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REGULAR GRAPHS AND DISCRETE SUBGROUPS OF PROJECTIVE LINEAR GROUPS

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Abstract. The homothety classes of lattices in a two dimensional vector space over a nonarchimedean local field form a regular tree $\mathcal{T}$ of degree $q+1$ on which the projective linear group acts naturally where q is the order of the residue field. We show that for any finite regular combinatorial graph of even degree $q+1$, there exists a torsion free discrete subgroup Γ of the projective linear group such that $\mathcal{T}/\Gamma$ is isomorphic to the graph.

1. Introduction

Discrete subgroups of semisimple (real, p-adic, or adelic) Lie groups play important roles in geometry and number theory and have been studied intensively. (See [1] and references there.) Such a Lie group G often acts transitively on a space and we may identify the space with $K\backslash G$ where K is the stabilizer of a point. For example, we have

$$\mathcal{H}^+ \cong \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$$

where $\mathcal{H}^+$ is the complex upper half plane on which $\mathrm{SL}_2(\mathbb{R})$ acts by linear fractional transformations and $\mathrm{SO}(2)$ is the stabilizer of i under this action. For a discrete subgroup Γ, like $\mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{R})$, the quotient space $\mathcal{H}^+/\Gamma$ (or its compactification) and functions, differential forms, etc. on it are important objects of number theory [5].

We have a p-adic analogue of the above example. Let $\mathcal{T}$ be the tree of (homothety classes of) lattices in $\mathbb{Q}_p^2$. It is a regular tree of degree $p+1$. (See §4 for details.) Then $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ and $\mathrm{PGL}_2(\mathbb{Z}_p) = \mathrm{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p^*$ is the stabilizer of the vertex corresponding

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to the standard lattice $\mathbb{Z}_p^2$. Thus we have

$$\mathcal{T} \cong \mathrm{PGL}_2(\mathbb{Z}_p)\backslash \mathrm{PGL}_2(\mathbb{Q}_p)$$

For a torsion free discrete subgroup Γ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ with compact quotient, the quotient space $\mathcal{T}/\Gamma$ is a finite graph. In [2], an explicit algorithm to construct all such subgroups is given. We reproduce it in §5 for the convenience of readers. An initial input data of this algorithm is an integral symmetric matrix satisfying certain conditions: A-1, A-2, A-3 in §5.

Let $\mathcal{G}$ be a finite connected graph. Then the universal covering $\mathcal{U}$ of $\mathcal{G}$ (or more precisely, of a topological realization of $\mathcal{G}$) is a tree, on which the fundamental group $\pi_1(\mathcal{G})$ acts and we have $\mathcal{G} \cong \mathcal{U}/\pi_1(\mathcal{G})$. If $\mathcal{G}$ is a regular graph of degree $p+1$, then $\mathcal{U}$ is isomorphic to $\mathcal{T}$ above. And we may ask if we can realize the action of $\pi_1(\mathcal{G})$ on $\mathcal{U}$ as that of (a subgroup of) $\mathrm{PGL}_2(\mathbb{Q}_p)$ on $\mathcal{T}$.

In this paper, we show that for any regular finite connected combinatorial graph $\mathcal{G}$ of degree $p+1$ with p an odd prime, there exists a torsion free discrete subgroup Γ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ such that $\mathcal{G} \cong \mathcal{T}/\Gamma$ (Theorem 6.1). A point of this proof is that if we order the vertices suitably (applying the breadth-first search algorithm, to be more precise) then the adjacency matrix with respect to this ordering of vertices satisfies the above mentioned conditions and the algorithm of [2] can be applied. We also give an example of Γ such that $\mathcal{G} \cong \mathcal{T}/\Gamma$ when $\mathcal{G}$ is a complete regular graph in §7.

In §2, §3 and §4, we review quickly necessary terms on graph theory and facts on groups acting on trees as well as on trees of (homothety classes of) lattices. Our reference for these is [4].

