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k SPACED SUM OF GENERALIZED PELL SEQUENCES

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Abstract. The purpose of work is to study the sums of k distanced numbers in Pell sequence $\{P_i\}$ as well as in the generalized λ-Pell sequence $\{P_{\lambda,i}\}$.

1. Introduction

When we say a λ-Pell sequence, we mean a generalized Pell sequence $\{P_{\lambda,i} \mid i \geq 0\}$ satisfying $P_{\lambda,i+2} = \lambda P_{\lambda,i+1} + P_{\lambda,i}$ with two initials for any $\lambda > 0$. If $\lambda = 1$ then $P_{\lambda,i}$ equals either the Fibonacci sequence F_i with initials $F_0 = 0$, $F_1 = 1$ or the Lucas sequence L_i with $L_0 = 2$, $L_1 = 1$. Similarly if $\lambda = 2$ then $P_{\lambda,i}$ equals either the Pell sequence P_i with $P_0 = 0$, $P_1 = 1$ or the Pell-Lucas sequence Q_i with $Q_0 = Q_1 = 2$.

A study of sum of numbers of certain sequence has been an important subject in mathematics for a long time. In fact, it was proved that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, $\sum_{i=1}^{n} L_i = L_{n+2} - 3$, $\sum_{i=1}^{n} P_i = \frac{1}{2}(P_{n+1} + P_n - 1)$ ([4], [5]), and even $\sum_{i=1}^{n} T_i = \frac{1}{2}(T_{n+2} + T_n - 1)$ with tribonacci sequence $\{T_i\}$ ([3], [6]). Moreover the sum of $n+1$ consecutive Pell numbers were discovered in [2] that $\sum_{i=0}^{n} P_{k+i} = \frac{1}{2}Q_{k+n} + P_{k+n} - \frac{1}{2}Q_k$ and $\sum_{i=0}^{n} (-1)^i P_{k+i} = (-1)^n \frac{1}{2}Q_{k+n} - \frac{1}{2}Q_k + P_k$. On the other hand, [8] proved the sum of the first $4n+1$ Pell numbers is a perfect square $(P_{2n} + P_{2n+1})^2$, and [1] characterized the sum of the first n Pell numbers in terms of squares of Pell numbers.

In this work, as a variation of sums of consecutive Pell numbers, we concern about the sum $\sum_{i=0}^{t} P_{ki+r}$ of k distanced $t+1$ numbers of $\{P_i\}$

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starting from P_r. The sum is called the k spaced sum and denoted by $s_t^{(k,r)}$. In particular if $k=1$ then $s_t^{(1,r)}$ is the sum of $t+1$ consecutive numbers from P_r to P_{t+r}, and also if $r=1$ then $s_t^{(1,1)} = \sum_{i=0}^{t} P_i$ ([4]). For the purpose, we display all Pell numbers P_i $(i>0)$ in order inside a rectangle-like table with k columns. Then each column is composed of k distanced numbers, so if $1 \le r \le k$ then $s_t^{(k,r)}$ is the sum of $t+1$ numbers in rth column of the table. We shall find some recurrence formulas of $s_t^{(k,r)}$, and then as a generalization, we extend the investigation to λ-Pell sequence $\{P_{\lambda,i}\}$ and the sum $s_{\lambda,t}^{(k,r)} = \sum_{i=0}^{t} P_{\lambda,ki+r}$.

2. k spaced sum of Pell numbers

By a Pell table (P-table, for short) with k columns, we mean a rectangle table with k columns consisting of all P_i in order (left below). So if $1 \le r \le k$, the sum $s_t^{(k,r)} = \sum_{i=0}^{t} P_{ki+r}$ of k spaced $t+1$ numbers is the sum of $t+1$ entries in rth column, hence under the connection of P-table with k columns, sometimes we call $s_t^{(k,r)}$ a rth partial sum of $t+1$ Pell numbers. The table composed of all $s_t^{(k,r)}$ will be called a k spaced sum table (right below).

P-table with k columns				
P_1	$\cdots$	P_r	$\cdots$	P_k
P_{k+1}	$\cdots$	P_{k+r}	$\cdots$	P_{2k}
P_{tk+1}	$\ddots$	P_{tk+r}	$\ddots$	$P_{(t+1)k}$

t	k spaced sum table				
0	$s_0^{(k,1)}$	$\cdots$	$s_0^{(k,r)}$	$\cdots$	$s_0^{(k,k)}$
1	$s_1^{(k,1)}$	$\cdots$	$s_1^{(k,r)}$	$\cdots$	$s_1^{(k,k)}$
t	$s_t^{(k,1)}$	$\ddots$	$s_t^{(k,r)}$	$\ddots$	$s_t^{(k,k)}$

On the other hand, if $r>k$ then with $r = ak+u$ $(a \ge 0,\ 1 \le u \le k)$ we have

$$\begin{aligned} s_t^{(k,r)} &= \sum_{i=0}^{t} P_{k(a+i)+u} = \sum_{j=a}^{a+t} P_{kj+u} = \sum_{j=0}^{a+t} P_{kj+u} - \sum_{j=0}^{a-1} P_{kj+u} \\ &= s_{a+t}^{(k,u)} - s_{a-1}^{(k,u)}, \end{aligned}$$

so we are enough to consider the k spaced sum $s_t^{(k,r)}$ with $1 \le r \le k$. In the same sense, the Fibonacci or Lucas tables with k columns are considered. In the work $\{P_n\}$ is assumed to have all integer subscripts $n \in \mathbb{Z}$. We begin with some identities of Pell numbers for next use.

Lemma 2.1. *$P_{n+1}-P_{n-1}=2P_n$, $P_{n+2}+P_{n-2}=6P_n$, $P_{n+3}-P_{n-3}=14P_n$, $P_{n+4}+P_{n-4}=34P_n$, $P_{n+5}-P_{n-5}=82P_n$ and $P_{n+6}+P_{n-6}=198P_n$.*

Proof. Since $P_{n+1}-P_{n-1}=(2P_n+P_{n-1})-P_{n-1}=2P_n$, we have

$P_{n+2}+P_{n-2}=(2P_{n+1}+P_n)+(P_n-2P_{n-1})=2P_n+2(P_{n+1}-P_{n-1})=6P_n$.

The rest can be proved analogously. □

Theorem 2.2. *The sum $s_t^{(k,r)}$ with $k=2,3$ satisfies $s_t^{(2,r)}=7P_{2(t-1)+r}+s_{t-3}^{(2,r)}$ and $s_t^{(3,r)}=15P_{3(t-1)+r}+2s_{t-2}^{(3,r)}-s_{t-3}^{(3,r)}$.*

Proof. The identity $P_{n+4}+P_n=6P_{n+2}$ in Lemma 2.1 shows

$$\begin{aligned} s_t^{(2,r)} &= s_{t-1}^{(2,r)}+P_{2t+r}=(s_{t-2}^{(2,r)}+P_{2(t-1)+r})+P_{2t+r}\\ &=(s_{t-3}^{(2,r)}+P_{2(t-2)+r})+P_{2(t-1)+r}+P_{2t+r}\\ &=s_{t-3}^{(2,r)}+(P_{2(t-2)+r}+P_{2t+r})+P_{2(t-1)+r}=7P_{2(t-1)+r}+s_{t-3}^{(2,r)}. \end{aligned}$$

