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CLOSED AND DENSE ELEMENTS OF BE-ALGEBRAS

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Abstract. The notions of closed elements and dense elements are introduced in BE-algebras. Characterization theorems of closed elements and closed filters are obtained. The notion of dense elements is introduced in BE-algebras. Dense BE-algebras are characterized with the help of maximal filters and congruences. The concept of D-filters is introduced in BE-algebras. A set of equivalent conditions is derived for every D-filter to become a closed filter.

1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [5]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras by K. Iseki and S. Tanaka [4]. Some properties of filters of BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by J.L. Meng in [6]. In [10], A. Walendziak discussed some relationships between congruence relations and normal filters of a BE-algebra. In [3], Gispert and Torrens defined the Boolean center and the Boolean skeleton of a bounded BCK-algebra and they used the Boolean skeleton to obtain a representation of bounded BCK-algebras. In [7], C. Muresan studied some properties of dense elements and the radical of residuated lattices. Later in 2011, D. Piciu and D. Tascau [8] developed a theory of localization for bounded commutative BCK-algebras.
In this paper, the notions of closed elements is introduced in BE-algebras. A set of equivalent conditions is derived for every element of a BE-algebra to become closed. The notion of closed filters is introduced in BE-algebras. Closed filters are characterized in terms of closed elements

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of BE-algebras. The notions of dense elements and dense BE-algebras are introduced. Some characterization theorems of dense BE-algebras are derived in terms of maximal filters and congruences. The concept of D-filters is introduced in BE-algebras. A set of equivalent conditions is obtained for every D-filter of a BE-algebra to become a closed filter.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [2], [3] and [5] for the ready reference of the reader.

Definition 2.1. [5] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called implicative if $(x * y) * x = x$ for all $x, y \in X$. A BE-algebra X is called commutative if $(x*y)*y = (y*x)*x$ for all $x, y \in X$. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive and symmetric. If X is commutative, then $\leq$ is anti-symmetric and hence a partial order on X. Throughout this article, X stands for a partially ordered set.

Theorem 2.2. [6] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,
(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 2.3. [1] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$ is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. Let $(X, *, 0, 1)$ be a bounded BE-algebra, where 0 is the smallest element of X with respect to the ordering $\leq$. Hence $0 * x = 1$ for all $x \in X$. For any $x \in X$, define a unary operation N on X as $xN = x * 0$, where xN is called the pseudo-complement of x. It is easily seen that $0N = 1$ and $1N = 0$.

Proposition 2.4. [2] *Let X be a transitive BE-algebra and $x, y \in X$. Then the following properties hold.*

(1) $x \leq xNN$,
(2) $x \leq y$ *implies* $yN \leq xN$,
(3) $xNNN = xN$,
(4) $x * yN = xNN * yN$,
(5) $(x * yNN)NN = x * yNN$,
(6) $(x * y)NN \leq xNN * yNN$.

Theorem 2.5. [3] *Let X be a BE-algebra and $a, b \in X$. Then $a * c = 1$ and $b * c = 1$ imply $c = 1$ for all $c \in X$ if and only if $\langle a \rangle \cap \langle b \rangle = \{1\}$.*

An element x of X is called Boolean [3] if $\langle x \rangle \cap \langle xN \rangle = \{1\}$. Let us denote the set of all Boolean elements of a bounded BE-algebra X by $\mathcal{B}(X)$. Clearly $0, 1 \in \mathcal{B}(X)$.

Proposition 2.6. [3] *Let X be a transitive BE-algebra. Then for every $a \in \mathcal{B}(X)$ and $x, y \in X$, the following conditions hold.*

(1) $aNN = a$,
(2) $a * (a * x) = a * x$,
(3) $a * (x * y) = (a * x) * (a * y)$.

3. Closed elements of BE-algebras

In this section, the notion of closed elements is introduced and studied their properties. A set of equivalent conditions is established for every element of a BE-algebra to become a Boolean element. A set of equivalent conditions is derived for every element of a BE-algebra to become a closed element.

Definition 3.1. An element a of a BE-algebra is a *closed element* if $aNN = a$.

