JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 1, February 2019
http://dx.doi.org/10.14403/jcms.2019.32.1.23

ON MINIMIZERS FOR THE INTERACTION ENERGY WITH MILDLY REPULSIVE POTENTIAL

Hwa Kil Kim

Abstract. In this paper, we consider an interaction energy with attractive-repulsive potential. We survey recent results on the structure of global minimizers for the mildly repulsive interaction energy. We introduce a theorem which is important to the proof of the above results, and give a detailed proof of the theorem.

1. Introduction

Given a Borel measurable function $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the interaction energy $E : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E(\mu) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} W(x-y)d\mu(x)d\mu(y) \qquad \text{for all } \mu \in \mathcal{P}(\mathbb{R}^n), \tag{1.1}$$

where $\mathcal{P}(\mathbb{R}^n)$ is the set of Borel probability measures. Recently, the study of local and global minimizers of the energy of the form (1.1) has received much attention from people in the area of applied mathematics. One of main reason for this interest is that E is a Lyapunov functional for the continuity equation

$$\partial_t \mu + \nabla \cdot (-(\nabla W * \mu)\mu) = 0, \tag{1.2}$$

called the aggregation equation. More precisely, the equation (1.2) is a gradient flow in *2-Wasserstein space* whose meaning will be clear soon. This equation describes the behavior of particles interacting through the potential W, and is a very important mathematical model in many applications including mathematical biology, physics, granular media and economics, see [1], [2] and the references therein.

Received October 24, 2018; Accepted November 19, 2018.
2010 Mathematics Subject Classification: Primary 70F45, 49J27; Secondary 92B99.
Key words and phrases: mildly repulsive interaction energy, attractive-repulsive potential.
Hwa Kil Kim's work is supported by 2018 Hannam University Research Fund.

In these interacting particle models, the potential W is usually repulsive in the short range so that particles do not collide, and attractive in the long range for the particles to gather to form a group or a structure. Therefore, we naturally consider radially symmetric interaction potential of the form

$$W(x) = \omega(|x|),$$

where $\omega : [0,\infty) \to [0,\infty)$ is decreasing on some interval $[0, r_0)$ and increasing on (r_0, ∞). We refer to such potential as being *repulsive-attractive.*

It is interesting that the geometric structure of minimizers directly related to the repulsiveness of the potential at the origin. Motivated to this property, we refer to W as *strongly repulsive* at the origin if

$$\Delta W(x) \sim -\frac{1}{|x|^\alpha}, \quad \text{as } |x| \to 0,$$

with $0 < \alpha < d$. On the other hand, if the potential satisfies

$$\omega(|x|) \sim -|x|^\alpha, \qquad \text{as } |x| \to 0$$

with $\alpha > 2$ then we refer to W as *mildly repulsive* at the origin.

In this paper, we introduce a few most recent results on the geometry of global minimizers for the interaction energy with mildly repulsive potential. Furthermore, we discuss one important technical theorem which plays a crucial role to get those recent results.

2. preliminaries

We remind $\mathcal{P}(\mathbb{R}^d)$ is the set of Borel probability measures. First, we introduce the *Wasserstein* metric in subsets of $\mathcal{P}(\mathbb{R}^d)$. For $p \in \mathbb{N}$, we denote

$$\mathcal{P}_p(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|^p d\mu(\mathbf{x}) < \infty\}.$$

For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we define *p-Wasswerstein distance* between μ and ν which is denoted by $d_p(\mu, \nu)$ as follows

$$d_p^p(\mu, \nu) = \min_{\pi \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y),$$

where $\Gamma(\mu, \nu) := \{\gamma \in \mathcal{P}(\mathbb{R}^d) : \pi_1 \# \gamma = \mu, \ \pi_2 \# \gamma = \nu\}$ with π_1 and π_2 being projections onto the first and second marginal, respectively. For $p = \infty$, we define $d_\infty(\mu, \nu)$ by

$$d_\infty(\mu, \nu) = \inf_{\pi \in \Gamma(\mu,\nu)} \sup_{(x,y) \in \text{spt}\pi} |x - y|.$$

Now we remind a notion on quantification of convexity which is so called ϕ-uniform convex.