Similarly by means of $P_{n+6}-P_n=14P_{n+3}$, we have

$$\begin{aligned} s_t^{(3,r)} &= s_{t-1}^{(3,r)}+P_{3t+r}=(s_{t-2}^{(3,r)}+P_{3(t-1)+r})+P_{3t+r}\\ &=s_{t-2}^{(3,r)}+P_{3(t-1)+r}+(14P_{3(t-1)+r}+P_{3(t-2)+r})\\ &=s_{t-2}^{(3,r)}+15P_{3(t-1)+r}+P_{3(t-2)+r}\\ &=s_{t-2}^{(3,r)}+15P_{3(t-1)+r}+(s_{t-2}^{(3,r)}-s_{t-3}^{(3,r)})\\ &=15P_{3(t-1)+r}+2s_{t-2}^{(3,r)}-s_{t-3}^{(3,r)}. \end{aligned}$$

□

From the following P-tables with 2 and 3 columns,

P-table		$s_t^{(2,1)}$	$s_t^{(2,2)}$
1	2	1	2
5	12	6	14
29	70	35	84
169	408	204	492

P-table			$s_t^{(3,1)}$	$s_t^{(3,2)}$	$s_t^{(3,3)}$
1	2	5	1	2	5
12	29	70	13	31	75
169	408	985	182	439	1060
2378	5741	13860	2560	6180	14920

it is easy to observe $s_3^{(2,1)}=204=7(29)+1=7P_5+s_0^{(2,1)}$ and $s_3^{(2,2)}=492=7(70)+2=7P_6+s_0^{(2,2)}$, and also

$$\begin{aligned} s_3^{(3,1)} &= 2560=15(169)+25=15(169)+2(13)-1\\ &=15P_7+2s_1^{(3,1)}-s_0^{(3,1)},\\ s_3^{(3,2)} &= 6180=15(408)+60=15(408)+2(31)-2\\ &=15P_8+2s_1^{(3,2)}-s_0^{(3,2)}, \end{aligned}$$

and so on.

THEOREM 2.3. *$s_t^{(k,r)}$ with $k=4,5,6$ satisfies $s_t^{(4,r)} = 35P_{4(t-1)+r} + s_{t-3}^{(4,r)}$, $s_t^{(5,r)} = 83P_{5(t-1)+r} + 2s_{t-2}^{(5,r)} - s_{t-3}^{(5,r)}$ and $s_t^{(6,r)} = 199P_{6(t-1)+r} + s_{t-3}^{(6,r)}$.*

Proof. Since $P_{n+8} + P_n = 34P_{n+4}$ in Lemma 2.1, we have

$$\begin{aligned} s_t^{(4,r)} &= s_{t-1}^{(4,r)} + P_{4t+r} = s_{t-2}^{(4,r)} + P_{4(t-1)+r} + P_{4t+r} \\ &= (s_{t-3}^{(4,r)} + P_{4(t-2)+r}) + P_{4(t-1)+r} + P_{4t+r} \\ &= s_{t-3}^{(4,r)} + P_{4(t-1)+r} + (P_{4(t-2)+r} + P_{4t+r}) \\ &= s_{t-3}^{(4,r)} + P_{4(t-1)+r} + 34P_{4(t-1)+r} = 35P_{4(t-1)+r} + s_{t-3}^{(4,r)}. \end{aligned}$$

On the other hand, the identity $P_{n+10} - P_n = 82P_{n+5}$ shows

$$\begin{aligned} s_t^{(5,r)} &= s_{t-1}^{(5,r)} + P_{5t+r} = s_{t-2}^{(5,r)} + P_{5(t-1)+r} + P_{5t+r} \\ &= s_{t-2}^{(5,r)} + P_{5(t-1)+r} + (82P_{5(t-1)+r} + P_{5(t-2)+r}) \\ &= 83P_{5(t-1)+r} + s_{t-2}^{(5,r)} + P_{5(t-2)+r} \\ &= 83P_{5(t-1)+r} + s_{t-2}^{(5,r)} + (s_{t-2}^{(5,r)} - s_{t-3}^{(5,r)}) = 83P_{5(t-1)+r} + 2s_{t-2}^{(5,r)} - s_{t-3}^{(5,r)}. \end{aligned}$$

Similarly from $P_{n+12} + P_n = 198P_{n+6}$ for all n, we have

$$\begin{aligned} s_t^{(6,r)} &= s_{t-3}^{(6,r)} + P_{6(t-2)+r} + P_{6(t-1)+r} + P_{6t+r} \\ &= s_{t-3}^{(6,r)} + (P_{6(t-2)+r} + P_{6t+r}) + P_{6(t-1)+r} \\ &= s_{t-3}^{(6,r)} + 198P_{6(t-1)+r} + P_{6(t-1)+r} = 199P_{6(t-1)+r} + s_{t-3}^{(6,r)}. \end{aligned}$$ □

From the P-tables with 4 or 5 columns

P-table with 4 col.

1	2	5	12
29	70	169	408
985	2378	5741	13860
33461	80782	···	

P-table with 5 col.

1	2	5	12	29
70	169	408	985	2378
5741	13860	33461	80782	195025
470832	1136689	2744210	···	

the spaced sum tables are

$s_t^{(4,r)} (1 \le r \le 4)$

1	2	5	12
30	72	174	420
1015	2450	5915	14280
34476	83232	···	

$s_t^{(5,r)} (1 \le r \le 5)$

1	2	5	12	29
71	171	413	997	2407
5812	14031	33874	81779	197432
476644	1150720	2778084	···	

so we see $s_3^{(4,1)} = 34476 = 35(985) + 1 = 35P_9 + s_0^{(4,1)}$, $s_3^{(4,2)} = 83232 = 35(2378) + 2 = 35P_{10} + s_0^{(4,2)}$ etc, and also

$$s_3^{(5,1)} = 83(5741) + 141 = 83(5741) + 2(71) - 1 = 83P_{11} + 2s_1^{(5,1)} - s_0^{(5,1)},$$

$$\begin{aligned} s_3^{(5,2)} &= 83(13860) + 340 = 83(13860) + 2(171) - 2 \\ &= 83P_{12} + 2s_1^{(5,2)} - s_0^{(5,2)}, \end{aligned}$$

and so on. Similarly $s_3^{(6,1)} = 6858740 = 199(33461) + 1 = 99P_{13} + s_0^{(6,1)}$

and $s_4^{(6,1)} = 199(6625109) + 170 = 199P_{19} + s_1^{(6,1)}$ from the 6 spaced sum table.

In order to generalize Theorem 2.2 and 2.3 to any $s_t^{(k,r)}$ $(k \geq 1)$, consider the Pell-Lucas sequence $\{Q_i\}$ satisfying $Q_{i+2} = 2Q_{i+1} + Q_i$ with $Q_0 = Q_1 = 2$. Then $\{Q_i\}_{i \geq 0} = \{2, 2, 6, 14, 34, 82, 198, 478, \cdots\}$ satisfy the next lemma ([2]).

Lemma 2.4. *$Q_n = P_{n-1} + P_{n+1}$ and $P_{n+2k} = Q_k P_{n+k} + (-1)^{k-1} P_n$ for any n, k.*

Comparing to the well known identities $L_n = F_{n-1} + F_{n+1}$ and $F_{n+2k} = L_k F_{n+k} + (-1)^{k-1} F_n$, we may say the Pell-Lucas is to Pell sequence what Lucas is to Fibonacci sequence. And the coefficients $7, 15, 35, 83, 199$ in Theorem 2.2 and 2.3 are explained by $\{Q_i\}$, and can be generalized as follows.

