# BLACK-SCHOLES EQUATION 

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#### Abstract

The purpose of this paper is to present approximation of $C_{0}$-sequentially equicontinuous semigroups on a sequentially complete locally convex space $X$.


## 1. Introduction

In 1973, Black and Scholes showed that under certain natural assumptions about the financial market, the price of a European option $V$, as a function of time $\tau$ and the current value of the underlying asset $s$, satisfies the final value problem

$$
\left\{\begin{array}{l}
\frac{\partial V}{\partial \tau}=-\frac{1}{2} \sigma^{2} s^{2} \frac{\partial^{2} V}{\partial s^{2}}-r s \frac{\partial V}{\partial s}+r V, \quad 0 \leq \tau \leq \bar{\tau}, 0 \leq s<\infty \\
V(s, \bar{\tau})=\bar{h}(s)
\end{array}\right.
$$

where $\sigma$ is the volatility, $r$ is the risk-free interest rate and $\bar{\tau}$ is the expiry date. By introducing new variables

$$
t=\frac{1}{2}(\bar{\tau}-\tau) \sigma^{2}, \quad x=\ln s \text { and } u(x, \tau)=V(s, \tau)
$$

we have the following initial value problem

$$
\left\{\begin{aligned}
\frac{\partial u}{\partial t} & =\frac{\partial^{2} u}{\partial x^{2}}+(\gamma-1) \frac{\partial u}{\partial x}-\gamma u \\
& u(x, 0)=h(x)
\end{aligned}\right.
$$

where $\gamma=2 r / \sigma^{2}<1$ is a constant and $h(x)=\bar{h}\left(e^{x}\right)$.
Rewriting this equation in terms of a differential operator $A$, we can interpret this equation as an abstract Cauchy problem $u^{\prime}(t)=A u(t)$, $u(0)=f$. It is well-known that the solution of the abstract Cauchy problem for a linear operator $A: D(A) \rightarrow X$ on a Banach space $X$ and

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an initial value $f \in D(A)$ is given by $u(t) S(t) f$ if $A$ is the generator of a $C_{0}$ semigroup $\{S(t): t \geq 0\}$. And thus the price oft he option is obtained from the semigroup. Therefore it is crucial to determine whether the Black-Scholes operator generates a $C_{0}$ semigroup or not.

In this paper we will show that the differential operator

$$
\frac{d}{d x^{2}}+(\gamma-1) \frac{d}{d x}-\gamma I
$$

is a perturbation of the square of a generator of a $C_{0}$ group and it generates a $C_{0}$ semigroup on a suitable Banach space $X$ and then we will present the approximation of this semigroup by discrete semigroups.

## 2. Approximation

We recall some definitions of the $C_{0}$ semigroup. For more information about the $C_{0}$ semigroup, see $[\mathrm{P}]$. Let $X$ be a Banach space.

Definition 2.1. A family $\{T(t): t \geq 0\}$ of bounded linear operators from $X$ into itself is called a $C_{0}$ semigroup on $X$ if
(i) $T(0)=I$, the identity operator on $X$ and $T(t+s)=T(t) T(s)$ for $t, s \geq 0$
(ii) $\lim _{t \rightarrow 0} T(t) x=x$ for all $x \in X$.
$\{T(t): t \geq 0\}$ is called a $C_{0}$ semigroup of contractions if $\|T(t)\| \leq 1$ for all $t \geq 0$. If the properties (i) and (ii) hold for all $t \in \mathbf{R}$, we call $\{T(t): t \in \mathbf{R}\}$ a $C_{0}$ group.

The generator of $\{T(t): t \geq 0\}$ is the linear operator $A$, given by

$$
A x=\lim _{h \rightarrow 0} \frac{1}{h}(T(h) x-x)
$$

with $D(A)=\left\{x \in X: \lim _{h \rightarrow 0} \frac{1}{h}(T(h) x-x) \in X\right\}$.
In order to trnsform Black-Scholes equation into the abstract Cauchy problem, consider the Banach space of continuous functions vanishing at infinity

$$
X=C_{0}(\mathbf{R})=\left\{f \in C(\mathbf{R}): \lim _{|x| \rightarrow \infty} f(x)=0\right\}
$$

with the usual supremum norm $\|f\|=\sup _{x \in \mathbf{R}}|f(x)|$.
Let $(T(t) f)(x)=f(t+x)$ for $t \in \mathbf{R}, f \in X$ and $x \in \mathbf{R}$. Then it is not difficult to show that $\{T(t): t \in \mathbf{R}\}$ is a $C_{0}$ group on $X$. Let $B$ be
its generator. Then for $f \in D(B)$ and $x \in \mathbf{R}$, we have

$$
\begin{aligned}
(B f)(x) & =\lim _{h \rightarrow 0} \frac{1}{h}(T(t) f(x)-f(x)) \\
& =\lim _{h \rightarrow 0} \frac{1}{h}(f(x+h)-f(x))=f^{\prime}(x)
\end{aligned}
$$

