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A FIXED POINT APPROACH TO THE STABILITY OF THE QUADRATIC AND CUBIC TYPE FUNCTIONAL EQUATION

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Abstract. In this paper, I investigate the stability of the functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-3f(y)+3f(-y)=0$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

In 1940, the problem of stability of group homomorphism was first raised by S. M. Ulam [9]. In the next year, D. H. Hyers [6] gave a partial solution to Ulam's question for the case of additive mappings. Hyers' result has greatly influenced the study of the stability problem of the functional equation. His result was generalized by Th. M. Rassias [7] for linear mappings.

In 2004, L. Cădariu and V. Radu [2] to prove stability theorems of *the Cauchy functional equation:*

$$f(x+y)-f(x)-f(y)=0 \tag{1.1}$$

and in 2003, they [1] obtained the stability of *the quadratic functional equation:*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1.2}$$

by using the fixed point method. Throughout this paper, let V be a (real or complex) linear space and Y a Banach space. We call a solution $f:V\to W$ of (1.1) an additive mapping and call a solution of (1.2) *a*

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quadratic mapping. Notice that a mapping $f : V \to W$ is called a cubic mapping if f is a solution of the cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0. \tag{1.3}$$

A mapping f is called a quadratic-cubic mapping if f is represented by sum of a quadratic mapping and a cubic mapping. Now we consider the functional equation:

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y) = 0 \tag{1.4}$$

which is called *the quadratic and cubic type functional equation.* The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^3 + bx^2 + c$ is a solution of this functional equation, where a, b, c are real constants.

In 2010, W. Towanlong and P. Nakmahachalasint, [8] proved the stability of the quadratic and cubic type functional equation by handling the odd part and the even part of the given function f, respectively. In this paper, instead of splitting the given function $f : X \to Y$ into two parts, we will prove the stability of the functional equation (1.3) at once by using the fixed point theory and we will show that every quadratic-cubic mapping is a solution of the functional equation(1.3).

2. Main results

We need the following Margolis and Diaz's fixed point theorem to prove the main theorem.

THEOREM 2.1. ([4]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:
(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

For a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned} f_e(x) :=& \frac{f(x)+f(-x)}{2}, \quad f_o(x) := \frac{f(x)-f(-x)}{2}, \\ Cf(x,y) :=& f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y), \\ Qf(x,y) :=& f(x+y) + f(x-y) - 2f(x) - 2f(y), \\ Df(x,y) :=& f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y) \end{aligned}$$

for all $x, y \in V$. As we stated in the previous section, solutions of $Qf \equiv 0$ and $Cf \equiv 0$ are called a quadratic mapping and a cubic mapping, respectively. Now we will show that f is a quadratic-cubic mapping if f is a solution of the functional equation $Df(x,y) = 0$ for all $x, y \in V$ with $f(0) = 0$.

THEOREM 2.2. *A mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$ with $f(0) = 0$ if and only if f is a quadratic-cubic mapping.*

Proof. If a mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$, then $Df_o(x,y) = 0$ and $Df_e(x,y) = 0$ for all $x, y \in V$. Therefore the equalities $Cf_o(x,y) = Df_o(x,y)$ and $Qf_e(x,y) = 0$ can be obtained from $Cf_o(x,y) = Df_o(x,y)$ and $12Qf_e(x,y) = Df_e(0,x+y) + Df_e(0,x-y) - Df_e(2x,y) - Df_e(2x,-y) - 4Df_e(y,x) - 4Df_e(-y,x) + 6D_ef(0,x)$ for all $x, y \in V$. So f_o is a cubic mapping and f_e is a quadratic mapping. It follows from the equality $f = f_o + f_e$ that f is a quadratic-cubic mapping.