Theorem 2.5. $$s_t^{(k,r)} = \begin{cases} (Q_k + 1)P_{k(t-1)+r} + s_{t-3}^{(k,r)} & \text{if } 2 \mid k \\ (Q_k + 1)P_{k(t-1)+r} + 2s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)} & \text{if } 2 \nmid k \end{cases}$$

Proof. When $2 \leq k \leq 6$, Theorem 2.2 and 2.3 show that the first coefficient equals $Q_k + 1$. Now if $2 \mid k$ then $P_{k(t-2)+r} + P_{kt+r} = Q_k P_{k(t-1)+r}$ in Lemma 2.4 proves

$$\begin{aligned} s_t^{(k,r)} &= s_{t-3}^{(k,r)} + P_{k(t-2)+r} + P_{k(t-1)+r} + P_{kt+r} \\ &= s_{t-3}^{(k,r)} + (P_{k(t-2)+r} + P_{kt+r}) + P_{k(t-1)+r} \\ &= s_{t-3}^{(k,r)} + Q_k P_{k(t-1)+r} + P_{k(t-1)+r} = s_{t-3}^{(k,r)} + (Q_k + 1)P_{k(t-1)+r}. \end{aligned}$$

When $2 \nmid k$, using $P_{kt+r} = Q_k P_{k(t-1)+r} + P_{k(t-2)+r}$, we also have

$$\begin{aligned} s_t^{(k,r)} &= s_{t-2}^{(k,r)} + P_{k(t-1)+r} + P_{kt+r} = (Q_k+1)P_{k(t-1)+r} + s_{t-2}^{(k,r)} + P_{k(t-2)+r} \\ &= (Q_k + 1)P_{k(t-1)+r} + s_{t-2}^{(k,r)} + (s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}) \\ &= (Q_k + 1)P_{k(t-1)+r} + 2s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}. \end{aligned}$$ □

The Pell number can be retrieved from $s_t^{(k,r)}$ $(1 \leq r \leq k)$ that

$$P_{kt+r} = \begin{cases} \frac{1}{(Q_k+1)}(s_{t+1}^{(k,r)} - s_{t-2}^{(k,r)}) & \text{if } 2 \mid k \\ \frac{1}{(Q_k+1)}(s_{t+1}^{(k,r)} - 2s_{t-1}^{(k,r)} + s_{t-2}^{(k,r)}) & \text{if } 2 \nmid k. \end{cases}$$

3. Recurrence formula of $s_t^{(k,r)}$ by $s_{t-i}^{(k,r)}$

In this section we shall express $s_t^{(k,r)}$ as a linear combinations of $s_{t-i}^{(k,r)}$. We begin with an easy lemma about $s_t^{(2,r)}$ for next use.

Lemma 3.1. *(1)* $s_t^{(2,2)} - s_{t-2}^{(2,2)} = 2(P_{2t+1} + P_{2t})$ *and* $P_{2t-2} = 5P_{2t} - 2P_{2t+1}$.

(2) $s_t^{(2,1)} - s_{t-2}^{(2,1)} = 2(P_{2t} + P_{2t-1})$ *and* $P_{2t-3} = 5P_{2t-1} - 2P_{2t}$.

Proof. We shall only prove (1) that

$$\begin{aligned} s_t^{(2,2)} - s_{t-2}^{(2,2)} &= (s_{t-1}^{(2,2)} + P_{2t+2}) - (s_{t-1}^{(2,2)} - P_{2(t-1)+2}) \\ &= P_{2t+2} + P_{2t} = (2P_{2t+1} + P_{2t}) + P_{2t} = 2(P_{2t+1} + P_{2t}), \end{aligned}$$

and $P_{2t-2} = P_{2t} - 2P_{2t-1} = P_{2t} - 2(P_{2t+1} - 2P_{2t}) = 5P_{2t} - 2P_{2t+1}$. □

Theorem 3.2. $s_t^{(2,1)} = 6s_{t-1}^{(2,1)} - s_{t-2}^{(2,1)}$ *and* $s_t^{(2,2)} = 6s_{t-1}^{(2,2)} - s_{t-2}^{(2,2)} + 2$.

Proof. By observing the 2 spaced sum table in Theorem 2.2, we have $s_3^{(2,1)} = 6(35) - 6 = 6s_2^{(2,1)} - s_1^{(2,1)}$ and $s_4^{(2,1)} = 6(204) - 35 = 6s_3^{(2,1)} - s_2^{(2,1)}$.

Assume the induction hypothesis $s_t^{(2,1)} = 6s_{t-1}^{(2,1)} - s_{t-2}^{(2,1)}$ for $t > 0$. Then

$$\begin{aligned} 6s_t^{(2,1)} - s_{t-1}^{(2,1)} &= 6s_t^{(2,1)} - (6s_{t-2}^{(2,1)} - s_{t-3}^{(2,1)}) \\ &= 6s_t^{(2,1)} - 6s_{t-2}^{(2,1)} + (s_{t-2}^{(2,1)} - P_{2(t-2)+1}) \\ &= 6s_t^{(2,1)} - 5s_{t-2}^{(2,1)} - P_{2(t-2)+1} = s_t^{(2,1)} + 5(s_t^{(2,1)} - s_{t-2}^{(2,1)}) - P_{2t-3}. \end{aligned}$$

Thus due to Lemma 3.1, it follows that

$$\begin{aligned} 6s_t^{(2,1)} - s_{t-1}^{(2,1)} &= s_t^{(2,1)} + 5 \cdot 2(P_{2t} + P_{2t-1}) - (5P_{2t-1} - 2P_{2t}) \\ &= s_t^{(2,1)} + 12P_{2t} + 5P_{2t-1} = s_t^{(2,1)} + 6(2P_{2t} + P_{2t-1}) - P_{2t-1} \\ &= s_t^{(2,1)} + 6P_{2t+1} - P_{2(t-1)+1}. \end{aligned} \tag{3-1}$$

On the other hand, by Theorem 2.2, we also have

$$\begin{aligned} s_{t+1}^{(2,1)} &= 7P_{2t+1} + s_{t-2}^{(2,1)} = 7P_{2t+1} + (s_{t-1}^{(2,1)} - P_{2(t-1)+1}) \\ &= 7P_{2t+1} - P_{2(t-1)+1} + (s_t^{(2,1)} - P_{2t+1}) \\ &= s_t^{(2,1)} + 6P_{2t+1} - P_{2(t-1)+1}. \end{aligned} \tag{3-2}$$

Hence (3-1) and (3-2) proves $s_t^{(2,1)} = 6s_{t-1}^{(2,1)} - s_{t-2}^{(2,1)}$ for all $t > 0$.