We denote by $\mathcal{C}(X)$ the set of all closed elements of a BE-algebra X. Obviously $0, 1 \in \mathcal{C}(X)$. In the following, a characterization of closed elements is derived.

THEOREM 3.2. *The following are equivalent in a transitive BE-algebra X:*

(1) *Every element is closed;*
(2) *for $x, y \in X$, $xN = yN$ implies $x = y$;*
(3) *for $x, y \in X$, $xN * yN = y * x$.*

Proof. (1) $\Leftrightarrow$ (2): Assume that every element of X is closed. Let $x, y \in X$ be such that $xN = yN$. Hence, it implies that $x = xNN = yNN = y$. Conversely, assume the condition (2). Let $x \in X$. By Proposition 2.4 (3), we have $xNNN = xN$. From the condition (2), we get $xNN = x$. Hence x is closed.
(2) $\Rightarrow$ (3): Assume that the condition (2) holds. Let $x, y \in X$ be two arbitrary elements. Hence $xN * yN = (x * 0) * (y * 0) = y * ((x * 0) * 0) = y * xNN = y * x$.
(3) $\Rightarrow$ (1): Assume that the condition (3) holds. Let $a \in X$. Then, we get $aNN = aN * 0 = aN * 1N = 1 * a = a$. Therefore a is closed. $\square$

It is observed from Proposition 2.6 (1) that every Boolean element of a BE-algebra is a closed element. It is evident from the following example that every closed element of a BE-algebra need not be Boolean.

EXAMPLE 3.3. *Let $X = \{1, a, b, c, d, 0\}$ be a non-empty set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra with smallest element 0. Observe that $aNN = dN = a; bNN = cN = b; cNN = bN = c$ and $dNN = aN = d$. Therefore $\mathcal{C}(X) = X$. But the elements a and d of the BE-algebra X are not Boolean, because of $\langle a\rangle \cap \langle aN\rangle = \langle a\rangle \cap \langle d\rangle = \{1, a\} \cap \{1, d, a, c\} = \{1, a\} \neq \{1\}$ also $\langle d\rangle \cap \langle dN\rangle \neq \{1\}$.

THEOREM 3.4. *In an implicative BE-algebra, every closed element is Boolean.*

Proof. Let X be an implicative BE-algebra. Let $a \in \mathcal{C}(X)$. Let $x \in X$ and suppose $a * x = 1$ and $aN * x = 1$. Then, it infers that $xN \leq aNN = a \leq x$. Since X is implicative, we get $x = (x * 0) * x = xN * x = 1$. Hence by Theorem 2.5, we get that $\langle a \rangle \cap \langle aN \rangle = \{1\}$. Therefore a is a Boolean element. □

REMARK 3.5. For any transitive BE-algebra X, define the set $X^* = \{x \in X \mid x = aN$ for some $a \in X\}$. Since $xNNN = xN$ for all $x \in X$, it is clear that $X^* = \mathcal{C}(X)$.

THEOREM 3.6. *Let X be a transitive BE-algebra. Then $\mathcal{C}(X)$ is closed under $*$.*

Proof. From Proposition 2.4 (5), it is clear. □

For any two BE-algebras $(X_1, *, 0, 1)$ and $(X_2, *, 0', 1')$, it is clear that their product $X_1 \times X_2$ is a BE-algebra in which the pseudo-complement of any $(a, b) \in X_1 \times X_2$ is defined as (aN, bN). Then the following result is an easy consequence.

THEOREM 3.7. *Let $(X_1, *, 0, 1)$ and $(X_2, *, 0, 1)$ be two bounded BE-algebras. Then a_1 and a_2 are closed elements of X_1 and X_2 respectively if and only if (a_1, a_2) is a closed element of $X_1 \times X_2$.*