Notations: In this paper, the identity element of a group will be denoted by 1. For a subset S of a group, we let $S^{-1} = \{s^{-1} \mid s \in S\}$.

2. Graphs

We recall basic definitions to fix notations. We refer [4] for more detail as well as for notions unexplained below. A *graph* $\mathcal{G}$ consists of a set $V = V(\mathcal{G})$ of vertices and a set $E = E(\mathcal{G})$ of edges equipped with two maps $E \to V \times V, e \mapsto (o(e), t(e))$ and $E \to E, e \mapsto \bar{e}$ such that $o(\bar{e}) = t(e), t(\bar{e}) = o(e), \bar{\bar{e}} = e$ and $\bar{e} \neq e$ for each $e \in E$. For an edge $e \in E$, we call $o(e), t(e)$ and $\bar{e}$ the *origin*, the *terminus* and the *inverse* of e, respectively.

An *orientation* of a graph $\mathcal{G}$ is a subset $E_+ \subset E = E(\mathcal{G})$ such that E is the disjoint union of E_+ and $\overline{E}_+$ where $\overline{E}_+$ is the set of $\bar{e}$ with $e \in E_+$. A morphism between graphs and an automorphism of a graph are defined in an obvious way.

Remark 2.1. The above definition of graphs allows loops (i.e. an edge e with $o(e) = t(e)$) and multiple edges (i.e. edges $e \neq e'$ with $o(e) = o(e'), t(e) = t(e')$. A graph without loops nor multiple edges (or equivalently, without a *circuit* of length ≤ 2) is called *combinatorial.*

Example 2.2. Let Γ be a group and $S \subset \Gamma$ be a set of generators for G. The (oriented) graph $\mathcal{G} = \mathcal{G}(\Gamma, S)$ is defined by $V(\mathcal{G}) = \Gamma$ and $E(\mathcal{G})_+ = \Gamma \times S$ with $o(e) = g$ and $t(e) = gs$ for an edge $e = (g, s)$. It is not difficult to see the following for $\mathcal{G} = \mathcal{G}(\Gamma, S)$.

1. It is a combinatorial graph if and only if $S \cap S^{-1} = \emptyset$.
2. It is a regular graph of degree $2|S|$.
3. It is a tree if and only if Γ is a free group with basis S.
4. The multiplication (from the left) by elements of Γ gives orientation preserving automorphisms of $\mathcal{G}$.

3. Groups acting on trees

A *tree* is a connected graph without circuits. In particular, a tree is a combinatorial graph. A group action on a graph is defined in an obvious way: a group acts on the sets of vertices and edges preserving the incidence relation. We say a group Γ acts on a graph $\mathcal{G}$ *freely* if the action on $V(\mathcal{G})$ is free and the action on $E(\mathcal{G})$ is without inversion (i.e. for all $g \neq 1 \in \Gamma$ and $v \in V, e \in E$ we have $gv \neq v$ and $ge \neq \bar{e}$). For example, the action of Γ on the graph $\mathcal{G}(\Gamma, S)$ in Example 2.2 (4) is free.

Suppose a group Γ acts on a tree $\mathcal{T}$ freely. Let $\mathcal{T}'$ be a maximal subtree of the quotient graph $\mathcal{G} = \mathcal{T}/\Gamma$, i.e. a subgraph of $\mathcal{G}$ which is a tree with $V(\mathcal{T}') = V(\mathcal{G})$. Let $\mathcal{T}''$ be a lift of $\mathcal{T}'$ as a subtree of $\mathcal{T}$, which always exists by [4, §3.1, Proposition 14]. Choose an orientation E_+ of $\mathcal{T}$. Let S be the set of $g \neq 1 \in G$ such that there exists an edge in E_+ from x to gy with $x, y \in V(\mathcal{T}'')$.