So $f$ is differentiable and $B f \in X$, that is,

$$
D(B) \subseteq C_{0}^{1}(\mathbf{R})=\left\{f \in C^{1}(\mathbf{R}): f, f^{\prime} \in X\right\}
$$

Conversely, let $f \in C_{0}^{1}(\mathbf{R})$. For $x \in \mathbf{R}$,

$$
\begin{aligned}
& \left|\frac{1}{h}(T(h) f(x)-f(x))-f^{\prime}(x)\right| \\
& \quad=\left|\frac{1}{h}(f(x+h)-f(x))-f^{\prime}(x)\right|=\left|\frac{1}{h} \int_{0}^{h}\left(f^{\prime}(x+\tau)-f^{\prime}(x)\right) d \tau\right| \\
& \quad \leq \sup _{0 \leq|\tau| \leq|h|}\left|f^{\prime}(x+\tau)-f^{\prime}(x)\right| \rightarrow 0
\end{aligned}
$$

as $h \rightarrow 0$ uniformly in $x \in \mathbf{R}$, since $f^{\prime} \in C_{0}(\mathbf{R})$ is uniformly continuous. Therefore $B=d / d x$ is a generator of a $C_{0} \operatorname{semigroup}\{T(t): t \in \mathbf{R}\}$ of contractions.

Theorem 2.2. Let $A$ be the operator defined by

$$
A=B^{2}+(\gamma-1) B=\frac{d^{2}}{d x^{2}}+(\gamma-1) \frac{d}{d x}
$$

with $D(A)=D\left(B^{2}\right)=\{f \in D(B): B f \in D(B)\}$. Then $A$ is the generator of a contraction $C_{0}$ semigroup $\{S(t): t \geq 0\}$ on $X$.

Proof. Note that $B$ is the generator of a contraction $C_{0} \operatorname{group}\left\{T_{2}(t)\right.$ : $t \in \mathbf{R}\}$ on $X$, where $T_{1}(t) f(x)=f(x+t)$ for $x \in \mathbf{R}$. By Corollary 3.7.5 in $[\mathrm{A}], B^{2}$ is the generator of a bounded holomorphic $C_{0}$ semigroups $\left\{T_{1}(t): t \geq 0\right\}$ on $X$, which is given by

$$
T_{2}(t) f=\frac{1}{\sqrt{4 \pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^{2}}{4 t}} T_{1}(s) f d s
$$

For $t \geq 0$, we have

$$
\left\|T_{2}(t) f\right\| \leq \frac{1}{\sqrt{4 \pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^{2}}{4 t}}\left\|T_{1}(s) f\right\| d s \leq\|f\|
$$

So $\left\{T_{2}(t): t \geq 0\right\}$ is a contraction $C_{0}$ semigroups on $X$.

By Lemma 2.8 of chapter 1 in $[\mathrm{P}]$, for $f \in D\left(B^{2}\right)$ and $\varepsilon>0$

$$
\begin{aligned}
\|B f\| & \leq 2\left(\left\|B^{2} f\right\|\|f\|\right)^{1 / 2}=2\left(\varepsilon\left\|B^{2} f\right\|\right)^{1 / 2}\left(\frac{1}{\varepsilon}\|f\|\right)^{1 / 2} \\
& \leq \varepsilon\left\|B^{2} f\right\|+\frac{1}{\varepsilon}\|f\|
\end{aligned}
$$

By Corollary 3.3 of chapter 3 in $[\mathrm{P}], B^{2}+(\gamma-1) B$ is the generator of a contraction $C_{0}$ semigroup $\{S(t): t \geq 0\}$.

REmARK 2.3. $A-\gamma I=d^{2} / d x^{2}+(\gamma-1) d / d x-\gamma I$ generates a contraction $C_{0}$ semigroup $\left\{e^{-\gamma t} S(t): t \geq 0\right\}$ on $X$.