Proof. Let $a_1 \in X_1$ and $a_2 \in X_2$. Assume that a_1 and a_2 are closed elements of X_1 and X_2 respectively. Clearly (a_1, a_2) is a closed element of $X_1 \times X_2$. Conversely assume that (a_1, a_2) is a closed element of $X_1 \times X_2$. Consider the projections $\Pi_i : X_1 \times X_2 \longrightarrow X_i$ for $i = 1, 2$. Let $\Pi_i(a_1, a_2) = a_i$ for $i = 1, 2$. We now prove that a_1 and a_2 are closed elements of X_1 and X_2 respectively. Since (a_1, a_2) is a closed element of $X_1 \times X_2$, we get the following:

$$\begin{aligned} a_1NN = \Pi_1(a_1NN, 1) &= \Pi_1(a_1NN, 1NN) \\ &= \Pi_1(a_1, 1)NN = \Pi_1(a_1, 1) = a_1. \end{aligned}$$

Therefore a_1 is a closed element of X_1. Similarly a_2 is a closed element of X_2. □

The following corollary is a direct consequence of the above theorem.

COROLLARY 3.8. *For any two BE-algebras X_1 and X_2, $\mathcal{C}(X_1 \times X_2) = \mathcal{C}(X_1) \times \mathcal{C}(X_2)$.*

THEOREM 3.9. *Let $(X, *, 0, 1)$ and $(Y, *, 0', 1')$ be two BE-algebras and $\alpha : X \to Y$ a BE-morphism. If a is a closed element of X, then $\alpha(a)$ is a closed element of Y.*

Proof. Let $a \in \mathcal{C}(X)$. Then $\alpha(a)NN = \alpha(aNN) = \alpha(a)$. Therefore $\alpha(a) \in \mathcal{C}(Y)$. □

In the following, the notion of *closed filters* of BE-algebras is introduced.

Definition 3.10. A filter F of a BE-algebra X is called a closed filter if $xNN \in F$ implies $x \in F$ for any $x \in X$.

If every element of a BE-algebra is closed, then clearly every filter is a closed filter. However, in the following, closed filters of BE-algebras are characterized.

Theorem 3.11. *A filter F of a transitive BE-algebra X is closed if and only if for any $x, y \in X$, $xN = yN$ and $x \in F$ imply $y \in F$.*

Proof. Assume that F is a closed filter of X. Let $x, y \in X$ be such that $xN = yN$ and $x \in F$. Since $x \in F$ and $x \leq xNN$, we get $yNN = xNN \in F$. Since F is closed, it yields that $y \in F$. Conversely, assume that the condition holds. Let $xNN \in F$ for $x \in X$. Since $xNNN = xN$, we get $x \in F$. Therefore F is a closed filter of X. □

Proposition 3.12. *Every maximal filter of a transitive BE-algebra is closed.*

Proof. Let F be a maximal filter of a transitive BE-algebra X. Let $x, y \in X$ be such that $xN = yN$ and $x \in F$. Suppose $y \notin F$. Then $\langle F \cup \{y\} \rangle = X$. Hence $0 \in \langle F \cup \{y\} \rangle$, which implies that $y^n * 0 \in F$ for some positive integer n. Hence

$$\begin{aligned} y^n * 0 \in F &\Rightarrow \underbrace{y * (y * (\cdots (y}_{\text{n times}} * 0) \cdots)) \in F \\ &\Rightarrow \underbrace{y * (y * (\cdots (y}_{\text{n-1 times}} * (y * 0)) \cdots)) \in F \\ &\Rightarrow \underbrace{y * (y * (\cdots (y}_{\text{n-1 times}} * (x * 0)) \cdots)) \in F \\ &\Rightarrow x * (\underbrace{y * (y * (\cdots (y}_{\text{n-1 times}} * 0)) \cdots)) \in F \\ &\quad \cdots \\ &\quad \cdots \\ &\Rightarrow x^n * 0 \in F. \end{aligned}$$

Since $x \in F$, we get that $0 \in F$, which is a contradiction. Hence, it infers $y \in F$. Therefore F is closed. □

For any filter F of a self-distributive BE-algebra X, it was observed in [9] that θ_F defined by $(x, y) \in \theta_F \Leftrightarrow x * y \in F$ and $y * x \in F$ is the unique congruence whose kernel is F. If X is bounded, then the quotient algebra $X/F = \{F_x \mid x \in X\}$(where F_x is the congruence class of x modulo θ_F) is also a bounded BE-algebra with smallest element F_0 in which $F_x * F_y = F_{x*y}$ and $F_x N = F_{xN}$ for all $x, y \in X$.