Now for the second identity, we notice $s_3^{(2,2)} = 6(84) - 14 + 2 = 6s_2^{(2,2)} - s_1^{(2,2)} + 2$ and $s_4^{(2,2)} = 6(492) - 84 + 2 = 6s_3^{(2,2)} - s_2^{(2,2)} + 2$. If we assume $s_t^{(2,2)} = 6s_{t-1}^{(2,2)} - s_{t-2}^{(2,2)} + 2$ as induction hypothesis, then it follows that

$$\begin{aligned} 6s_t^{(2,2)} - s_{t-1}^{(2,2)} + 2 &= 6s_t^{(2,2)} - (6s_{t-2}^{(2,2)} - s_{t-3}^{(2,2)} + 2) + 2 \\ &= 6s_t^{(2,2)} - 6s_{t-2}^{(2,2)} + s_{t-3}^{(2,2)} = 6s_t^{(2,2)} - 6s_{t-2}^{(2,2)} + (s_{t-2}^{(2,2)} - P_{2(t-2)+2}) \\ &= 6s_t^{(2,2)} - 5s_{t-2}^{(2,2)} - P_{2t-2} = s_t^{(2,2)} + 5(s_t^{(2,2)} - s_{t-2}^{(2,2)}) - P_{2t-2} \\ &= s_t^{(2,2)} + 10(P_{2t+1} + P_{2t}) - (5P_{2t} - 2P_{2t+1}) = s_t^{(2,2)} + 12P_{2t+1} + 5P_{2t} \end{aligned}$$

$$= s_t^{(2,2)} + 6(2P_{2t+1} + P_{2t}) - P_{2t} = s_t^{(2,2)} + 6P_{2t+2} - P_{2t}. \qquad (3\text{-}3)$$

by Lemma 3.1. On the other hand, Theorem 2.2 shows that

$$s_{t+1}^{(2,2)} = 7P_{2t+2} + s_{t-2}^{(2,2)} = 7P_{2t+2} + (s_{t-1}^{(2,2)} - P_{2(t-1)+2})$$
$$= 7P_{2t+2} + (s_t^{(2,2)} - P_{2t+2}) - P_{2t} = s_t^{(2,2)} + 6P_{2t+2} - P_{2t}. \qquad (3\text{-}4)$$

Hence (3-3) and (3-4) yield $s_t^{(2,2)} = 6s_{t-1}^{(2,2)} - s_{t-2}^{(2,2)} + 2$ for all $t > 0$. □

We now generalize Theorem 3.2 to give recurrence rules in any rth column of table of $s_t^{(k,r)}$ $(1 \le r \le k)$.

Theorem 3.3. *The $s_t^{(k,r)}$ is a linear combination of $s_{t-1}^{(k,r)}$, $s_{t-2}^{(k,r)}$ and $s_{t-3}^{(k,r)}$ with coefficients $a_k = Q_k + 1$, $b_k = -(Q_k + (-1)^k)$ and $c_k = (-1)^k$. Moreover $a_k = 2a_{k-1} + a_{k-2} - 2$ and $b_k = -a_k$ if $2 \mid k$, otherwise $b_k = -a_k + 2$.*

Proof. When k is even, Theorem 2.5 shows

$$s_t^{(k,r)} = (Q_k + 1)P_{k(t-1)+r} + s_{t-3}^{(k,r)} = (Q_k + 1)(s_{t-1}^{(k,r)} - s_{t-2}^{(k,r)}) + s_{t-3}^{(k,r)},$$

so $s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + c_k s_{t-3}^{(k,r)}$ with $a_k = -b_k = Q_k + 1$ and $c_k = 1$. On the other hand if k is odd then

$$s_t^{(k,r)} = (Q_k + 1)P_{k(t-1)+r} + 2s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}$$
$$= (Q_k + 1)(s_{t-1}^{(k,r)} - s_{t-2}^{(k,r)}) + 2s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}$$
$$= (Q_k + 1)s_{t-1} - (Q_k - 1)s_{t-2} - s_{t-3}^{(k,r)}.$$

Furthermore, we also have

$$2a_{k-1} + a_{k-2} - 2 = 2(Q_{k-1} + 1) + (Q_{k-2} + 1) - 2 = Q_k + 1 = a_k.$$

□

In fact, $s_t^{(2,r)} = 7s_{t-1}^{(2,r)} - 7s_{t-2}^{(2,r)} + s_{t-3}^{(2,r)}$, $s_t^{(3,r)} = 15s_{t-1}^{(3,r)} - 13s_{t-2}^{(3,r)} - s_{t-3}^{(3,r)}$, $s_t^{(4,r)} = 35s_{t-1}^{(4,r)} - 35s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)}$, $s_t^{(5,r)} = 83s_{t-1}^{(5,r)} - 81s_{t-2}^{(5,r)} - s_{t-3}^{(5,r)}$, $s_t^{(6,r)} = 199s_{t-1}^{(6,r)} - 199s_{t-2}^{(6,r)} + s_{t-3}^{(6,r)}$ and, $s_t^{(7,r)} = 479s_{t-1}^{(7,r)} - 477s_{t-2}^{(7,r)} - s_{t-3}^{(7,r)}$ since $a_7 = 2(199) + 83 - 2 = 479$. Theorem 3.3 also yield an expression of P_n by k spaced three Pell numbers.

Theorem 3.4. *$P_n = a_k P_{n-k} + b_k P_{n-2k} + (-1)^k P_{n-3k}$ with a_k, b_k in Theorem 3.3.*

Proof. Write $n = kt + r$ $(1 \le r \le k,\ t \ge 0)$. Then Theorem 3.4 proves that

$$P_n = P_{kt+r} = s_t^{(k,r)} - s_{t-1}^{(k,r)}$$
$$= (a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + (-1)^k s_{t-3}^{(k,r)}) - (a_k s_{t-2}^{(k,r)} + b_k s_{t-3}^{(k,r)} + (-1)^k s_{t-4}^{(k,r)})$$
$$= a_k(s_{t-1}^{(k,r)} - s_{t-2}^{(k,r)}) + b_k(s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}) + (-1)^k(s_{t-3}^{(k,r)} - s_{t-4}^{(k,r)})$$

$= a_k P_{k(t-1)+r} + b_k P_{k(t-2)+r} + (-1)^k P_{k(t-3)+r}$,
this equals $a_k P_{n-k} + b_k P_{n-2k} + (-1)^k P_{n-3k}$. □

For instance, for the 26th Pell number, if we take $k = 8$ then
$P_{26} = a_8 P_{18} + b_8 P_{10} + P_2 = 1155(2744210) - 1155(2378) + 2 = 3166815962$,
since $a_8 = Q_8 + 1 = 1155$ and $b_8 = -(Q_8 + 1) = -1155$. Similarly by taking another $k = 5$ then P_{26} still can be obtained from
$P_{26} = a_5 P_{21} + b_5 P_{16} - P_{11} = 83(38613965) - 81(470832) - 5741$,
with $a_5 = Q_5 + 1 = 83$ and $b_5 = -Q_5 + 1 = -81$. Now for the interrelations of $s_t^{(k,r)}$ for all $1 \le r \le k$, we have the followings.

While Theorem 3.3 is about a recurrence in any rth column in table of $s_t^{(k,r)}$, the next theorem shows a recurrence in any tth row.

THEOREM 3.5. *$s_t^{(k,j-1)} + 2s_t^{(k,j)} = s_t^{(k,j+1)}$ for $1 < j < k$. And $s_t^{(k,k-1)} + 2s_t^{(k,k)} = s_{t+1}^{(k,1)} - 1$, also $s_t^{(k,k)} + 2s_{t+1}^{(k,1)} = s_{t+1}^{(k,2)}$.*

Proof. In particular if $k = 2$ then it is easy to see that

$$\begin{aligned} s_t^{(2,1)} - 2s_{t-1}^{(2,2)} &= (P_1 + P_3 + \cdots + P_{2t+1}) - 2(P_2 + P_4 + \cdots + P_{2(t-1)+2}) \\ &= P_1 + [(P_3 - 2P_2) + (P_5 - 2P_4) + \cdots + (P_{2t+1} - 2P_{2t})] \\ &= P_1 + (P_1 + P_3 + \cdots + P_{2t-1}) = 1 + s_{t-1}^{(2,1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} s_t^{(2,2)} - 2s_t^{(2,1)} &= (P_2 - 2P_1) + (P_4 - 2P_3) + \cdots + (P_{2t+2} - 2P_{2t+1}) \\ &= P_0 + P_2 + \cdots + P_{2t} = s_{t-1}^{(2,2)}. \end{aligned}$$