Conversely, assume that f is a quadratic-cubic mapping. Then there are mappings f_1 and f_2 satisfying the equalities $f := f_1 + f_2$, $Qf_1(x,y) = 0$, and $Cf_2(x,y) = 0$ for all $x, y \in V$. Since f_1 and f_2 are quadratic mapping and cubic mapping, respectively, the equalities $f_1(x) = f_1(-x)$, $f_2(x) = -f_2(-x)$, $f_1(2x) = 4f_1(x)$, and $f_2(2x) = 8f_2(x)$ hold for all $x \in V$. From these equalities, we obtain the equalities

$$\begin{aligned} Df_1(x,y) &= Qf_1(x+y,y) - Qf_1(x,y), \\ Df_2(x,y) &= Cf_2(x,y) \end{aligned}$$

for all $x, y \in V$, which mean that

$$Df(x,y) = Df_1(x,y) + Df_2(x,y) = 0$$

for all $x, y \in V$ as we desired. □

LEMMA 2.3. *If $f : V \to Y$ is a mapping satisfying the equality $Df(x,y) = 0$ for all $x, y \in V \backslash \{0\}$ and $f(0) = 0$, then $Df(x,y) = 0$ for all $x, y \in V$*

Proof. Since $f(0) = 0$, we easily obtain the equalities

$$Df(0, y) = -Df(y, -y) = 0$$

for all $x \in V\backslash\{0\}$ and $f(x, 0) = 0$ for all $x \in V$. □

Now we can prove some stability results of the functional equation (1.3) by using the fixed point theory.

THEOREM 2.4. *Let $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$ be a given function. Suppose that the mapping $f : V \to Y$ satisfies*

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.1}$$

for all $x, y \in V\backslash\{0\}$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(2x, 2y) \le 4L\varphi(x, y) \tag{2.2}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique quadratic-cubic mapping $F : V \to Y$ such that

$$\|f(x) - f(0) - F(x)\| \le \frac{3\psi(x)}{16(1 - L)} \tag{2.3}$$

for all $x \in V\backslash\{0\}$, where $\psi(x) := \varphi(x, -x) + \varphi(-x, x)$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \left(\frac{f_e(2^n x)}{2^{2n}} + \frac{f_o(2^n x)}{2^{3n}} \right) \tag{2.4}$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{4}$, $\varphi(x, y)$ is continuous, and $f(0) = 0$, then f is itself a quadratic-cubic mapping.

Proof. If we consider the mapping $\tilde{f} = f - f(0)$, then $\tilde{f} : V \to Y$ satisfies $\tilde{f}(0) = 0$ and

$$\|D\tilde{f}(x, y)\| = \|Df(x, y)\| \le \varphi(x, y)$$

for all $x, y \in V\backslash\{0\}$. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$ and introduce a generalized metric on S by

$$d(g, h) = \inf\{K \in \mathbb{R}^+ | \|g(x) - h(x)\| \le K\psi(x) \ \text{ for all } \ x \in V\backslash\{0\}\}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{g(2x) - g(-2x)}{16} + \frac{g(2x) + g(-2x)}{8}$$

for all $x \in V$. Notice

$$J^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2 \cdot 8^n} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}$$

for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &\leq \frac{3}{16}\|g(2x) - h(2x)\| + \frac{1}{16}\|g(-2x) - h(-2x)\| \\ &\leq \frac{1}{4}K\psi(2x) \leq KL\psi(x)\end{aligned}$$

for all $x \in V\backslash\{0\}$, which implies that $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1), we see that

$$\|\tilde{f}(x) - J\tilde{f}(x)\| = \frac{1}{16}\| - 3Df(x, -x) - Df(-x, x)\| \leq \frac{3}{16}\psi(x)$$

for all $x \in V\backslash\{0\}$. It means that $d(\tilde{f}, J\tilde{f}) \leq \frac{3}{16} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n\tilde{f}\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(\tilde{f}, g) < \infty\}$, which is represented by

$$F(x) = \lim_{n\to\infty} \Big(\frac{\tilde{f}_e(2^n x)}{2^{2n}} + \frac{\tilde{f}_o(2^n x)}{2^{3n}}\Big)$$