Theorem 3.1 (Theorem 4' in §3.3, [4]). *A group Γ acting freely on a tree $\mathcal{T}$ is a free group. More precisely, the set S obtained as above is a basis for Γ.*

4. Tree of lattices

Let $\mathfrak{K}$ be a local field (e.g. a finite extension of $\mathbb{Q}_p$) and let $\mathfrak{O} = \{x \in \mathfrak{K} \mid |x| \leq 1\}$ where $|\ |$ is the nonarchimedean norm on $\mathfrak{K}$. Also let ω be a prime element and let $\mathfrak{p} = \omega\mathfrak{O}$, $\mathfrak{k} = \mathfrak{O}/\mathfrak{p}$, $q = |\mathfrak{O}/\mathfrak{p}|$ be the maximal ideal, the residue field of $\mathfrak{O}$ and its order, respectively.

A *lattice* in the vector space $\mathfrak{K}^2$ is a sub $\mathfrak{O}$-module generated by a basis for $\mathfrak{K}^2$. Two lattices L, L' in $\mathfrak{K}^2$ are said to be equivalent if there exists $\lambda \in \mathfrak{K}^*$ such that $L' = \lambda L$.

DEFINITION 4.1. The graph $\mathcal{T} = \mathcal{T}(\mathfrak{K}^2)$ is defined as follows:

- Its vertices are the equivalence classes of lattices in $\mathfrak{K}^2$.
- Two vertices v, v' are adjacent if there exist lattices L, L' with $v = [L], v' = [L']$ such that $L \subset L'$ and $|L'/L| = q$.

PROPOSITION 4.2 ([4]). *The graph $\mathcal{T}$ is a regular tree of degree $q+1$ on which the group $G = \mathrm{PGL}_2(\mathfrak{K})$ acts in an obvious way.*

We suppose it acts from the right viewing elements of $\mathfrak{K}^2$ as row vectors. This action is transitive on vertices. Since the stabilizer of the vertex $\mathfrak{o} = [\mathfrak{O}^2]$ is $K = \mathrm{GL}_2(\mathfrak{O})/\mathfrak{O}^*$, we have $V(\mathcal{T}) \cong K\backslash G$ as G-sets.

REMARK 4.3. The above action of $\mathrm{PGL}_2(\mathfrak{K})$ has inversion. For example, if e is the edge from $[\mathfrak{O} \oplus \mathfrak{O}]$ to $[\mathfrak{O} \oplus \omega\mathfrak{O}]$, then $\left(\begin{smallmatrix} 0 & 1 \\ \omega & 0 \end{smallmatrix}\right)$ maps e to its inverse. On the other hand, the action of $\mathrm{PSL}_2(\mathfrak{K})$ on $\mathcal{T}$ has no inversion. But its action on the set of vertices is not transitive: there are two orbits.

5. Discrete subgroups of projective linear groups

In this section, we review part of [2] on construction of torsion free discrete subgroups of $G = \mathrm{PGL}_2(\mathfrak{K})$. We keep the notations of the last section. We suppose p, the residual characteristic of $\mathfrak{K}$ is odd.

THEOREM 5.1 (Theorem 1 in [2]). *A torsion free discrete subgroup of $\mathrm{PGL}_2(\mathfrak{K})$ is free.*

Proof. Suppose Γ is a torsion free discrete subgroup of G. By Theorem 3.1, it is enough to show that Γ acts on $\mathcal{T}$ freely. For any $x \in G$, $\Gamma \cap x^{-1}Kx$ is finite since it is discrete and compact. Since Γ is torsion free, this implies $\Gamma \cap x^{-1}Kx = \{1\}$. Hence the action of Γ on the vertex set of $\mathcal{T}$ is free since $x^{-1}Kx$ is the stabilizer of the vertex $\mathfrak{o}x$. If the action of $\gamma \in \Gamma$ is an inversion. Then γ^2 fixes a vertex. Hence $\gamma^2 = 1$, which is a contradiction. $\square$

The original proof of the above theorem gives an algorithm to find a basis for a given Γ. By reversing this procedure carefully, we can construct all torsion free discrete subgroups of G. An explicit algorithm is given in [2, §4], which we reproduce below.