Now for any $1 < j < k$, we have

$$s_t^{(k,j-1)} + 2s_t^{(k,j)} = \sum_{i=0}^{t} (P_{ki+j-1} + 2P_{ki+j}) = \sum_{i=0}^{t} P_{ki+(j+1)} = s_t^{(k,j+1)}.$$

Moreover

$$s_t^{(k,k-1)} + 2s_t^{(k,k)} = \sum_{i=0}^{t} (P_{ki+k-1} + 2P_{ki+k}) = \sum_{j=0}^{t+1} P_{kj+1} - P_1 = s_{t+1}^{(k,1)} - 1.$$

Furthermore by using $P_1 = 1$ and $P_2 = 2$, we have

$$\begin{aligned} s_t^{(k,k)} + 2s_{t+1}^{(k,1)} &= \sum_{i=0}^{t} P_{(i+1)k} + 2(P_1 + \sum_{i=1}^{t+1} P_{ki+1}) \\ &= \sum_{i=0}^{t} P_{(i+1)k} + 2\sum_{i=0}^{t} P_{(i+1)k+1} + 2P_1 = \sum_{i=0}^{t} (P_{(i+1)k} + 2P_{(i+1)k+1}) + 2P_1 \\ &= \sum_{i=0}^{t} P_{(i+1)k+2} + 2P_1 = \sum_{j=0}^{t+1} P_{jk+2} - P_2 + 2P_1 = s_{t+1}^{(k,2)}. \end{aligned}$$

□

In fact $s_3^{(2,1)} = 204 = 2s_2^{(2,2)} + s_2^{(2,1)} + 1$ and $s_3^{(2,2)} = 492 = 2s_3^{(2,1)} + s_2^{(2,2)}$. In particular if $k = 1$ then $s_t^{(1,r)}$ is the sum from P_1 to P_t. We denote $s_n = s_n^{(1,1)} = \sum_{i=1}^{n} P_i$. Then $\{s_n\} = \{1, 3, 8, 20, 49, 119, 288, 696, \cdots\}$.

THEOREM 3.6. *For any $n > 0$, we have $s_n = \frac{1}{4}(P_{n+2} + P_n - 2)$.*

Proof. Clearly $4s_2 = 12 = P_4 + P_2 - 2$, $4s_3 = 32 = P_5 + P_3 - 2$ and $4s_4 = 80 = P_6 + P_4 - 2$. Hence if we assume $4s_n = P_{n+2} + P_n - 2$ for some n, then

$$\begin{aligned}4s_{n+1} &= 4(s_n + P_{n+1}) = 4s_n + 4P_{n+1} = (P_{n+2} + P_n - 2) + 4P_{n+1} \\ &= P_{n+2} + 2P_{n+1} + (2P_{n+1} + P_n) - 2 = P_{n+2} + 2P_{n+1} + P_{n+2} - 2 \\ &= (2P_{n+2} + P_{n+1}) + P_{n+1} - 2 = 2P_{n+3} + P_{n+1} - 2. \qquad \square\end{aligned}$$

For example, $4s_{10} = 4(P_1 + \cdots + P_{10}) = 4 \cdot 4059 = 16236 = P_{12} + P_{10} - 2$.

4. λ-Pell sequence and its k spaced sum

Let λ be a positive integer. A λ-Pell sequence $\{P_{\lambda,i}\}$ satisfies $P_{\lambda,i} = \lambda P_{\lambda,i-1} + P_{\lambda,i-2}$ with initials $P_{\lambda,0} = 0$, $P_{\lambda,1} = 1$, while the λ-Pell Lucas sequence $\{Q_{\lambda,i}\}$ holds $Q_{\lambda,i} = \lambda Q_{\lambda,i-1} + Q_{\lambda,i-2}$ with $Q_{\lambda,0} = 2$, $Q_{\lambda,1} = \lambda$.

Let $s_{\lambda,t}^{(k,r)} = \sum_{i=0}^{t} P_{\lambda,ki+r}$ be the sum of k spaced $(t+1)$ λ-Pell numbers starting from $P_{\lambda,r}$ $(1 \le r \le k)$. If $\lambda = 2$ then $\{P_{2,i}\} = \{P_i\}$ and $\{Q_{2,i}\} = \{Q_i\}$, so $s_{2,t}^{(k,r)}$ were already discussed. On the other hand if $\lambda = 1$ then $\{P_{1,i}\} = \{F_i\}$ and $\{Q_{1,i}\} = \{L_i\}$. Write $s_{1,t}^{(k,r)} = s_{F,t}^{(k,r)} = \sum_{i=0}^{t} F_{ki+r}$ the sum of k spaced $t+1$ Fibonacci numbers starting from F_r $(1 \le r \le k)$.

LEMMA 4.1. *The sum $s_{F,t}^{(k,r)}$ satisfies $s_{F,i}^{(2,1)} = F_{2(i+1)}$ and $\sum_{i=0}^{t} s_{F,i}^{(2,1)} = s_{F,t}^{(2,2)}$. Moreover $s_{F,t}^{(2,2)} - s_{F,t}^{(2,1)} = s_{F,t-1}^{(2,2)}$ and $s_{F,t}^{(2,2)} + s_{F,t}^{(2,1)} = s_{F,t+1}^{(2,1)} - 1$.*

Proof. The Fibonacci table with 2 columns and its 2 spaced sum table shows $s_{F,0}^{(2,1)} = F_2$, $s_{F,1}^{(2,1)} = F_4$, $s_{F,2}^{(2,1)} = F_6$ and $s_{F,3}^{(2,1)} = F_8$. Hence we have $s_{F,i}^{(2,1)} = F_{2(i+1)}$, so $\sum_{i=0}^{t} s_{F,i}^{(2,1)} = \sum_{i=0}^{t} F_{2i+2} = s_{F,t}^{(2,2)}$.

Moreover $s_{F,t}^{(2,2)} - s_{F,t}^{(2,1)} = \sum_{i=0}^{t} s_{F,i}^{(2,1)} - s_{F,t}^{(2,1)} = \sum_{i=0}^{t-1} s_{F,i}^{(2,1)} = s_{F,t-1}^{(2,2)}$, and

$$s_{F,t+1}^{(2,1)} - s_{F,t}^{(2,1)} = F_{2t+3} = \sum_{i=0}^{t} F_{2(i+1)} + 1 = \sum_{i=0}^{t} s_{F,i}^{(2,1)} + 1 = s_{F,t}^{(2,2)} + 1. \quad \square$$

It corresponds to the known identities $\sum_{i=1}^{t} F_{2i} = F_{2t+1} - 1$ and $\sum_{i=1}^{t} F_{2i-1} = F_{2t}$.

THEOREM 4.2. *Any rth column of table of $s_{F,t}^{(k,r)}$ satisfies the followings.*

(1) $s_{F,t}^{(2,r)} = 3s_{F,t-1}^{(2,r)} - s_{F,t-2}^{(2,r)} = 4s_{F,t-1}^{(2,r)} - 4s_{F,t-2}^{(2,r)} + s_{F,t-3}^{(2,r)}$.

(2) $s_{F,t}^{(3,r)} = 5s_{F,t-1}^{(3,r)} - 3s_{F,t-2}^{(3,r)} - s_{F,t-3}^{(3,r)}$ *and* $s_{F,t}^{(4,r)} = 8s_{F,t-1}^{(4,r)} - 8s_{F,t-2}^{(4,r)} + s_{F,t-3}^{(4,r)}$.