DEFINITION 2.1. Let $\phi : [0,\infty) \to \mathbb{R}$ be a measurable function. We say that ϕ is *modulus of convexity* if ϕ satisfies

1. $\phi : [0,\infty) \to R$ is continuous, $\phi(0) = 0$, and $\phi(x) \neq 0 \ for\ x > 0$
2. $\phi(x) \geq -kx$ for some $k < \infty$.

DEFINITION 2.2. (ϕ-uniform convexity) Let $u : [0,\infty) \to \mathbb{R}$ be a measurable function. We say that u is *ϕ-uniformly convex* on (a,b) if there exists a modulus of convexity ϕ such that

$$u\big(tx + (1-t)y\big) \leq tu(x) + (1-t)u(y) - t(1-t)\int_0^{|x-y|} \phi(s)ds. \tag{2.1}$$

for all $x, y \in (a,b)$, $t \in [0,1]$.

2.1. recent progress

Recently, there have been important progresses in revealing the geometry of global minimizers of mildly repulsive interaction energy. In [2], authors(Carrillo et al) proved that a global minimizer μ is compactly supported and the support of μ consists of finitely many points under the assumptions

(H1) W is of class C^2 and radially symmetric.

(H2) $W(0) = 0$, there exists $R > 0$ such that $W(x) < 0$ if $0 < |x| < R$, and $W(x) \geq 0$ if $|x| \geq R$.

(H3) There exists $\alpha > 2$ and $M > 0$ such that $\omega'(r)r^{1-\alpha} \to -M$ as $r \to 0$.

THEOREM 2.3. [2, Theorem 1.1. and Lemma 2.6.] Suppose (H1), (H2), and (H3) hold. If $\mu \in \mathcal{P}(\mathbb{R}^d)$ is a d_∞-local minimizer of the interaction energy (1.1) then every point in the support of μ is isolated. Furthermore, if μ is a global minimizer then μ is compactly supported and the support of μ consists of finitely many points.

For $d = 1$, they were also able to give a quantitative estimation on the cardinality of points in the support of the global minimizer.

THEOREM 2.4. [2, Theorem 4.5.] Suppose (H1), (H2), and (H3) hold. Let $d = 1$ and μ be a global minimizer of the interaction energy (1.1). Suppose that there exists $r \in (0,R]$ such that $\sqrt{-\omega}$ is strictly convex on $(0,r)$. Then we have $\#(\text{spt}(\mu)) \leq 2\lceil \frac{R}{r} \rceil + 1$, where $\#$ denotes the cardinality.

Later, under a stronger convexity assumption on the potential function, the above result was extended to the case of general dimension $d \geq 1$ by Kang et al in [3].

THEOREM 2.5. [3, Theorem 1.1.] Suppose (H1), (H2), and (H3) hold. Let $\alpha \geq 4$, $K(r) := \sup_{0\leq\rho\leq 2r} \frac{\sqrt{-\omega(\rho)}}{\rho^{\alpha/2}}$, and μ be a global minimizer of the interaction energy. Assume further that there exists $r_0 > 0$ such that $\sqrt{-\omega}$ is ϕ-uniformly convex on $(0, 2r_0)$ with $\phi(t) = Ct^{\frac{\alpha}{2}-1}$, where C is a constant satisfying $C \geq \frac{\alpha}{2}K(r_0)$. Then,

$$\#(\text{supp}(\mu)) \leq N(r_0, R)2^d,$$

where $N(r_0, R) \in \mathbb{N}$ is the optimal number of radius r_0-balls to cover a radius R-ball.

3. Main theorem

There is one important property that is necessary to achieve all the above results .i.e. Theorem 2.3, Theorem 2.4 and Theorem 2.5. We remark that this property was already mentioned in [2, Lemma 2.4 and Remark 2.5]. However, in the proof of [2, Lemma 2.4], there is one unclear point in quoting Proposition 2 of [1]. Hence, we give a detailed proof of the key property without using Proposition 2 of [1].