First, a preliminary set up: Choose $\pi_0, \pi_1, \cdots, \pi_q \in G$ such that $\mathfrak{o}\pi_0, \cdots, \mathfrak{o}\pi_q$ are the vertices adjacent to $\mathfrak{o} = [\mathfrak{O} \oplus \mathfrak{O}]$ in $\mathcal{T}$. For each $0 \leq i \leq q$, $\mathfrak{o}\pi_0\pi_i, \mathfrak{o}\pi_1\pi_i, \cdots, \mathfrak{o}\pi_q\pi_i$ are the vertices adjacent to $\mathfrak{o}\pi_i$. Hence there exists a unique $\varphi(i)$ such that $\mathfrak{o}\pi_{\varphi(i)}\pi_i = \mathfrak{o}$, i.e. $\pi_{\varphi(i)}\pi_i \in K$.

Second, the input data of the construction: Let $A = (a_{ij})$ be a symmetric $h \times h$ matrix satisfying the following conditions. For each $i > 1$, let $\rho(i)$ be such that $a_{i\rho(i)}$ is the first non-zero entry in the i-th row of A.

A-1 All the entries of A are non-negative integers and diagonal entries are even.
A-2 For each i, $\sum_{j=1}^{h} a_{ij} = q + 1$.
A-3 For each $i > 1$, $\rho(i) < i$ and $\rho(2) \leq \rho(3) \leq \cdots \leq \rho(h)$.

The actual construction consists of several steps. For each $1 \leq i, j \leq h$, we can choose $P_{ij} \subset \{0, 1, \cdots, q\}$ satisfying the following conditions (i), $\cdots$, (iv): (i) $|P_{ij}| = a_{ij}$ and (ii) $P_{1j} \cup P_{2j} \cup \cdots \cup P_{qj}$ is a partition of $\{0, \cdots, q\}$ for each j. For each $1 \leq i \leq h$, let μ_i be the least element in $P_{i\rho(i)}$. We require that (iii) $\varphi(\mu_i) \in P_{\rho(i)i}$ and (iv) $\mu_i > \mu_{i'}$ whenever $\rho(i) = \rho(i')$ and $i > i'$.

For each $1 \leq i, j \leq h$, we choose a bijection $\sigma_{ij} : P_{ij} \to P_{ji}$ such that (i) $\sigma_{ji}\sigma_{ij} = 1$, (ii) σ_{ii} is without a fixed point and (iii) for each $1 < i \leq h$, $\sigma_{i\rho(i)}(\mu_i) = \varphi(\mu_i)$. Define $x_1, x_2, \cdots, x_h \in G$ inductively by

$$(1) \qquad x_1 = 1, \qquad x_i = \pi_{\mu_i} x_{\rho(i)} \quad \text{for } i > 1 \, .$$

For each i, j and $\nu \in P_{ij}$, we can choose

$$(2) \qquad x_{ij}^{(\nu)} \in K\pi_\nu \cap \pi_{\sigma_{ij}\nu}^{-1} K$$

such that $x_{ji}^{(\sigma_{ij}\nu)} = (x_{ij}^{(\nu)})^{-1}$ and $x_{i\rho(i)}^{(\mu_i)} = \pi_{\mu_i}$. For each $1 \leq i, j \leq h$, let

$$\mathbf{P}_{ij} = \{x_{ij}^{(\nu)} \mid \nu \in P_{ij}\}$$
$$S_{ij} = x_i^{-1} \mathbf{P}_{ij} x_j$$

(We set $\mathbf{P}_{ij} = S_{ij} = \emptyset$ for $a_{ij} = 0$.) Then $1 \in S_{i\rho(i)}$ for each $i > 1$. We also have $S_{ji} = S_{ij}^{-1}$, and $S_{ii} = S_{ii}^{-1}$ in particular. Choose T_i such that S_{ii} is the disjoint union of T_i and T_i^{-1}.