Next we will present the approximation of the $C_{0}$ semigroup $\{S(t)$ : $t \geq 0\}$ generated by $A$.

Let $X_{n}$ be the space of all bounded real sequences $\left\{c_{k}\right\}_{-\infty}^{\infty}$ satisfying $\lim _{|k| \rightarrow \infty} c_{k}=0$ with the usual supremum norm. Define linear operators $P_{n}: X \rightarrow X_{n}$ and $E_{n}: X_{n} \rightarrow X$ by

$$
P_{n} f(x)=\{f(k / n)\}_{k=-\infty}^{\infty} \text { and } E_{n}\left(\left\{c_{k}\right\}_{-\infty}^{\infty}\right)=g(x),
$$

where $g(k / n)=c_{k}$ and $g(x)$ is linear between two consecutive points $k / n$ and $(k+1) / n$. Then $\left\{E_{n}\right\}$ and $\left\{P_{n}\right\}$ satisfy the assumption 6.1 in chapter 3 in $[\mathrm{P}]$.

Theorem 2.4. Let $A$ be the generator of a $C_{0}$ semigroup $\{S(t): t \geq$ $0\}$ in Theorem. We define a linear operator $F\left(\rho_{n}\right): X_{n} \rightarrow X_{n}$ by

$$
\begin{aligned}
F\left(\rho_{n}\right)(c)=\{ & \left(\frac{1}{2}-2 n^{2} \rho_{n}\right) c_{k}+n^{2} \rho_{n}\left(c_{k+1}+c_{k-1}\right) \\
& \left.+\left(\frac{1}{2}+(\gamma-1) n \rho_{n}\right) c_{k}-(\gamma-1) n \rho_{n} c_{k-1}\right\}_{k=-\infty}^{\infty}
\end{aligned}
$$

for $c=\left\{c_{k}\right\}_{k=-\infty}^{\infty}$ in $X_{n}$ and some $\rho_{n}>0$ such that $4 n^{2} \rho_{n}<1$. Then we have

$$
\lim _{n \rightarrow \infty}\left\|F\left(\rho_{n}\right)^{k_{n}} P_{n} f-P_{n} S(t) f\right\|=0
$$

for $f \in X$ and a sequence $\left\{k_{n}\right\}$ of positive integers such that $\lim _{n \rightarrow \infty} k_{n} \rho_{n}=$ $t$.

Proof. First we will show that $F\left(\rho_{n}\right)$ is a contraction. For $c=$ $\left\{c_{k}\right\}_{k=-\infty}^{\infty}$ in $X_{n}$

$$
\begin{aligned}
\left\|F\left(\rho_{n}\right)(c)\right\| \leq & \sup _{k}\left\{\left(\frac{1}{2}-2 n^{2} \rho_{n}\right)\left|c_{k}\right|+n^{2} \rho_{n}\left(\left|c_{k+1}\right|+\left|c_{k-1}\right|\right)\right. \\
& +\left(\frac{1}{2}+(\gamma-1) n \rho_{n}\left|c_{k}\right|+(1-\gamma) n \rho_{n}\left|c_{k-1}\right|\right\} \\
\leq & \left(\left(\frac{1}{2}-2 n^{2} \rho_{n}\right)+2 n^{2} \rho_{n}+\left(\frac{1}{2}+(\gamma-1) n \rho_{n}\right)+(1-\gamma) n \rho_{n}\right)\|c\| \\
= & \|c\| .
\end{aligned}
$$

So $F\left(\rho_{n}\right)$ is a contraction. And we have for $f \in D(A)$

$$
\begin{aligned}
& \left\|\frac{1}{\rho_{n}}\left(F\left(\rho_{n}\right)-I\right) P_{n} f-P_{n} A f\right\| \\
& \quad \leq \sup \left\lvert\, n^{2}\left(\left.f\left(\frac{k+1}{n}\right)-2 f\left(\frac{k}{n}\right)+f\left(\frac{k-1}{n}\right)-f^{\prime \prime}\left(\frac{k}{n}\right) \right\rvert\,\right.\right. \\
& \quad+\sup \left\lvert\, n\left(\left.f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)-f^{\prime}\left(\frac{k}{n}\right) \right\rvert\, .\right.\right.
\end{aligned}
$$

Since $D(A)$ is dense in $X, f^{\prime}(x)$ and $f^{\prime \prime}(x)$ are uniformly continuous on $\mathbf{R}$. By Theorem 6.7 of chapter 3 in [P], the result follows.

## References

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