THEOREM 3.13. *The following are equivalent in a self-distributive BE-algebra X:*

(1) *Every element of X is closed;*
(2) *for any filter F, $\mathcal{C}(X/F) = X/F$;*
(3) *every filter is closed.*

Proof. (1) $\Rightarrow$ (2): It is obvious.
(2) $\Rightarrow$ (3): Assume that the condition (2) holds. Let F be a filter of X and $x \in X$. Suppose $xNN \in F$. For this $x \in X$, we get $F_x \in X/F$. By the condition (2), we get $F_{xNN} = F_x NN = F_x$. Hence $(xNN, x) \in \theta_F$, which gives $xNN * x \in F$. Since $xNN \in F$, it yields that $x \in F$. Therefore F is closed.
(3) $\Rightarrow$ (1): Assume that every filter is closed. Let $x \in X$. By (3), we get $\langle xNN \rangle$ is closed and $xNN \in \langle xNN \rangle$. Hence $x \in \langle xNN \rangle$. Thus $xNN \leq x$. Therefore $xNN = x$. □

In the following theorem, properties of the homomorphic images and inverse images of closed filters of BE-algebras are studied.

THEOREM 3.14. *Let X and Y be two BE-algebras and $\psi : X \to Y$ a bounded BE-morphism. If F is a closed filter of Y, then $\psi^{-1}(F)$ is a closed filter of X. Moreover, if ψ is onto, then $\psi(F)$ is a closed filter for any closed filter F of X.*

Proof. Let F be a closed filter of Y. Clearly $\psi^{-1}(F)$ is a filter of X. Let $xNN \in \psi^{-1}(F)$. Then $\psi(x)NN = \psi(xNN) \in F$. Since F is a closed filter of Y, we get that $\psi(x) \in F$. Hence $x \in \psi^{-1}(F)$. Therefore $\psi^{-1}(F)$ is a closed filter of X. Suppose ψ is onto. Let F be a closed filter of X. Clearly $\psi(F)$ is a filter of Y. Let $xNN \in \psi(F)$ where $x \in Y$. Since ψ is onto, there exists $a \in X$ such that $\psi(a) = x$. Hence $\psi(aNN) = \psi(a)NN = xNN$. Hence $aNN \in F$. Since F is closed, it yields $a \in F$. Hence $x = \psi(a) \in \psi(F)$. Therefore $\psi(F)$ is a closed filter of Y. □

In the following theorem, a set of equivalent conditions is derived for every filter of a BE-algebra to become a closed filter, which leads to a characterization of $\mathcal{C}(X)$.

Theorem 3.15. *Let X be a self-distributive BE-algebra. Then the following conditions are equivalent.*

(1) $\mathcal{C}(X) = X$;
(2) *every element is closed;*
(3) *every filter is closed;*
(4) *every principal filter is closed.*

Proof. $(1) \Leftrightarrow (2), (2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious.
$(4) \Rightarrow (2)$: Assume that every principal filter is closed. Let $x \in X$. Clearly $xNN \in \langle xNN \rangle$. Since $\langle xNN \rangle$ is a closed filter, it yields that $x \in \langle xNN \rangle$. Hence $xNN \leq x$. Since $x \leq xNN$, we get $x = xNN$. Therefore every element of X is closed. □

4. Dense elements of BE-algebras

In this section, the notion of dense elements is introduced in BE-algebras. Some properties of the class of dense elements are studied. The concept of dense BE-algebras is introduced and characterized. The notion of D-filters is introduced and characterized with the help of closed elements.

Definition 4.1. An element x of a BE-algebra X is called *dense* if $xN = 0$.

It is obvious that 1 is a dense element of X. Let us denote the class of all dense elements of a BE-algebra X by $\mathcal{D}(X)$. Then clearly $\mathcal{D}(X)$ is a subalgebra of X.

