\left(F'\left(\frac{-x}{2^n}\right), f\left(\frac{-x}{2^n}\right)\right) \\
\leq & (16^n+2^n) \sum_{k=0}^{\infty} \left(\frac{16^k+2^k}{2} \varphi\left(\frac{x}{2^{k+n+1}}, 0\right) + \frac{16^k-2^k}{2} \varphi\left(\frac{-x}{2^{k+n+1}}, 0\right)\right) \\
& + (16^n-2^n) \sum_{k=0}^{\infty} \left(\frac{16^k+2^k}{2} \varphi\left(\frac{-x}{2^{k+n+1}}, 0\right) + \frac{16^k-2^k}{2} \varphi\left(\frac{x}{2^{k+n+1}}, 0\right)\right) \\
\leq & 2 \cdot 16^n \sum_{k=0}^{\infty} 16^k \left(\varphi\left(\frac{x}{2^{k+n+1}}, 0\right) + \varphi\left(\frac{-x}{2^{k+n+1}}, 0\right)\right)
\end{aligned}
$$

$$\leq \sum_{k=n+1}^{\infty} 16^k \left(\varphi\left(\frac{x}{2^k}, 0\right) + \varphi\left(\frac{-x}{2^k}, 0\right)\right) \to 0 \quad \text{as } n \to \infty,$$

which means that $F'(x) = \lim_{n\to\infty} \frac{16^n+2^n}{2} f\left(\frac{x}{2^n}\right) \oplus \frac{16^n-2^n}{2} f\left(\frac{-x}{2^n}\right))$ for all $x \in X$. Therefore $F(x) = F'(x)$ holds for all $x \in X$. This completes the proof of the theorem. □

References

[1] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, in: Lec. Notes in Math. **580**, Springer, Berlin, 1977.

[2] H. Y. Chu, A. Kim, and S. Y. Yoo, *On the stability of the generalized cubic set-valued functional equation*, Appl. Math. Lett. **37** (2014), 7-14.

[3] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[4] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[5] S. Y. Jang, C. Park, and Y. Cho, *Hyers-Ulam stability of a generalized additive set-valued functional equation*, J. Inequal. Appl. **101** (2013), 6 pages.

[6] H. A. Kenary, H. Rezaei et al., *On the stability of set-valued functional equations with the fixed point alternative*, Fixed Point Theory Appl. **81** (2012), 17 pages.

[7] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.

[8] Y. -H. Lee, *On the stability of the monomial functional equation*, Bull. Korean Math. Soc. **45** (2008), 397–403.

[9] Y. H. Lee and K. W. Jun, *On the Stability of Approximately Additive Mappings*, Proc. Amer. Math. Soc. **128** (2000), 1361–1369.

[10] C. Park, D. O'Regan, and R. Saadati, *Stability of some seset-valued functional equations*, Appl. Math. Lett. **24** (2011), 1910–1914.

[11] M. Piszczek, *The properties of functional inclusions and Hyers-Ulam stability*, Aequat. Math. **85** (2013), 111–118.

[12] H. Rädström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.

[13] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[14] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Korea
E-mail: lyhmzi@pro.gjue.ac.kr