for all $x \in V$. Notice that

$$d(\tilde{f}, F) \leq \frac{1}{1-L}d(\tilde{f}, J\tilde{f}) \leq \frac{3}{16(1-L)}$$

which implies (2.3) and (2.4). By the definitions of F, together with (2.1) and (2.2), we have that

$$\begin{aligned}\|DF(x, y)\| &= \lim_{n\to\infty} \Big\|\frac{D\tilde{f}_o(2^n x, 2^n y)}{8^n} + \frac{D\tilde{f}_e(2^n x, 2^n y)}{4^n}\Big\| \\ &\leq \lim_{n\to\infty} \frac{2^n + 1}{2 \cdot 8^n} (\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)) \\ &\leq \lim_{n\to\infty} \frac{(2^n + 1)L^n}{2 \cdot 2^n} (\varphi(x, y) + \varphi(-x, -y)) = 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. By Lemma 2.3, F satisfies $DF(x, y) = 0$ for all $x, y \in V$. Notice that if F' is another solution of the functional equation (1.3), then the equality $F'(x) - JF'(x) = \frac{-3DF'(x,-x)-DF'(-x,x)}{16} = 0$ implies that F' is a fixed point of J. Hence F is unique mapping satisfying (2.4).

Moreover, if $0 < L < \frac{1}{4}$ and φ is continuous, then

$$\lim_{n\to\infty} \varphi(2^n x, 2^n y) \leq \lim_{n\to\infty} (4L)^n\varphi(x, y) = 0$$

for all $x, y \in V\backslash\{0\}$. Since φ is continuous, we get

$$\begin{aligned}
&\lim_{n\to\infty} \varphi((2^n a_1 + a_2)x, (2^n b_1 + b_2)y) \\
&\le \lim_{n\to\infty} (4L)^n \varphi\left(\left(a_1 + \frac{a_2}{2^n}\right)x, \left(b_1 + \frac{b_2}{2^n}\right)y\right) \\
&= 0 \cdot \varphi\left(a_1 x, b_1 y\right) = 0
\end{aligned}$$

for all $x, y \in V\backslash\{0\}$ and for any fixed integers a_1, a_2, b_1, b_2 with $a_1, b_1 \neq 0$. Therefore, we obtain

$$\begin{aligned}
&3\|f(x) - F(x)\| \\
&\le \lim_{n\to\infty} (\|Df((2^n+1)x, -2^n x) - DF((2^n+1)x, -2^n x)\| \\
&\quad + \|(F-f)((1-2^n)x)\| + 3\|(F-f)((2^n+1)x)\| \\
&\quad + \|(f-F)((2^{n+1}+1)x)\| + 3\|(f-F)(-2^n x)\| \\
&\quad + 3\|(F-f)(2^n x)\|) \\
&\le \lim_{n\to\infty} \varphi((2^n+1)x, 2^n x) + \frac{3}{16(1-L)} \lim_{n\to\infty} \big(\psi((1-2^n)x) \\
&\quad + 3\psi((2^n+1)x) + \psi((2^{n+1}+1)x) + 3\psi(-2^n x) + 3\psi(-2^n x)\big) \\
&= 0
\end{aligned}$$

for all $x \in V\backslash\{0\}$. This completes the proof of this theorem. □

We continue our investigation with the next result.

Theorem 2.5. *Let $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$. Suppose that $f : V \to Y$ satisfies the inequality $\|Df(x, y)\| \le \varphi(x, y)$ for all $x, y \in V\backslash\{0\}$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(2x, 2y) \ge 8\varphi(x, y) \tag{2.5}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique quadratic-cubic mapping $F : V \to Y$ such that

$$\|f(x) - f(0) - F(x)\| \le \frac{L\psi(x)}{8(1-L)} \tag{2.6}$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \left(8^n f_o\left(\frac{x}{2^n}\right) + 4^n f_e\left(\frac{x}{2^n}\right)\right) \tag{2.7}$$

for all $x \in V$.