Example 4.2. *Let $X = \{1, a, b, 0\}$ be a set and $*$ a binary operation defined on X as follows:*

$*$	1	a	b	0
1	1	a	b	0
a	1	1	1	a
b	1	a	1	0
0	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra with smallest element 0. Observe that $aN = a; bN = 0$. Therefore b and 1 are dense but a is not a dense element.

Proposition 4.3. *For any transitive BE-algebra $X, \mathcal{D}(X)$ is a closed filter of X.*

Proof. Clearly $1 \in \mathcal{D}(X)$. Let $x, x * y \in \mathcal{D}(X)$. Then $xN = 0$ and $(x * y)N = 0$. By Proposition 2.4 (6), we get $1 = 0N = (x * y)NN \leq xNN * yNN = 0N * yNN = yNN$. Thus $yN = 0$, which yields $y \in \mathcal{D}(X)$. Therefore $\mathcal{D}(X)$ is a filter of X. Let $xNN \in \mathcal{D}(X)$. Then $xN = xNNN = 0$, which yields $x \in \mathcal{D}(X)$. Therefore $\mathcal{D}(X)$ is closed. □

THEOREM 4.4. *The following conditions hold in a BE-algebra X:*

(1) *$a \in \mathcal{D}(X)$ implies $aN \in \mathcal{B}(X)$,*
(2) *$a \in \mathcal{D}(X)$ if and only if $aNN \in \mathcal{D}(X)$,*
(3) *$\mathcal{D}(X)$ is a subalgebra of X,*
(4) *if a is a dense element in X, then $f(a)$ is a dense element in Y where $f : X \to Y$ is a bounded BE-morphism.*

Proof. (1). Let $a \in \mathcal{D}(X)$. Then $\langle aN \rangle \cap \langle aNN \rangle = \langle 0 \rangle \cap \langle 1 \rangle = \{1\}$. Therefore $aN \in \mathcal{B}(X)$.
(2). Since $aNNN = aN$, it follows immediately.
(3). Let $a, b \in \mathcal{D}(X)$. Then from (2), it immediately infers $a * b \in \mathcal{D}(X)$.
(4). Let $a \in \mathcal{D}(X)$. Then $f(a)N = f(aN) = f(0) = 0$. Therefore $f(a) \in \mathcal{D}(Y)$. □

In the following, the notion of dense BE-algebras is introduced.

DEFINITION 4.5. A BE-algebra X is called a *dense BE-algebra* if every non-zero element of X is dense (*i.e.* $xN = 0$ for all $0 \neq x \in X$).

Clearly the two-element bounded BE-algebra $\{0, 1\}$ is a dense BE-algebra. The bounded BE-algebra given in Example 4.2 is not a dense BE-algebra.

EXAMPLE 4.6. *Let $X = \{1, a, b, c, 0\}$ be a non-empty set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	b	b	0
b	1	a	1	a	0
c	1	1	1	1	0
0	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra with smallest element 0. Observe that $aN = bN = cN = 0$. Hence $\mathcal{D}(X) = X - \{0\}$. Therefore X is a dense BE-algebra.

It is observed that the set $\mathcal{D}(X)$ is a filter in a transitive BE-algebra but it is not sure that $\mathcal{D}(X)$ is a filter in a BE-algebra. However, in the following theorem, a necessary and sufficient condition is derived for $\mathcal{D}(X)$ to become a maximal filter which leads to a characterization of dense BE-algebras.

THEOREM 4.7. *A BE-algebra X is dense if and only if $\mathcal{D}(X)$ is a maximal filter.*

Proof. Assume that X is dense. Clearly $1 \in \mathcal{D}(X)$. Let $a, a * b \in \mathcal{D}(X)$. Suppose $b = 0$. Since $aN = 0$, we get $0 = (a * b)N = (a * 0)N = 0N = 1$. which is a contradiction. Therefore $b \neq 0$. Since X is dense, it yields $bN = 0$. Hence $b \in \mathcal{D}(X)$. Therefore $\mathcal{D}(X)$ is a filter of X. Suppose P is a proper filter of X such that $\mathcal{D}(X) \subset P$. Choose $x \in P - \mathcal{D}(X)$. Clearly $x \neq 0$. Since X is dense, it gives $xN = 0$. Hence $x \in \mathcal{D}(X)$, which is contradiction. Therefore $\mathcal{D}(X)$ is a maximal filter of X.

