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ON A CLASS OF NONCOOPERATIVE FOURTH-ORDER ELLIPTIC SYSTEMS WITH NONLOCAL TERMS AND CRITICAL GROWTH

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Abstract. In this paper, we consider a class of noncooperative fourth-order elliptic systems involving nonlocal terms and critical growth in a bounded domain. With the help of Limit Index Theory due to Li [32] combined with the concentration compactness principle, we establish the existence of infinitely many solutions for the problem under the suitable conditions on the nonlinearity. Our results significantly complement and improve some recent results on the existence of solutions for fourth-order elliptic equations and Kirchhoff type problems with critical growth.

1. Introduction

In this paper, we are interested in the existence of nontrivial solutions for the following fourth-order elliptic systems

$$(1.1)\qquad \begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = |u|^{2_*-2}u + F_u(x,u,v) & \text{in } \Omega,\\ -\Delta^2 v + M\left(\int_\Omega |\nabla v|^2\,dx\right)\Delta v = |v|^{2_*-2}u + F_v(x,u,v) & \text{in } \Omega,\\ u = \Delta u = 0,\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a smooth bounded domain, $2_* = \frac{2N}{N-4}$, $\Delta^2(\cdot) = \Delta(\Delta\cdot)$ is the biharmonic operator, $M : \mathbb{R}_0^+ := [0,+\infty) \to \mathbb{R}$ is a increasing and continuous function, $\nabla F = (F_u, F_v)$ is the gradient of a C^1-function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}_0^+$ with respect to the variable $w = (u,v) \in \mathbb{R}^2$. Let us assume throughout this paper that

$(\mathcal{M}_1)$ There exists $m_0 > 0$ such that

$$M(t) \geq m_0, \quad \forall t \in \mathbb{R}_0^+;$$

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($\mathcal{M}_2$) There exists $\sigma \in \left(\frac{2}{2_*}, 1\right]$ such that

$$\widehat{M}(t) \geq \sigma M(t)t, \quad \forall t \in \mathbb{R}_0^+,$$

where $\widehat{M}(t) = \int_0^t M(\tau)\, d\tau$.

We can see that there are many functions satisfying conditions ($\mathcal{M}_1$)-($\mathcal{M}_2$), for example $M(t) = m_0 + bt^{\frac{1}{\sigma}-1}$ with $\sigma \leq 1$, $m_0 > 0$ and $b \geq 0$. The energy functional corresponding to problem (1.1) is

$$\begin{aligned} J(u,v) = &\frac{1}{2}\int_\Omega |\Delta u|^2\, dx - \frac{1}{2}\int_\Omega |\Delta v|^2\, dx \\ &+ \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\, dx\right) - \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\, dx\right) \\ &- \frac{1}{2_*}\int_\Omega |u|^{2_*}\, dx - \frac{1}{2_*}\int_\Omega |v|^{2_*}\, dx - \int_\Omega F(x,u,v)\, dx \end{aligned}$$

which is strongly indefinite in the sense that J is unbounded from below and from above on any subspace of finite codimension. Problem (1.1) is related to extensible beam equations and stationary Berger plate equations. More precisely, Woinowsky-Krieger [30] studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where L is the length of the beam in the rest position, E is the Young modulus of the material, I is the cross-sectional moment of interia, ρ is the mass density, H is the tension in the rest position and A is the cross-sectional area. This model was proposed to modify the theory of the dynamic Euler-Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation of the gradient. In [3], Berger studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - \left(Q + \int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x, u_t, x), \tag{1.3}$$

which describes large deflection of plate, where the parameter Q describes in-plane forces applied to the plate and the function f represents transverse loads which may depend on the displacement u and the velocity u_t. Problem (1.1) is a generalization of the stationary problem associated with problem (1.2) in dimension one or problem (1.3) in dimension two. For important details about the physical motivation of equations (1.2) and (1.3), interested readers are referred to [2, 31].