THEOREM 5.2 (Theorem 3 in [2]). *Given an $h \times h$ symmetric matrix $A = (a_{ij})$ satisfying A-1, A-2, A-3 above, the sets S_{ij} for $i > j > \rho(i)$, $S_{i\rho(i)} - \{1\}$ and T_i generate a torsion free discrete subgroup Γ of G. The above sets are disjoint and form a basis for the free group Γ. Hence we have*

$$\mathrm{rank}(\Gamma) = \frac{(q-1)h}{2} + 1 \; .$$

Moreover, $\{x_1, \cdots, x_h\}$ is a set of representatives for $K\backslash G/\Gamma$. Every torsion free discrete subgroup of G can be constructed in this way.

6. Regular graphs as quotients of $\mathcal{T}$

Let $\mathcal{G}$ be a finite combinatorial regular graph of degree $q+1$. Our goal in this section is to construct a torsion free discrete subgroup Γ of $G = \mathrm{PGL}_2(\mathfrak{K})$ such that $\mathcal{G} \cong \mathcal{T}/\Gamma$. Let $h = |V(\mathcal{G})|$.

Let $\mathcal{T}'$ be the spanning tree of $\mathcal{G}$ obtained by the *breadth-first search* algorithm starting from a vertex v_1. And we reindex vertices $v_2, v_3, \cdots$ in the order visited during the search. More precisely, starting from v_1 we append vertices $v_2, \cdots$ adjacent to v_1 and edges from v_1 to them to $\mathcal{T}'$. These vertices are called children of v_1 and 1st generation descendents of v_1. Among the remaining vertices, we append children of (i.e. vertices adjacent to) v_2, then those of v_3, ... to $\mathcal{T}'$. Children of 1st generation descendents are called 2nd generation descendents. In the end, we obtain a spanning tree $\mathcal{T}'$ of $\mathcal{G}$ and the generation measures the distance from v_1 in $\mathcal{T}'$. (See [3] for detail.) Let $A = (a_{ij})$ be the adjacency matrix of $\mathcal{G}$: $a_{ij} = 1$ if v_i and v_j is adjacent, and $a_{ij} = 0$ otherwise.

THEOREM 6.1. *Let $\mathcal{G}$ be a finite combinatorial regular graph of degree $q+1$. Then there exists a torsion free discrete subgroup Γ of $\mathrm{PGL}_2(\mathfrak{K})$ such that $\mathcal{G} \cong \mathcal{T}/\Gamma$ where $\mathcal{T}$ is the tree of homothety classes of lattices in $\mathfrak{K}^2$.*

More precisely, if we index vertices of $\mathcal{G}$ as above, then the adjacency matrix A satisfies the conditions A-1, A-2, A-3 required in Theorem 5.2. And for any subgroup Γ constructed as in the same theorem starting from A, we have $\mathcal{G} \cong \mathcal{T}/\Gamma$.

Proof. The conditions A-1 and A-2 are satisfied obviously. And A-3 follows from the ordering of vertices $v_1, v_2, \cdots, v_h$. Really, suppose v_i is a child of v_j in the breadth-first search of vertices of $\mathcal{G}$ as above. Then by the ordering of vertices, we have $i > j$ and $a_{ij} = 1$. This shows $\rho(i) < i$. Moreover, we have $\rho(i) = j$ since otherwise v_i must be visited earlier during the search. If v_{i+1} is also a child of v_j, then we have $\rho(i) = \rho(i+1) = j$. Otherwise, v_{i+1} is a child of $v_{j'}$ with $j' > j$, thus $\rho(i) = j < j' = \rho(i+1)$.