Proof. Let the mapping $\tilde{f}$ and the set (S, d) be as in the proof of Theorem 2.2. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := 8g_o\left(\frac{x}{2}\right) + 4g_e\left(\frac{x}{2}\right)$$

for all $g \in S$ and $x \in V$. Notice that

$$J^n g(x) = 8^n g_o\left(\frac{x}{2^n}\right) + 4^n g_e\left(\frac{x}{2^n}\right)$$

and $J^0 g(x) = g(x)$ for all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= 6\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + 2\left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\leq 8K\psi\left(\frac{x}{2}\right) \leq LK\psi(x) \end{aligned}$$

for all $x \in V \backslash \{0\}$. So $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\|\tilde{f}(x) - J\tilde{f}(x)\| = \left\|-D\tilde{f}\left(-\frac{x}{2}, \frac{x}{2}\right)\right\| \leq \psi\left(\frac{x}{2}\right) \leq \frac{L}{8}\psi(x)$$

for all $x \in V \backslash \{0\}$, which implies that $d(\tilde{f}, J\tilde{f}) \leq \frac{L}{8} < \infty$. Therefore according to Theorem 2.1, the sequence $\{J^n \tilde{f}\}$ converges to the unique fixed point F of J in the set $T := \{g \in S | d(\tilde{f}, g) < \infty\}$, which is represented by

$$F(x) = \lim_{n\to\infty} 8^n \tilde{f}_o\left(\frac{x}{2^n}\right) + 4^n \tilde{f}_e\left(\frac{x}{2^n}\right)$$

for all $x \in V$. Notice that

$$d(\tilde{f}, F) \leq \frac{1}{1-L} d(\tilde{f}, J\tilde{f}) \leq \frac{L}{8(1-L)}$$

which implies (2.6) and (2.7). From the definition of $F(x)$, (2.1), and (2.5), we have

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n\to\infty} \left\|8^n D\tilde{f}_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 4^n D\tilde{f}_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} \frac{8^n + 4^n}{2}\left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} \frac{(2^n+1)L^n}{2 \cdot 2^n}\left(\varphi(x, y) + \varphi(-x, -y)\right) = 0 \end{aligned}$$

for all $x, y \in V \backslash \{0\}$. By Lemma 2.3, F satisfies $DF(x, y) = 0$ for all $x, y \in V$. Notice that if F is a solution of the functional equation (1.3), then the equality $F(x) - JF(x) = -DF\left(-\frac{x}{2}, \frac{x}{2}\right)$ implies that F is a fixed point of J. □

Remark 2.6. In Theorem 2.4 and Theorem 2.4, if φ satisfies the equality $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V\backslash\{0\}$, the inequalities (2.3) and (2.6) can be replaced by

$$\|f(x) - F(x)\| \le \frac{\varphi(x,-x)}{4(1-L)} \quad \text{and} \quad \|f(x) - F(x)\| \le \frac{L\varphi(x,-x)}{8(1-L)}$$

for all $x \in V\backslash\{0\}$, respectively.

Since the equality

$$Df(0,x) = -Df(x,-x)$$

holds for all $x \in V$, we can easily prove the following theorems by using the same method in the proofs of the above theorems.

Theorem 2.7. *Let $\varphi : V^2 \to [0,\infty)$ be a given function. Suppose that the mapping $f : V \to Y$ satisfies (2.1) for all $x, y \in V$. If there exists a constant $0 < L < 1$ such that φ has the property (2.2) for all $x, y \in V$, then there exists a unique quadratic-cubic mapping $F : V \to Y$ such that*

$$\|f(x) - f(0) - F(x)\| \le \frac{3(\varphi(0,-x) + \varphi(0,x))}{16(1-L)} \tag{2.8}$$

for all $x \in V$. In particular, F is represented by (2.4) for all $x \in V$.