Conversely, assume that $\mathcal{D}(X)$ is a maximal filter of X. Suppose X is non-dense. Then there exists $0 \neq x \in X$ such that $xN \neq 0$. Hence $x \notin \mathcal{D}(X)$. Then $\mathcal{D}(X) \subset \langle \mathcal{D}(X) \cup \{x\} \rangle$. Since $\mathcal{D}(X)$ is maximal, we get $\langle \mathcal{D}(X) \cup \{x\} \rangle = X$. Hence $x * (d * 0) = d * (x * 0) = 1$ for some $d \in \mathcal{D}(X)$. Hence $xN = 1$. Thus $x = 0$, which is a contradiction. Therefore X is a dense BE-algebra. □

THEOREM 4.8. *Let X be a transitive BE-algebra. Then $\mathcal{D}(X)$ is contained in the intersection of all maximal filters of X.*

Proof. Let $x \in \mathcal{D}(X)$ and M a maximal filter of X such that $x \notin M$. Then $\langle M \cup \{x\} \rangle = X$. Thus $0 \in \langle M \rangle \cup \{x\} \rangle$. Hence $x^n * 0 \in M$ for some $n \in \mathbb{Z}^+$. Since $x * 0 = xN = 0$, we get $x^n * 0 = x^{n-1} * (x * 0) = x^{n-1} * 0 = x^{n-2} * (x * 0) = \ldots x * 0 = 0$. Hence $0 = xN = x * 0 = x^n * 0 \in M$, which is a contradiction. Thus $x \in M$ for all maximal filters of X. Therefore $\mathcal{D}(X) \subseteq M$ for all maximal filters of X. □

The following result is an immediate consequence of the above two results.

THEOREM 4.9. *The following are equivalent in a transitive and dense BE-algebra X.*

(1) *X has a unique maximal filter;*
(2) *$\mathcal{D}(X)$ is maximal;*
(3) *$\mathcal{D}(X)$ is the intersection of all maximal filters.*

In the following theorem, a set of equivalent conditions is derived for every BE-algebra to become a dense BE-algebra.

THEOREM 4.10. *The following are equivalent in a transitive BE-algebra X.*

(1) *X is dense;*
(2) $(a * b)N = (b * a)N$ *for all* $0 \neq a, b \in X$*;*
(3) $a * bNN = b * aNN$ *for all* $0 \neq a, b \in X$*;*
(4) $aNN * bNN = bNN * aNN$ *for all* $0 \neq a, b \in X$.

Proof. (1) $\Rightarrow$ (2): Assume that X is dense. Let $0 \neq a, b \in X$. Suppose $a * b = 0$. Then we get $b \leq a * b = 0$, which is a contradiction. Hence, it yields $a * b \neq 0$. Similarly $b * a \neq 0$. Then there exist $0 \neq c \in X$ and $0 \neq d \in X$ such that $a * b = c$ and $b * a = d$. Since X is dense and $c \neq 0$, we get $(a * b)N = cN = 0$. Similarly, we can obtain $(b * a)N = dN = 0$. Therefore $(a * b)N = (b * a)N$ for all $0 \neq a, b \in X$.
(2) $\Rightarrow$ (3): Assume that $(a * b)N = (b * a)N$ for all $0 \neq a, b \in X$. Then

$$\begin{aligned} 1 &= (b * bNN)NN \\ &= (bNN * b)NN \\ &\leq ((a * bNN) * (a * b))NN \\ &\leq ((a * bNN) * (a * b)NN)NN \\ &= (a * bNN) * (a * b)NN \end{aligned}$$