In recent years, there have been many papers concerning elliptic equations with nonlocal terms. In [8–11, 24, 33], the authors have studied the existence and multiplicity of solutions for Kirchhoff type problems with subcritical or critical growth conditions, we refer to [7, 17] for further information about this type of problems. In [27], Wang and An considered the following fourth elliptic equation

$$
(1.4) \qquad \begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}_0^+ \to \mathbb{R}$ are continuous functions and f has subcritical growth. By assuming that M is bounded on $\mathbb{R}_0^+$ and the nonlinear term f satisfies the Ambrosetti-Rabinowitz type condition, Wang et al. obtained in [27] at least one nontrivial solution for problem (1.4) using the mountain pass theorem. Moreover, the authors also showed the existence at least two solutions in the case when f is asymptotically linear at infinity. After that, Wang et al. [28] studied problem (1.4) in the case when M is unbounded function, i.e., $M(t) = a + bt$, where $a > 0$, $b \geq 0$ by using the mountain pass techniques and the truncation method. Some extensions regarding these results can be found in [1, 13, 25] in which the authors considered problem (1.4) in $\mathbb{R}^N$. Relatively speaking, problem (1.4) with critical growth condition have rarely been considered, we refer to some interesting papers [5,15,26]. There, the authors have established the existence and multiplicity of solutions for the problem using variational methods combined with the concentration compactness principle due to Lions [21,22].

In this paper, we are interested in the existence of solutions for a class of noncooperative fourth-order elliptic systems involving nonlocal terms and critical growth. Unlike as in [5,15,26], our main tool used here is Limit index theory firstly introduced by Li [32] for local problems with subcritical growth condition in bounded domains. Huang et al. [16] developed the method of Li to noncooperative elliptic systems in $\mathbb{R}^N$ using the principle of symmetric criticality and it was also extended by Cai et al. [6] to the case when the energy functional may not locally Lipschitz continuous in Banach spaces. In [20], Lin et al. considered noncooperative elliptic systems with critical exponents of the form

$$
(1.5) \qquad \begin{cases} \Delta u = |u|^{2_*-2}u + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = |v|^{2_*-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}
$$

where Ω is an bounded domain in $\mathbb{R}^N$, $N \geq 5$ and $2_* = \frac{2N}{N-4}$. There, the authors established the existence of infinitely many solutions for problem (1.5) without using Concentration Compactness Principle. Some similar results for p-Laplacian or $p(x)$-Laplacian problems were obtained by Fang et al. [14] and S. Liang et al. [18, 19]. Motivated by the contribution cited above, we shall study the existence of solutions for (1.1). We can see that there are three main difficulties in considering our problem. Firstly, problem (1.1) involves nonlocal terms $M\left(\int_\Omega |\nabla u|^2\, dx\right)$ and $M\left(\int_\Omega |\nabla v|^2\, dx\right)$ which prevents us from applying the methods as before. The second difficulty is that the energy functional associated to the problem is strongly indefinite in the sense that it is neither unbounded from below or from above on any subspace of finite codimension. Therefore, one cannot apply the symmetric mountain pass theorem on the energy functional. Finally, one of our difficulties comes from the lack of compactness of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2_*}(\Omega)$. To overcome this difficulty, we use the Concentration Compactness Principle due to Lions [21, 22]. It is worth emphasizing that our situation here is different from those presented in the papers [12, 23, 33]. We believe that with the same arguments as presented in this paper, we can obtain some similar results for the problem involving the p-biharmonic operator $\Delta\left(|\Delta \cdot|^{p-2}\Delta\cdot\right)$.

In order to state the main results concerning problem (1.1), we introduce the following hypotheses

($\mathcal{F}_1$) $F(x,s,t) = F(x,-s,-t)$ for all $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$;
($\mathcal{F}_2$) $\lim_{|s|\to+\infty} \frac{F_s(x,s,t)}{|s|^{2_*-1}} = 0$ uniformly in $x \in \overline{\Omega}$ and $t \in \mathbb{R}$;
($\mathcal{F}_3$) $F_t(x,s,t)t \geq 0$ for all $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$;