Theorem 2.8. *Let $\varphi : V^2 \to [0,\infty)$. Suppose that $f : V \to Y$ satisfies the inequality (2.1) for all $x, y \in V$. If there exists $0 < L < 1$ such that the mapping φ has the property (2.5) for all $x, y \in V$, then there exists a unique quadratic-cubic mapping $F : V \to Y$ such that*

$$\|f(x) - f(0) - F(x)\| \le \frac{L}{8(1-L)}\left(\varphi(0,-x) + \varphi(0,x)\right)$$

for all $x \in V$. In particular, F is represented by (2.7) for all $x \in V$.

3. Applications

For a given mapping $f : V \to Y$, we use the following abbreviations

$$Cf(x,y) := f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),$$
$$Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y)$$

for all $x, y \in V$. Using Theorem 2.4 and Theorem 2.5, we will show the stability results of the cubic functional equation $Cf \equiv 0$ and the quadratic functional equation $Qf \equiv 0$ in the following corollaries.

Corollary 3.1. *Let $f_i : V \to Y, i = 1, 2,$ be functions for which there exist functions $\phi_i : V^2 \to [0, \infty), i = 1, 2,$ such that*

$$\|Cf_i(x, y)\| \le \phi_i(x, y) \tag{3.1}$$

for all $x, y \in V$. If $f_i(0) = 0, i = 1, 2,$ and there exists $0 < L < 1$ such that

$$\phi_1(2x, 2y) \le 4L\phi_1(x, y) \quad \text{and} \quad L\phi_2(2x, 2y) \ge 8\phi_2(x, y) \tag{3.2}$$

for all $x, y \in V$, then there exist unique cubic mappings $F_i : V \to Y, i = 1, 2,$ such that

(3.3)
$$\|f_1(x) - F_1(x)\| \le \frac{3(\phi_1(x, -x) + \phi_1(-x, x) + \phi_1(0, x) + \phi_1(0, -x))}{32(1 - L)},$$
$$\|f_2(x) - F_2(x)\| \le \frac{L(\phi_2(x, -x) + \phi_2(-x, x) + \phi_2(0, x) + \phi_2(0, -x))}{16(1 - L)}$$

for all $x \in V$. In particular, the mappings $F_i, i = 1, 2,$ are represented by

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{8^n} \quad \text{and} \quad F_2(x) = \lim_{n\to\infty} 8^n f_2\left(\frac{x}{2^n}\right) \tag{3.4}$$

for all $x \in V$.

Proof. Notice that

$$Df_i(x, y) = \frac{1}{2}Cf_i(x, y) - \frac{1}{2}Cf_i(x + y, -y)$$

for all $x, y \in V$ and $i = 1, 2$. Put

$$\varphi_i(x, y) := \frac{1}{2}\phi_i(x, y) + \frac{1}{2}\phi_i(x + y, -y)$$

for all $x, y \in V$ and $i = 1, 2$, then φ_1 satisfies (2.2) and φ_2 satisfies (2.5). Therefore $\|Df_i(x, y)\| \le \varphi_i(x, y)$ for all $x, y \in V$ and $i = 1, 2$. According to Theorem 2.4, there exists a unique mapping $F_1 : V \to Y$ satisfying

(3.3), which is represented by (2.4). Observe that, by (3.1) and (3.2),

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_{1e}(2^nx)}{4^n}\right\| &= \lim_{n\to\infty}\frac{1}{3\cdot 4^{n+1}}\|Cf_1(2^nx,-2^nx)+Cf_1(0,2^nx)\|\\ &\le \lim_{n\to\infty}\frac{1}{3\cdot 4^{n+1}}(\phi_1(2^nx,-2^nx)+\phi_1(0,-2^nx))\\ &\le \lim_{n\to\infty}\frac{L^n}{12}(\phi_1(x,-x)+\phi_1(0,-x))=0\end{aligned}$$

as well as $\lim_{n\to\infty}\left\|\frac{f_1(2^nx)+f_1(-2^nx)}{2\cdot 8^n}\right\|=0$ for all $x\in V$. From this and (2.4), we get (3.4). Moreover, we have

$$\left\|\frac{Cf_1(2^nx,2^ny)}{8^n}\right\|\le\frac{\phi_1(2^nx,2^ny)}{8^n}\le\frac{L^n}{2^n}\phi_1(x,y)$$

for all $x,y\in V$. Taking the limit as $n\to\infty$ in the above inequality, we get $CF_1(x,y)=0$ for all $x,y\in V$.