Hence, it yields that $a * bNN \leq (a * b)NN$. Again, since $b \leq bNN$, we get $(a * b)NN \leq (a * bNN)NN = a * bNN$. From the above two observations, it is concluded that $(a * b)NN = a * bNN$. Therefore $a * bNN = (a * b)NN = (b * a)NN = b * aNN$ for all $0 \neq a, b \in X$.
(3) $\Rightarrow$ (4): Assume that the condition (3) holds. Let $a, b \in X - \{0\}$. Then by Proposition 2.4 (4), it follows that $aNN * bNN = a * bNN = b * aNN = bNN * aNN$.
(4) $\Rightarrow$ (1): Assume that the condition (4) holds. Let $0 \neq a \in X$. Then by Proposition 2.4 (4), it implies that $aNN = 1 * aNN = 1NN * aNN = aNN * 1NN = a * 1NN = a * 1 = 1$. Hence, it yields that $aN = 0$. Therefore X is dense. □

If X is dense, then it can be routinely verified that X/F is also dense. Though the converse of this statement is not true, in the following, a necessary and sufficient condition is derived for the quotient algebra X/F to become dense.

THEOREM 4.11. *If F is a maximal filter of a self-distributive BE-algebra X, then X/F is dense.*

Proof. Let F be a maximal filter of a self-distributive BE-algebra X. Clearly $a \in F$ for all $0 \neq a \in X$. Hence $aNN \in F$ for all $0 \neq a \in X$. Let $F_0 \neq F_x \in X/F$ be a non-zero element. Then clearly $x \neq 0$. Since F is maximal, we get $x \in F$ and hence $xNN \in F$. Since $1 \in F$ and F is a congruence class, it yields $F_{xNN} = F_1$. It implies $F_xN = F_{xN} = F_0$. Thus F_x is a dense element in X/F. Therefore X/F is dense. □

For any $x \in F$, it can be seen that $F_x = F_{xNN}$. For, consider $x \in X$. Then clearly $x * xNN \in F$. Since $x \in F$, we get $xNN \in F$. Therefore it concludes that $(x, xNN) \in \theta_F$. In the following theorem, a set of equivalent conditions is derived for every closed filter of a self-distributive BE-algebra to become a maximal filter.

Theorem 4.12. *Let X be a self-distributive BE-algebra. Then the following conditions are equivalent.*

(1) *X is dense;*
(2) *for every filter $F, X/F$ is dense;*
(3) *every closed filter is maximal;*
(4) *$\mathcal{D}(X)$ is maximal.*

Proof. (1) ⇒ (2): It is obvious.
(2) ⇒ (3): Assume that the condition (2) holds. Let F be a closed filter of X. Then by the hypothesis, we get X/F is dense. Let $0 \neq x \in X$. It is enough to show that $x \in F$. Since $x \neq 0$, we get $F_x \neq F_0$. Since X/F is dense, it yields $F_{xN} = F_xN = F_0$. Hence $F_{xNN} = F_{0N} = F_1$ and so $(xNN, 1) \in \theta_F$. Thus $xNN = 1 * xNN \in F$. Since F is a closed filter of X, it yields that $x \in F$. Therefore F is a maximal filter of X.
(3) ⇒ (4): Since $\mathcal{D}(X)$ is closed, it follows immediately.
(4) ⇒ (1): Assume that $\mathcal{D}(X)$ is maximal. By Theorem 4.10, it is clear. □

The notion of D-filters of BE-algebras is now introduced in the following:

Definition 4.13. A proper filter F of a BE-algebra is called *D-filter* if $\mathcal{D}(X) \subseteq F$.

In any BE-algebra X, clearly $\mathcal{D}(X)$ is a D-filter and every closed filter is a D-filter. Since every maximal filter of a transitive BE-algebra is closed it is also a D-filter. In the following theorem, some equivalent conditions are derived for every filter of a transitive BE-algebra to become D-filter.

Theorem 4.14. *The following conditions are equivalent in a transitive BE-algebra.*

(1) *Every filter is a D-filter;*
(2) *every principal filter is a D-filter;*
(3) *X has a unique dense element.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3): Assume that every principal filter is a D-filter. Hence, we get that $\langle 1 \rangle$ is a D-filter of X. Thus, it implies that $\mathcal{D}(X) \subseteq \langle 1 \rangle$. Hence, it concludes that $\mathcal{D}(X) = \{1\}$. Therefore X has the unique dense element, precisely 1.
(3) $\Rightarrow$ (1): It is obvious. □

For any transitive BE-algebra X, define a mapping $\nu : X \to X/\mathcal{D}(X)$ by $\nu(x) = \mathcal{D}(X)_x$ for all $x \in X$. Then clearly ν is an epimorphism. In the following theorem, we derive a set of equivalent conditions for ν to become an isomorphism.