THEOREM 3.1. Suppose (H1), (H2), and (H3) hold. Let μ be a global minimizer of the interaction energy with (1.1). Then, we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} W(x-y)d\nu(x)d\nu(y) \geq 0 \tag{3.1}$$

for any signed measure ν such that supp $\nu \subset$ supp μ and $\nu(\mathbb{R}^n) = 0$.

Proof. Suppose ν be a signed measure such that $\nu(\mathbb{R}^n) = 0$ and supp $\nu \subset$ supp μ. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. With out loss of generality, we may assume

$$\nu^+(\mathbb{R}^n) = \nu^-(\mathbb{R}^n) = 1.$$

First, we approximate ν^+ and ν^- by point measures as follows

$$\nu^+ = \lim_{n\to\infty} \nu_n^+ := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \qquad \nu^- = \lim_{n\to\infty} \nu_n^- := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \delta_{y_i}, \tag{3.2}$$

where $x_i \in$supp ν^+ and $y_i \in$supp ν^- for $i = 1, \dots, n$. Then, for each n, we again approximate ν_n^+ and ν_n^- as follows

$$\nu_{n,m}^+ := \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_{|B(x_i,\frac{1}{m})}}{\mu(B(x_i,\frac{1}{m}))}, \qquad \nu_{n,m}^- := \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_{|B(y_i,\frac{1}{m})}}{\mu(B(y_i,\frac{1}{m}))}, \tag{3.3}$$

and define

$$\nu_{n,m} := \nu_{n,m}^+ - \nu_{n,m}^-.$$

Notice that $\nu_{n,m}(\mathbb{R}^d) = 0$ and

$$\mu + \varepsilon\nu_{n,m} \geq 0 \qquad \text{if} \quad 0 \leq \varepsilon < \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\mu(B(y_i,\frac{1}{m}))}. \tag{3.4}$$

Now, we compute

(3.5)

$$\begin{aligned} E(\mu + \varepsilon\nu_{n,m}) &= \iint W(x-y) d\big(\mu + \varepsilon\nu_{n,m}\big)(x)\ d\big(\mu + \varepsilon\nu_{n,m}\big)(y) \\ &= \iint W(x-y) d\mu(x) d\mu(y) + 2\varepsilon \iint W(x-y) d\mu(x) d\nu_{n,m}(y) \\ &+ \varepsilon^2 \iint W(x-y) d\nu_{n,m}(x) d\nu_{n,m}(y). \end{aligned}$$

Since μ is a global minimizer of interaction energy E, exploiting (3.5), we have

$$\frac{d^2}{d\varepsilon^2} E(\mu + \varepsilon\nu_{n,m})_{|\varepsilon=0} = \iint W(x-y) d\nu_{n,m}(x) d\nu_{n,m}(y) \geq 0. \tag{3.6}$$

Now, from (3.6), we have

$$\begin{aligned} 0 &\leq \lim_{m\to\infty} \iint W(x-y) d\nu_{n,m}(x) d\nu_{n,m}(y) \\ &= \iint W(x-y) d\nu_n(x) d\nu_n(y), \end{aligned} \tag{3.7}$$

where the equality comes from facts that $\nu_{n,m}$ converges weakly to ν_n as $m \to \infty$, and the supports of $d\nu_{n,m}$ are uniformly bounded and W is a continuous function.

Similarly, we let $n \to \infty$ in (3.7) and have

$$\iint W(x-y) d\nu(x) d\nu(y) \geq 0, \tag{3.8}$$

which completes the proof. □

References

[1] D. Balague, J. A. Carrillo, T. Laurent, and G. Raoul, *Dimensionality of Local Minimizers of the Interaction Energy,* Arch. Rational Mech. Anal, **209** (2013), 1055-1088.

[2] J. A. Carrillo, A. Figalli, and F.S.Patacchini, *Geometry of minimizers for the interaction energy with mildly repulsive potentials,* Annales de l'Institut Henri Poincare (C) Non Linear Analysis, **34** (2017), 1299-1308.

[3] K. Kang, H. K. Kim, and G. Seo, *Cardinality estimation of global minimizer for interaction energy with mildly repulsive potentials,* Preprint.

Department of Mathematics Education
Hannam University
Daejon 34430, Republic of Korea
E-mail: hwakil@hnu.kr