On the other hand, according to Theorem 2.5, there exists a unique mapping $F_2:V\to Y$ satisfying (3.3) which is represented by (2.7). Observe that, by (3.1) and (3.2),

$$\begin{aligned}\lim_{n\to\infty}2^{3n}\left\|f_{2e}\left(\frac{x}{2^n}\right)\right\| &= \lim_{n\to\infty}\frac{2^{3n}}{12}\left\|Cf_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right)+Cf_2\left(0,\frac{x}{2^n}\right)\right\|\\ &\le \lim_{n\to\infty}\frac{2^{3n}}{12}\left(\phi_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right)+\phi_2\left(0,\frac{x}{2^n}\right)\right)\\ &\le \lim_{n\to\infty}\frac{L^n}{12}(\phi_2(x,-x)+\phi_2(0,x))=0\end{aligned}$$

as well as $\lim_{n\to\infty}2^{2n}\left\|f_{2e}\left(\frac{x}{2^n}\right)\right\|=0$ for all $x\in V$. From these and (2.7), we get (3.4). Moreover, we have

$$\left\|2^{3n}Cf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\|\le 2^{3n}\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\le L^n\phi_2(x,y)$$

for all $x,y\in V$. Taking the limit as $n\to\infty$ in the above inequality, we get $CF_2(x,y)=0$ for all $x,y\in V$. □

COROLLARY 3.2. *Let $f_i:V\to Y, i=1,2,$ be functions for which there exist functions $\phi_i:V^2\to[0,\infty), i=1,2,$ such that*

$$\|Qf_i(x,y)\|\le\phi_i(x,y)$$

for all $x,y\in V$. If $f_i(0)=0, i=1,2,$ and there exists $0<L<1$ such that the mapping ϕ_1 and $\phi_2$satisfy (3.2) for all $x,y\in V$, then there exist *unique quadratic mappings $F_i : V \to Y, i = 1, 2,$ such that*

(3.5)

$$\|f_1(x) - F_1(x)\| \leq \frac{3(\phi_1(x,x) + \phi_1(-x,-x) + 2\phi_1(0,x) + 2\phi_1(0,-x))}{16(1-L)},$$

$$\|f_2(x) - F_2(x)\| \leq \frac{L(\phi_2(x,x) + \phi_2(-x,-x) + 2\phi_2(0,x) + 2\phi_2(0,-x))}{8(1-L)}$$

for all $x \in V$. In particular, the mappings $F_i, i = 1, 2,$ are represented by

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{4^n} \quad \text{and} \quad F_2(x) = \lim_{n\to\infty} 4^n f_2\left(\frac{x}{2^n}\right) \tag{3.6}$$

for all $x \in V$.

Proof. Notice that

$$Df_i(x,y) = Qf_i(x+y,y) - Qf_i(x,-y) + Qf_i(0,y)$$

for all $x, y \in V$ and $i = 1, 2$. Put $\varphi_i(x,y) := \phi_i(x+y,y) + \phi_i(x,-y) + \phi_i(0,y)$ for all $x, y \in V$ and $i = 1, 2$, then φ_1 satisfies (2.2) and φ_2 satisfies (2.6). Moreover

$$\|Df_i(x,y)\| \leq \varphi_i(x,y)$$

for all $x, y \in V$ and $i = 1, 2$. According to Theorem 2.4, there exists a unique mapping $F_1 : V \to Y$ satisfying (3.5) which is represented by (2.4). Observe that