THEOREM 4.15. *The following are equivalent in a transitive BE-algebra X.*
(1) *Every filter is a D-filter;*
(2) *for any $x, y \in X$, $(x * y)N = 0, (y * x)N = 0$ implies $x = y$;*
(3) *X is isomorphic to $X/\mathcal{D}(X)$.*

Proof. (1) $\Rightarrow$ (2): Assume that every filter of X is a D-filter. Then by above theorem, X has a unique dense element. Let $x, y \in X$. Assume that $(x * y)N = 0$ and $(y * x)N = 0$. Hence $x * y \in \mathcal{D}(X) = \{1\}$ and $y * x \in \mathcal{D}(X) = \{1\}$. Hence $x * y = 1$ and $y * x = 1$. Therefore, it concludes that $x = y$.
(2) $\Rightarrow$ (3): Assume that condition (2) holds. Clearly $\nu : X \to X/\mathcal{D}(X)$ is an epimorphism. Let $x, y \in X$ be such that $\mathcal{D}(X)_x = \mathcal{D}(X)_y$. Hence $x*y \in \mathcal{D}(X)$ and $y*x \in \mathcal{D}(X)$, which yield $(x*y)N = 0$ and $(y*x)N = 0$. Then by condition (2), we get $x = y$. Therefore ν is one-one and so ν is an isomorphism.
(3) $\Rightarrow$ (1): Assume that X is isomorphic to $X/\mathcal{D}(X)$. Let $x, y \in \mathcal{D}(X)$. Clearly $x * y, y * x \in \mathcal{D}(X)$. Hence $\mathcal{D}(X)_x = \mathcal{D}(X)_y$. Since ν is an isomorphism, it yields $x = y$. Hence X has a unique dense element. Thus every filter of X is a D-filter. □

It is clear that every closed filter of a BE-algebra is a D-filter but the converse is not true. For consider the filter $F = \{1, b, c\}$ of the BE-algebra X given in the Example 4.6. Clearly F is a D-filter but not closed. However, in the following theorem, a set of equivalent conditions is derived for every D-filter to become closed.

THEOREM 4.16. *The following are equivalent in a self-distributive BE-algebra X.*

(1) *Every D-filter is closed;*
(2) $\mathcal{D}(X)_x = \mathcal{D}(X)_{xNN}$ *for all* $x \in X$*;*
(3) *for any D-filter F,* $xNN * x \in F$ *for all* $x \in X$.

Proof. (1) ⇒ (2): Assume that every D-filter is a closed. Let $x \in X$. Clearly $\langle \mathcal{D}(X) \cup \{xNN\} \rangle$ is a D-filter of X. By the hypothesis, $\langle \mathcal{D}(X) \cup \{xNN\} \rangle$ is closed. Since $xNN \in \langle \mathcal{D}(X) \cup \{xNN\} \rangle$, we get $x \in \langle \mathcal{D}(X) \cup \{xNN\} \rangle$. Therefore $xNN * x \in \mathcal{D}(X)$. Clearly $x * xNN = 1 \in \mathcal{D}(X)$. Hence $(x, xNN) \in \theta_{\mathcal{D}(X)}$. Therefore $\mathcal{D}(X)_x = \mathcal{D}(X)_{xNN}$.
(2) ⇒ (3): Assume that $\mathcal{D}(X)_x = \mathcal{D}(X)_{xNN}$ for all $x \in X$. Let F be a D-filter of X. Let $x \in X$. Then by (2), we get $(x, xNN) \in \theta_{\mathcal{D}(X)}$. Therefore $xNN * x \in \mathcal{D}(X) \subseteq F$.
(3) ⇒ (1): Assume that the condition (3) holds. Let F be a D-filter of X. Let $x \in X$ and $xNN \in F$. By condition (3), we get that $xNN * x \in F$. Since F is a filter, it yields that $x \in F$. Therefore F is closed. □

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