Let $x_1, \cdots, x_h \in G$ be given inductively by (1). We claim that the map $v_i \mapsto \mathfrak{o}x_i$ induces an isomorphism between $\mathcal{T}'$ and the subtree $\mathcal{T}''$ of $\mathcal{T}$ generated by $\mathfrak{o}x_1, \cdots, \mathfrak{o}x_h$. Really, suppose v_i is a child v_j. Then as proven above, we have $\rho(i) = j$ and $x_i = \pi_{\mu_i}x_j$. Since $\mathfrak{o}x_i = \mathfrak{o}\pi_{\mu_i}x_j$ and $\mathfrak{o}x_j$ are adjacent in $\mathcal{T}$, this implies the map $\mathcal{T}' \to \mathcal{T}, v_i \mapsto \mathfrak{o}x_i$ is an injective graph morphism whose image is a subtree of $\mathcal{T}$ with vertices $\mathfrak{o}x_1, \cdots, \mathfrak{o}x_h$. By definition, this subtree is $\mathcal{T}''$.

We proceed with the construction in Theorem 5.2. Choose sets P_{ij}, $\mathbf{P}_{ij} = \{x_{ij}\}$. (Since $|P_{ij}| = |\mathbf{P}_{ij}| = 1$ if not empty, we will write x_{ij} for $x_{ij}^{(\nu)}$ with $\nu \in P_{ij}$ to simplify notations.) And let $s_{ij} = x_i^{-1}x_{ij}x_j$. Then $\{s_{ij} \mid i > j > \rho(i),\ a_{ij} \neq 0\}$ is a basis for Γ. We have $s_{ji} = s_{ij}^{-1}$ and $s_{i\rho(i)} = x_i^{-1}\pi_{\mu_i}x_j = 1$.

By Theorem 5.2, $\{\mathfrak{o}x_1, \cdots, \mathfrak{o}x_h\}$ is a set of representatives for vertices of $\mathcal{T}/\Gamma$. Thus the above $\mathcal{T}''$ is a maximal subtree of $\mathcal{T}/\Gamma$. Fix j. Then by (2), $\mathfrak{o}x_{ij}$ for $1 \leq i \leq h$ with $a_{ij} \neq 0$ are the $q+1$ vertices adjacent to $\mathfrak{o}$. Hence $\mathfrak{o}x_{ij}x_j = \mathfrak{o}x_is_{ij}$ are the vertices adjacent to $\mathfrak{o}x_j$. Thus $\overline{\mathfrak{o}x_i}$ and $\overline{\mathfrak{o}x_j}$ are adjacent in $\mathcal{T}/\Gamma$ if $a_{ij} = 1$. (Let us denote by $\overline{v}$ the image of $v \in V(\mathcal{T})$ under $\mathcal{T} \to \mathcal{T}/\Gamma$.) Conversely, suppose $\overline{\mathfrak{o}x_k}$ and $\overline{\mathfrak{o}x_j}$ are adjacent in $\mathcal{T}/\Gamma$. The edge from $\overline{\mathfrak{o}x_j}$ to $\overline{\mathfrak{o}x_k}$ in $\mathcal{T}/\Gamma$ can be lifted to an edge e in $\mathcal{T}$ from $\mathfrak{o}x_j$, whose end point must be one of $q+1$ vertices $\mathfrak{o}x_is_{ij}$ $(1 \leq i \leq h,\ a_{ij} \neq 0)$ adjacent to $\mathfrak{o}x_j$ in $\mathcal{T}$. Sine $\overline{\mathfrak{o}x_is_{ij}} = \overline{\mathfrak{o}x_i}$ are distinct each other, the end point of e is $\mathfrak{o}x_ks_{kj}$. Thus $a_{kj} = 1$. This proves the adjacency matrix of $\mathcal{T}/\Gamma$ is equal to A. □

Corollary 6.2. *Let $\mathcal{G}$ be a finite regular combinatorial graph of degree $q+1$ where q can be any positive integer. Then the adjacency matrix of $\mathcal{G}$ is conjugate by a permutation matrix to a symmetric matrix satisfying A-1, A-2, A-3 of Theorem 5.2.*

Proof. We have shown in the proof of the last theorem that if we reindex the vertices by applying breadth-first search then the adjacency matrix with respect this ordering sarisfies the required conditions. □

7. Example

Let $\mathcal{G}$ be a complete regular graph of degree $p+1$ where p is an odd prime. We want to find a basis for a torsion free discrete subgroup Γ

of $\mathrm{PGL}_2(\mathfrak{K})$ such that $\mathcal{G} \cong \mathcal{T}/\Gamma$ where $\mathfrak{K} = \mathbb{Q}_p$ or $\mathbb{F}_p((z))$, the field of Laurent series in the indeterminate z with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\omega = p$ or z be a prime element of $\mathfrak{K}$ in each case.