$$\begin{aligned}\lim_{n\to\infty} \left\|\frac{f_{1o}(2^n x)}{2^{3n}}\right\| &= \lim_{n\to\infty} \frac{\|Qf_1(0,-2^n x)\|}{2^{3n+1}} \leq \lim_{n\to\infty} \frac{\phi_1(0,-2^n x)}{2^{3n+1}} \\ &\leq \lim_{n\to\infty} \frac{L^n}{2^{n+1}} \phi_1(0,-x) = 0\end{aligned}$$

as well as $\lim_{n\to\infty} \left\|\frac{f_1(2^n x) - f_1(-2^n x)}{2\cdot 4^n}\right\| = 0$ for all $x \in V$. From this and (2.4), we get (3.6) for all $x \in V$. Moreover, we have

$$\left\|\frac{Qf_1(2^n x, 2^n y)}{4^n}\right\| \leq \frac{\phi_1(2^n x, 2^n y)}{4^n} \leq L^n \phi_1(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_1(x,y) = 0$$

for all $x, y \in V$. On the other hand, according to Theorem 2.5, there exists a unique mapping $F_2 : V \to Y$ satisfying (3.5) which is represented by (2.7). Observe that

$$8^n\left\|f_{2_o}\left(\frac{x}{2^n}\right)\right\| = \frac{8^n}{2}\left\|Qf_2\left(0,-\frac{x}{2^n}\right)\right\| \leq \frac{8^n}{2}\phi_2\left(0,-\frac{x}{2^n}\right)$$
$$\leq \frac{L^n}{2}\phi_2(0,-x) = 0$$

for all $x \in V$. It leads us to get

$$\lim_{n\to\infty} 8^n f_{2_o}\left(\frac{x}{2^n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} 4^n f_{2_o}\left(\frac{x}{2^n}\right) = 0$$

for all $x \in V$. From these and (2.7), we obtain (3.6). Moreover, we have

$$\left\|4^n Qf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \leq 4^n\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right) \leq \frac{L^n}{2^n}\phi_2(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_2(x,y) = 0$$

for all $x, y \in V$. □

Now we can use Remark 2.6, Theorem 2.7 and Theorem 2.8 to show the stability of the Hyers-Ulam-Rassias stability of the functional equation (1.3) in the following theorems:

COROLLARY 3.3. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.7}$$

for all $x, y \in X$, where $\theta \geq 0$ and $p \in [0,2) \cup (3,\infty)$. Then there exists a unique quadratic-cubic mapping $F : X \to Y$ such that

$$\|f(x) - f(0) - F(x)\| \leq \begin{cases} \dfrac{\theta}{2^p - 8}\|x\|^p & \text{if } p > 3, \\ \dfrac{\theta}{4 - 2^p}\|x\|^p & \text{if } 0 \leq p < 2 \end{cases}$$

for all $x \in X$.

Proof. This corollary follows from Remark 2.6, Theorem 2.7 and Theorem 2.8, by putting $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$, $L := 2^{p-2} < 1$ when $p < 2$, and $L := 2^{3-p} < 1$ when $p > 3$. □

COROLLARY 3.4. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies $f(0) = 0$ and the inequality (3.7) for all $x, y \in X\backslash\{0\}$, where $\theta \geq 0$ and $p < 0$. Then f is itself a quadratic-cubic mapping.*

Proof. This corollary follows from Theorem 2.4, by putting $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $L := 2^{p-2} < \frac{1}{4}$. □

COROLLARY 3.5. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \le \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$, where $\theta \ge 0$, $p > 0$ and $p + q \in [0, 2) \cup (3, \infty)$. Then f is itself a quadratic-cubic mapping.

Proof. This corollary follows from Remark 2.6, Theorem 2.7 and Theorem 2.8, by putting $\varphi(x, y) := \theta\|x\|^p\|y\|^q$, $L := 2^{p+q-2} < 1$ when $p + q < 2$, and $L := 2^{3-p-q} < 1$ when $p + q > 3$. □

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