(3) $s_{F,t}^{(k,r)} = a_k s_{F,t-1}^{(k,r)} + b_k s_{F,t-2}^{(k,r)} + (-1)^k s_{F,t-3}^{(k,r)}$ *and* $F_n = a_k F_{n-k} + b_k F_{n-2k} + (-1)^k F_{n-3k}$, *where* $a_k = L_k + 1$ *and* $b_k = -(L_k + (-1)^k)$.

Proof. Clearly $s_{F,t+1}^{(k,r)} = s_{F,t}^{(k,r)} + F_{k(t+1)+r}$. Observe $21 = 3(8) - 3$ and $33 = 3(12) - 4$ from the 2 spaced sum table of Fibonacci numbers. If we assume $s_{F,t}^{(2,r)} = 3s_{F,t-1}^{(2,r)} - s_{F,t-2}^{(2,r)}$ for some t then

$$\begin{aligned} s_{F,t+1}^{(2,r)} &= s_{F,t}^{(2,r)} + F_{2(t+1)+r} = (3s_{F,t-1}^{(2,r)} - s_{F,t-2}^{(2,r)}) + (3F_{2t+r} - F_{2(t-1)+r}) \\ &= 3(s_{F,t-1}^{(2,r)} + F_{2t+r}) - (s_{F,t-2}^{(2,r)} + F_{2(t-1)+r}) = 3s_{F,t}^{(2,r)} - s_{F,t-1}^{(2,r)}, \end{aligned}$$

since $F_n = 3F_{n-2} - F_{n-4}$. And we also have

$$\begin{aligned} 4s_{F,t-1}^{(2,r)} - 4s_{F,t-2}^{(2,r)} + s_{F,t-3}^{(2,r)} &= (s_{F,t-1}^{(2,r)} + 3s_{F,t-1}^{(2,r)}) - (s_{F,t-2}^{(2,r)} + 3s_{F,t-2}^{(2,r)}) + s_{F,t-3}^{(2,r)} \\ &= s_{F,t-1}^{(2,r)} + s_{F,t}^{(2,r)} - s_{F,t-1}^{(2,r)} = s_{F,t}^{(2,r)}. \end{aligned}$$

Similarly from the 3 spaced sum table, we notice $72 = 5(17) - 3(4) - 1$, $116 = 5(27) - 3(6) - 1$ and $188 = 5(44) - 3(10) - 2$. Assume that $s_{F,t}^{(3,r)} = 5s_{F,t-1}^{(3,r)} - 3s_{F,t-2}^{(3,r)} - s_{F,t-3}^{(3,r)}$ for some t. Then

$$\begin{aligned} 5F_n - 3F_{n-3} - F_{n-6} &= 5F_n - 2F_{n-3} - (F_{n-3} + F_{n-6}) \\ &= 5F_n - 2F_{n-3} - 2F_{n-4} = 5F_n - 2F_{n-2} = F_n + (4F_n - 2F_{n-2}) \\ &= F_n + 2F_{n+1} = F_{n+3}, \end{aligned}$$

due to $F_n + F_{n-3} = 2F_{n-1}$. Hence the induction hypothesis shows

$$\begin{aligned} s_{F,t+1}^{(3,r)} &= s_{F,t}^{(3,r)} + F_{3(t+1)+r} \\ &= (5s_{F,t-1}^{(3,r)} - 3s_{F,t-2}^{(3,r)} - s_{F,t-3}^{(3,r)}) + (5F_{3t+r} - 3F_{3(t-1)+r} - F_{3(t-2)+r}) \\ &= 5(s_{F,t-1}^{(3,r)} + F_{3t+r}) - 3(s_{F,t-2}^{(3,r)} + F_{3(t-1)+r}) - (s_{F,t-3}^{(3,r)} + F_{3(t-2)+r}) \\ &= 5s_{F,t}^{(3,r)} - 3s_{F,t-1}^{(3,r)} - s_{F,t-2}^{(3,r)}. \end{aligned}$$

t	$s_{F,t}^{(3,1)}$	$s_{F,t}^{(3,2)}$	$s_{F,t}^{(3,3)}$	$s_{F,t}^{(4,1)}$	$s_{F,t}^{(4,2)}$	$s_{F,t}^{(4,3)}$	$s_{F,t}^{(4,4)}$
0	1	1	2	1	1	2	3
1	4	6	10	6	9	15	24
2	17	27	44	40	64	104	168
3	72	116	188	273	441	714	1155

Also the 4 spaced sum table shows that $273 = 8(40) - 8(6) + 1$, $441 = 8(64) - 8(9) + 1$, $714 = 8(104) - 8(15) + 2$ and $1155 = 8(168) - 8(24) + 3$, so we may assume $s_{F,t}^{(4,r)} = 8s_{F,t-1}^{(4,r)} - 8s_{F,t-2}^{(4,r)} - s_{F,t-3}^{(4,r)}$ for some t. By using the identities $F_n - F_{n-4} = 3F_{n-1} - F_n$ and $2F_n - F_{n-2} = F_{n+1}$, we have

$$\begin{aligned}
8F_n - 8F_{n-4} + F_{n-8} &= 8F_n - 7F_{n-4} - (F_{n-4} - F_{n-8}) \\
&= 8F_n - 7F_{n-4} - (3F_{n-5} - F_{n-4}) = 8F_n - 6F_{n-4} - 3F_{n-5} \\
&= 8F_n - 3F_{n-4} - 3(F_{n-4} + F_{n-5}) = 8F_n - 3F_{n-4} - 3F_{n-3} \\
&= 8F_n - 3F_{n-2} = 2F_n + 3(2F_n - F_{n-2}) = 2F_n + 3F_{n+1} \\
&= 2F_{n+2} + F_{n+1} = F_{n+4}.
\end{aligned}$$

Therefore the induction hypothesis gives rise to

$$\begin{aligned}
s_{F,t+1}^{(4,r)} &= (8s_{F,t-1}^{(4,r)} - 8s_{F,t-2}^{(4,r)} + s_{F,t-3}^{(4,r)}) + (8F_{4t+r} - 8F_{4(t-1)+r} + F_{4(t-2)+r}) \\
&= 8(s_{F,t-1}^{(4,r)} + F_{4t+r}) - 8(s_{F,t-2}^{(4,r)} + F_{4(t-1)+r}) + (s_{F,t-3}^{(4,r)} + F_{4(t-2)+r}) \\
&= 8s_{F,t}^{(4,r)} - 8s_{F,t-1}^{(4,r)} + s_{F,t-2}^{(4,r)}.
\end{aligned}$$

Now for any k, (3) follows analogously from Theorem 3.3. □

For $1 \le n \le 5$, the table of $P_{\lambda,n}$ and $Q_{\lambda,n}$ with $2 \le \lambda \le 4$ are

	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
$P_{\lambda,n}$	$1, 2, 5, 12, 29, \cdots$	$1, 3, 10, 33, 109, \cdots$	$1, 4, 17, 72, 305, \cdots$
$Q_{\lambda,n}$	$2, 6, 14, 34, 82, \cdots$	$3, 11, 36, 119, 393, \cdots$	$4, 18, 76, 322, 1364, \cdots$

,

and with any λ,

$P_{\lambda,n}$	1	λ	$\lambda^2 + 1$	$\lambda^3 + 2\lambda$	$\lambda^4 + 3\lambda^2 + 1$	$\lambda^5 + 4\lambda^3 + 3\lambda$	...
$Q_{\lambda,n}$	λ	$\lambda^2 + 2$	$\lambda^3 + 3\lambda$	$\lambda^4 + 4\lambda^2 + 2$	$\lambda^5 + 5\lambda^3 + 5\lambda$	...	