Under assumptions ($\mathcal{F}_1$) and ($\mathcal{F}_2$), we have

$$F_s(x,s,t)s = o(|s|^{2_*}),$$

which means that, for all $\epsilon > 0$ and fixed t, there exist $a(\epsilon), b(\epsilon) > 0$ such that

$$|F(x,s,0)| \leq a(\epsilon) + \epsilon|s|^{2_*}, \ |F_s(x,s,t)s| \leq b(\epsilon) + \epsilon|s|^{2_*}, \ \forall (x,s) \in \overline{\Omega} \times \mathbb{R}. \tag{1.6}$$

Hence, together with condition (1.6) and the mean value theorem for the number σ in ($\mathcal{M}_2$) and fixed t we have

$$\left|F(x,s,0) - \frac{\sigma}{2}F_s(x,s,t)s\right| \leq c(\epsilon) + \epsilon|s|^{2_*}, \quad \forall (x,s) \in \overline{\Omega} \times \mathbb{R}, \tag{1.7}$$

for some $c(\epsilon) > 0$.

In this paper, we denote by $E = H^2(\Omega) \cap H_0^1(\Omega)$ the Hilbert space equipped with the inner product

$$\langle u, v\rangle_E = \int_\Omega (\Delta u \Delta v + \nabla u \cdot \nabla v)\, dx$$

and the norm

$$\|u\|_E = \left(\int_\Omega |\Delta u|^2 + |\nabla u|^2\, dx\right)^{\frac{1}{2}}, \quad u \in E.$$

We then have that E is continuously embedded into the Lebesgue space $L^r(\Omega)$ endowed the norm $|u|_r = \left(\int_\Omega |u|^r\, dx\right)^{\frac{1}{r}}$, $1 \leq r \leq 2_*$. Moreover, the embedding is compact if $1 \leq r < 2_*$. Denote by $C_r > 0$ the best constant for this embedding, that is,

$$C_r|u|_r \leq \|u\|_E, \quad \forall u \in E. \tag{1.8}$$

In particular, if S is the best constant for the embedding $E \hookrightarrow L^{2_*}(\Omega)$, then it is defined by the formula

$$S := \inf_{u \in E \setminus \{0\}} \frac{\int_\Omega \left(|\Delta u|^2 + |\nabla u|^2\right)\, dx}{\left(\int_\Omega |u|^{2_*}\, dx\right)^{\frac{2}{2_*}}}. \tag{1.9}$$

For the sake of notation, we shall denote $c(\epsilon) = \widetilde{C}$ throughout this paper if $\epsilon = \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right)$, where $c(\epsilon)$ is given by (1.7). In order to state the main result of the paper, we assume further that $F(x,s,t)$ also fulfills the following hypothesis.

($\mathcal{F}_4$) There exist $\mu > \frac{2}{\sigma}$, $L > 0$ (where L will be determined latter) and a constant

$$\xi < |\Omega|^{-1} \min\left\{0, \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right) S^{\frac{N}{4}} - \widetilde{C}|\Omega|\right\}$$

such that

$$F(x,s,t) \geq L|s|^\mu - \xi, \quad (x,s,t) \in \overline{\Omega} \times \mathbb{R}^2.$$

We shall seek solutions of problem (1.1) in the space $H = E \times E$ which is a Hilbert space under the inner product

$$\langle w_1, w_2 \rangle_H = \int_\Omega \left(\Delta u_1 \Delta u_2 + \Delta v_1 \Delta v_2 + \nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2\right)\, dx,$$

$w_i = (u_i, v_i)$, $i = 1, 2$ and the norm

$$\|w\|_H = \|u\|_E + \|v\|_E, \quad w = (u,v) \in H.$$

Definition 1.1. We say that $w = (u,v) \in H$ is a weak solution of problem (1.1) if it holds that

$$\begin{aligned}
&\int_\Omega \Delta u \Delta \varphi\, dx - \int_\Omega \Delta v \Delta \psi\, dx + M\left(\int_\Omega |\nabla u|^2\, dx\right) \int_\Omega \nabla u \cdot \nabla \varphi\, dx \\
&- M\left(\int_\Omega |\nabla v|^2\, dx\right) \int_\Omega \nabla v \cdot \nabla \psi\, dx - \int_\Omega |u|^{2_*-2} u \varphi\, dx - \int_\Omega |v|^{2_*-2} v \psi\, dx \\
&- \int_\Omega \left(F_u(x,u,v)\varphi + F_v(x,u,v)\psi\right)\, dx = 0
\end{aligned}$$

for all $(\varphi, \psi) \in X$.