Let $\pi_0 = \left(\begin{smallmatrix} 0 & 1 \\ \omega & 0 \end{smallmatrix}\right)$. Then $\mathfrak{o}\pi_0 g$ with $g \in K = \mathrm{GL}_2(\mathfrak{O})/\mathfrak{O}^*$ are vertices adjacent to $\mathfrak{o}$ and vice versa. Choosing a set $\{\alpha_0, \cdots, \alpha_p\}$ of representatives for $\pi_0^{-1} K \pi_0 \cap K \backslash K$ as

$$(3) \qquad \alpha_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad \text{for } 0 \le i \le p-1\,, \quad \alpha_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can take $\pi_1, \cdots, \pi_p$ as

$$(4) \qquad \pi_i = \pi_0 \alpha_i \quad (0 \le i \le p) \quad \text{where } \pi_0 = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}.$$

Note that for any i, we have $K\pi_i^{-1} = K\pi_0$, hence $\varphi(i) = 0$.

Let $h = p + 2 = |V(\mathcal{G})|$. The adjacency matrix $A = (a_{ij})$ is the $h \times h$ matrix with a_{ij} equal to 0 if $i = j$ and 1 otherwise. Hence it satisfies the conditions of Theorem 5.2: we have $\rho(2) = \rho(3) = \cdots = \rho(h) = 1$. We define sets $P_{ij} \subset \{0, 1, \cdots, p\}$ as follows:

$$P_{ii} = \emptyset \quad (1 \le i \le h)$$
$$P_{21} = \{0\},\ P_{31} = \{1\}, P_{41} = \{2\}, \cdots, P_{h1} = \{p\}$$
$$P_{12} = P_{13} = \cdots = P_{1h} = \{0\}$$
$$P_{ij} = \begin{cases} \{i-j\} & \text{if } 2 \le j < i \le h \\ \{i-j+p+1\} & \text{if } 2 \le i < j \le h \end{cases}$$

Then we have

$$x_1 = 1,\ x_2 = \pi_0,\ x_3 = \pi_1,\ \cdots,\ x_h = \pi_p.$$

The maps $\sigma_{ij} : P_{ij} \to P_{ji}$ are obvious ones. We should choose for each pair (i,j) with $2 \le j < i \le h$ an element of $\mathrm{PGL}_2(\mathfrak{K})$

$$x_{ij} \in K\pi_{i-j} \cap \pi_{j-i+p+1}^{-1} K$$

We may put for (i,j) with $2 \le j < i \le h$

$$x_{ij} = \alpha_{j-i+p+1}^{-1} \pi_0 \alpha_{i-j}$$

Then $\{x_i^{-1} x_{ij} x_j \mid 2 \le j < i \le h\}$ is a basis for a desired subgroup Γ. By shifting indices, we obtain the following.

PROPOSITION 7.1. *Let $\mathfrak{K} = \mathbb{Q}_p$ or $\mathbb{F}_p((z))$. The following elements form a basis for a torsion free discrete subgroup Γ of $\mathrm{PGL}_2(\mathfrak{K})$ such that $\mathcal{T}/\Gamma$ is a complete regular graph of degree $p+1$:*

$$\alpha_i^{-1}\pi_0\alpha_{j-i+p+1}^{-1}\pi_0\alpha_{i-j}\pi_0\alpha_j$$

for (i,j) with $0 \le j < i \le p$ and α_i, π_i are given in (3) and (4).

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