LEMMA 4.3. *$Q_{\lambda,n} = P_{\lambda,n-1} + P_{\lambda,n+1}$ and $P_{\lambda,n+k} = Q_{\lambda,k}P_{\lambda,n} + (-1)^{k-1}P_{\lambda,n-k}$, where $\{Q_{\lambda,n}\}$ is the λ-Pell Lucas sequence.*

Proof. From the above tables, $Q_{\lambda,i} = P_{\lambda,i-1} + P_{\lambda,i+1}$ is true for $1 \le i \le 5$. If we assume the identity for $1 \le i \le n$ as an induction hypothesis then

$$\begin{aligned}
Q_{\lambda,n+1} &= \lambda Q_{\lambda,n} + Q_{\lambda,n-1} = \lambda(P_{\lambda,n-1} + P_{\lambda,n+1}) + (P_{\lambda,n-2} + P_{\lambda,n}) \\
&= (\lambda P_{\lambda,n-1} + P_{\lambda,n-2}) + (\lambda P_{\lambda,n+1} + P_{\lambda,n}) = P_{\lambda,n} + P_{\lambda,n+2}.
\end{aligned}$$

If $\lambda = 1, 2$ then $P_{\lambda,n}$ equals F_i or P_i, so we have the well known identities $F_{n+k} = L_kF_n + (-1)^{k-1}F_{n-k}$ and $P_{n+k} = Q_kP_n + (-1)^{k-1}P_{n-k}$. Now for $\lambda > 0$, if $k = 1$ then $Q_{\lambda,1}P_{\lambda,n} + P_{\lambda,n-1} = \lambda P_{\lambda,n} + P_{\lambda,n-1} = P_{\lambda,n+1}$. For $1 \le i \le k$, we assume $P_{\lambda,n+i} = Q_{\lambda,i}P_{\lambda,n} + (-1)^{i-1}P_{\lambda,n-i}$. Then

$$P_{\lambda,n+k+1} = \lambda P_{\lambda,n+k} + P_{\lambda,n+k-1}$$

$$= \lambda(Q_{\lambda,k}P_{\lambda,n} + (-1)^{k-1}P_{\lambda,n-k}) + (Q_{\lambda,k-1}P_{\lambda,n} + (-1)^k P_{\lambda,n-(k-1)})$$
$$= (\lambda Q_{\lambda,k} + Q_{\lambda,k-1})P_{\lambda,n} + (-1)^k(P_{\lambda,n-(k-1)} - \lambda P_{\lambda,n-k})$$
$$= Q_{\lambda,k+1}P_{\lambda,n} + (-1)^k P_{\lambda,n-k-1}. \qquad \square$$

THEOREM 4.4. *Let* $a_{(\lambda,k)} = Q_{\lambda,k} + 1$ *and* $b_{(\lambda,k)} = -Q_{\lambda,k} + (-1)^{k-1}$. *Then*

(1) $a_{(\lambda,1)} = \lambda+1$, $a_{(\lambda,2)} = \lambda^2+3$, *and* $a_{(\lambda,k)} = \lambda a_{(\lambda,k-1)} + a_{(\lambda,k-2)} - \lambda$.
(2) $b_{(\lambda,k)} = -a_{(\lambda,k)}$ *if* $2 \mid k$, *otherwise* $b_{(\lambda,k)} = -a_{(\lambda,k)} + 2$.
(3) Moreover $P_{\lambda,n+k} = a_{(\lambda,k)}P_{\lambda,n} + b_{(\lambda,k)}P_{\lambda,n-k} + (-1)^k P_{\lambda,n-2k}$.

Proof. It is easy to see that

$$a_{(\lambda,k)} = Q_{\lambda,k} + 1 = \lambda Q_{\lambda,k-1} + Q_{\lambda,k-2} + 1$$
$$= \lambda(Q_{\lambda,k-1} + 1) + (Q_{\lambda,k-2} + 1) - \lambda = \lambda a_{(\lambda,k-1)} + a_{(\lambda,k-2)} - \lambda.$$

Write $n = kt + r$ $(1 \le r \le k,\ t \ge 0)$. If $\lambda = 1, 2$ then Theorem 3.3 shows

$$P_{\lambda,n} = P_{\lambda,kt+r} = s^{(k,r)}_{\lambda,t} - s^{(k,r)}_{\lambda,t-1}$$
$$= (a_{(\lambda,k)}s^{(k,r)}_{\lambda,t-1} + b_{(\lambda,k)}s^{(k,r)}_{\lambda,t-2} + (-1)^k s^{(k,r)}_{\lambda,t-3})$$
$$- (a_{(\lambda,k)}s^{(k,r)}_{\lambda,t-2} + b_{(\lambda,k)}s^{(k,r)}_{\lambda,t-3} + (-1)^k s^{(k,r)}_{\lambda,t-4})$$
$$= a_{(\lambda,k)}(s^{(k,r)}_{\lambda,t-1} - s^{(k,r)}_{\lambda,t-2}) + b_{(\lambda,k)}(s^{(k,r)}_{\lambda,t-2} - s^{(k,r)}_{\lambda,t-3}) + (-1)^k(s^{(k,r)}_{\lambda,t-3} - s^{(k,r)}_{\lambda,t-4})$$
$$= a_{(\lambda,k)}P_{\lambda,k(t-1)+r} + b_{(\lambda,k)}P_{\lambda,k(t-2)+r} + (-1)^k P_{\lambda,k(t-3)+r}$$
$$= a_{(\lambda,k)}P_{\lambda,n-k} + b_{(\lambda,k)}P_{\lambda,n-2k} + (-1)^k P_{\lambda,n-3k}.$$

Consider any $\lambda > 0$. If $k = 1$ then $a_{(\lambda,1)} = Q_{\lambda,1} + 1 = \lambda + 1$ so

$$a_{(\lambda,1)}P_{\lambda,n} + b_{(\lambda,1)}P_{\lambda,n-1} - P_{\lambda,n-2} = (\lambda+1)P_{\lambda,n} + (-\lambda+1)P_{\lambda,n-1} - P_{\lambda,n-2}$$
$$= \lambda P_{\lambda,n} + P_{\lambda,n-1} + (P_{\lambda,n} - \lambda P_{\lambda,n-1} - P_{\lambda,n-2}) = \lambda P_{\lambda,n} + P_{\lambda,n-1}$$
$$= P_{\lambda,n+1}.$$

Now we assume $P_{\lambda,n+k} = a_{(\lambda,k)}P_{\lambda,n} + b_{(\lambda,k)}P_{\lambda,n-k} + (-1)^k P_{\lambda,n-2k}$ for some k as an induction hypothesis. Then owing to Lemma 4.3, we have

$$a_{(\lambda,k+1)}P_{\lambda,n} + b_{(\lambda,k+1)}P_{\lambda,n-(k+1)} + (-1)^{k+1}P_{\lambda,n-2(k+1)}$$
$$= (Q_{\lambda,k+1}+1)P_{\lambda,n} + (-Q_{\lambda,k+1} + (-1)^k)P_{\lambda,n-(k+1)} + (-1)^{k+1}P_{\lambda,n-2(k+1)}$$
$$= (Q_{\lambda,k+1} + 1)P_{\lambda,n} - (Q_{\lambda,k+1}P_{\lambda,n-(k+1)} + (-1)^k P_{\lambda,n-2(k+1)})$$
$$+ (-1)^k P_{\lambda,n-(k+1)}$$
$$= (Q_{\lambda,k+1} + 1)P_{\lambda,n} - P_{\lambda,n} + (-1)^k P_{\lambda,n-(k+1)}$$
$$= Q_{\lambda,k+1}P_{\lambda,n} + (-1)^k P_{\lambda,n-(k+1)} = P_{\lambda,n+(k+1)}. \qquad \square$$

For example, when $\lambda = 3$, the 3-P-tables with 3 or 4 columns

3 columns			4 columns			
1	3	10	1	3	10	33
33	109	360	109	360	1189	3927
1189	3927	12970	12970	42837	141481	497280
42837	141481	497280	1543321	5097243	16835050	55602393

show $42837 = 36(1189) + 33 = 37(1189) - 35(33) - 1$ which implies $P_{3,n} = 36P_{3,n-3} + P_{3,n-6} = 37P_{3,n-3} - 35P_{3,n-6} - P_{3,n-9}$. At the same time, $1543321 = 120(12970) - 120(109) + 1$ and $P_{3,n} = 120P_{3,n-4} - 120P_{3,n-8} + P_{3,n-12}$.

Similarly when $\lambda = 4$, the 4-P-tables with 3 or 4 columns are

1	4	17	1	4	17	72
72	305	1292	305	1292	5473	23184
5473	23184	98209	98209	416020	1762289	7465176
416020	1762289	7465176	31622993	133957148	567451585	.

Clearly $416020 = 77(5473) - 75(72) - 1$ and $31622993 = 323(98209) - 323(305) + 1$ with $77 = Q_{4,3} + 1$, $323 = Q_{4,4} + 1$. A generalization is as follows.

THEOREM 4.5. *$s^{(k,r)}_{\lambda,t} = a_{(\lambda,k)} s^{(k,r)}_{\lambda,t-1} + b_{(\lambda,k)} s^{(k,r)}_{\lambda,t-2} + (-1)^k s^{(k,r)}_{\lambda,t-3}$ for any k, t.*

Proof. When $\lambda = 1$, it is due to Lemma 4.1, and certainly $a_{(1,k)} = L_k + 1$ satisfies $a_{(1,k)} = a_{(1,k-1)} + a_{(1,k-2)} - 1$. Also if $\lambda = 2$ it is due to Theorem 3.2.

When $\lambda = 3$, the $3,4$ spaced sum tables of 3-Pell numbers are

$s^{(3,1)}_{3,t}$	$s^{(3,2)}_{3,t}$	$s^{(3,3)}_{3,t}$	$s^{(4,1)}_{3,t}$	$s^{(4,2)}_{3,t}$	$s^{(4,3)}_{3,t}$	$s^{(4,4)}_{3,t}$
1	3	10	1	3	10	33
34	112	370	110	363	1199	3960
1223	4039	13340	13080	43200	142680	471240
44060	145520	.	1556401	5140443	.	.

It shows $44060 = 37(1223) - 35(34) - 1$ and $145520 = 37(4039) - 35(112) - 3$. By assuming induction hypothesis, we have $s^{(3,r)}_{3,t} = 37s^{(3,r)}_{3,t-1} - 35s^{(3,r)}_{3,t-2} - s^{(3,r)}_{3,t-3}$ and $s^{(4,r)}_{3,t} = 120s^{(4,r)}_{3,t-1} - 120s^{(4,r)}_{3,t-2} + s^{(4,r)}_{3,t-3}$ with $37 = Q_{3,3} + 1$ and $120 = Q_{3,4} + 1$.

Now if $\lambda = 4$, the $3,4$ spaced sum table of 4-Pell numbers are

$s^{(3,1)}_{4,t}$	$s^{(3,2)}_{4,t}$	$s^{(3,3)}_{4,t}$	$s^{(4,1)}_{4,t}$	$s^{(4,2)}_{4,t}$	$s^{(4,3)}_{4,t}$	$s^{(4,4)}_{4,t}$
1	4	17	1	4	17	72
73	309	1309	306	1296	5490	23256
5546	23493	99518	98515	417316	1767779	7488432
421566	1785782	.	31721508	134374464	.	.

We notice $421566 = 77(5546) - 75(73) - 1$ where $77 = Q_{4,3} + 1$, and $31721508 = 323(98515) - 323(306) + 1$ where $323 = Q_{4,4} + 1$. Thus by induction we can prove $s^{(k,r)}_{4,t} = (Q_{4,k}+1)s^{(k,r)}_{4,t-1} + (-Q_{4,k} + (-1)^{k-1})s^{(k,r)}_{4,t-2} + (-1)^k s^{(k,r)}_{4,t-3}$.

Now let $\lambda > 0$, and assume $s^{(k,r)}_{\lambda,t} = a_{(\lambda,k)} s^{(k,r)}_{\lambda,t-1} + b_{(\lambda,k)} s^{(k,r)}_{\lambda,t-2} + (-1)^k s^{(k,r)}_{\lambda,t-3}$ holds for some k. Write $kt + r = n$. Then

$$\begin{aligned}
s^{(k,r)}_{\lambda,t+1} &= s^{(k,r)}_{\lambda,t} + P_{\lambda,n} \\
&= (a_{(\lambda,k)} s^{(k,r)}_{\lambda,t-1} + b_{(\lambda,k)} s^{(k,r)}_{\lambda,t-2} + (-1)^k s^{(k,r)}_{\lambda,t-3}) \\
&\quad + (a_{(\lambda,k)} P_{\lambda,n-k} + b_{(\lambda,k)} P_{\lambda,n-2k} + (-1)^k P_{\lambda,n-3k}) \\
&= a_{(\lambda,k)}(s^{(k,r)}_{\lambda,t-1} + P_{\lambda,n-k}) + b_{(\lambda,k)}(s^{(k,r)}_{\lambda,t-2} + P_{\lambda,n-2k}) \\
&\quad + (-1)^k (s^{(k,r)}_{\lambda,t-3} + P_{\lambda,n-3k}) \\
&= a_{(\lambda,k)} s^{(k,r)}_{\lambda,t} + b_{(\lambda,k)} s^{(k,r)}_{\lambda,t-1} + (-1)^k s^{(k,r)}_{\lambda,t-2}.
\end{aligned}$$

□

References

[1] B. Bradie, *Extensions and refinements of some properties of sums involving Pell numbers*, Missouri J. Math. Sci. **22** (2010), 37-43.

[2] Z. Cerin and G.M. Gianella, *On sums of squares of Pell-Lucas numbers*, Integers: Electronic J. Combinatorial Number theory **6** (2006), no. A15, 16 pages.

[3] E. Choi and J. Jo, *On partial sum of tribonacci numbers*, Int. J. of Math. and Mathematical Science, **2015** (2015), ID 301814, 8 pages.

[4] A. F. Horadam, *Pell identities*, Fibo. Quart. **9** (1971), 245-252.

[5] A. F. Horadam, *Jacobsthal representation numbers*, Fibo. Quart. **34** (1996), 40-54.

[6] E. Kilic, *Tribonacci sequences with certain indices and their sums*, Ars. Comb. **86** (2008), 13-22.

[7] R. Melham, *Sums involving Fibonacci and Pell numbers*, Portugaliae Mathematica, **56** (1999), 309-317.

[8] S. F. Santana and J. L. Daz-Barrero, *Some properties of sums involving Pell numbers*, Missouri J. Math. Sci. **18** (2006), 33-40.

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