Theorem 1.2. *Assume that the functions M and F satisfy the conditions ($\mathcal{M}_1$)-($\mathcal{M}_2$) and ($\mathcal{F}_1$)-($\mathcal{F}_4$). Then there exists an integer $k_0 > 1$ such that problem* (1.1) *has at least $k_0 - 1$ pairs nontrivial weak solutions.*

In the rest of this section, we consider problem (1.1) in the special case $M(t) = a + bt$, $t \in \mathbb{R}_0^+$, $a > 0$ and $b \geq 0$. Then, the problem becomes

$$\begin{cases} \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2\, dx\right) \Delta u = |u|^{2_*-2} u + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta^2 v + \left(a + b\int_\Omega |\nabla v|^2\, dx\right) \Delta v = |v|^{2_*-2} u + F_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = 0,\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$ $(N \in \{5,6,7\})$ is a smooth bounded domain and $2_* = \frac{2N}{N-4}$. A function $w = (u,v) \in H$ is said to be a weak solution of problem (1.10) if it holds that

$$\int_\Omega \Delta u \Delta \varphi \, dx - \int_\Omega \Delta v \Delta \psi \, dx + \left(a + b\int_\Omega |\nabla u|^2 \, dx\right) \int_\Omega \nabla u \cdot \nabla \varphi \, dx$$
$$- \left(a + b\int_\Omega |\nabla v|^2 \, dx\right) \int_\Omega \nabla v \cdot \nabla \psi \, dx - \int_\Omega |u|^{2_*-2} u\varphi \, dx - \int_\Omega |v|^{2_*-2} v\psi \, dx$$
$$- \int_\Omega \left(F_u(x,u,v)\varphi + F_v(x,u,v)\psi\right) \, dx = 0$$

for all $(\varphi, \psi) \in X$. In relation (1.7), we consider $\epsilon = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2_*}\right) > 0$ since $N \in \{5,6,7\}$ and set $\widehat{C} = c(\epsilon)$. From Theorem 1.2 we obtain a multiplicity result for (1.10) as follows.

Corollary 1.3. *Assume that the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions $(\mathcal{F}_1)$-$(\mathcal{F}_3)$ and*

$(\mathcal{F}_4')$ *There exist $\mu > 4$, $L > 0$ (where L will be determined latter) and a constant*

$$\xi < |\Omega|^{-1} \min\left\{0, \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2_*}\right) S^{\frac{N}{4}} - \widehat{C}|\Omega|\right\}$$

such that

$$F(x,s,t) \geq L|s|^\mu - \xi, \quad (x,s,t) \in \overline{\Omega} \times \mathbb{R}^2.$$

Then there exists an integer $k_0 > 1$ such that problem (1.10) has at least $k_0 - 1$ pairs nontrivial weak solutions.

2. Preliminaries

In this section, we shall recall the Limit Index Theory due to [32]. In order to do that, let us first introduce the following definitions, the interested readers can easily refer to the book due to Willem [29].

Definition 2.1. The action of a topological group G on a normed space $(Z, \|\cdot\|)$ is a continuous map $G \times Z \to Z : [g,z] \mapsto gz$ such that

$$1 \cdot z = z, \quad (gh)z = g(hz), \quad z \mapsto gz \text{ is linear}, \quad \forall g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \quad \forall g \in G, \ z \in Z,$$

and in this case Z is called a G-space.

The set of invariant points is defined by

$$\mathrm{Fix}(G) := \{z \in Z : \ gz = z, \ \forall g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to \mathbb{R}$ is invariant if $\varphi(gz) = \varphi(z)$ for every $g \in G$ and $z \in Z$. A map $f : Z \to Z$

is equivariant if $f(gz) = g(fz)$ for every $g \in G$ and $z \in Z$. Suppose Z is a G-Banach space, that is, there is a G-isometric action on Z. Let

$$\sum := \{A \subset Z : \ A \text{ is closed and } gA = A, \ \forall g \in G\}$$

be a family of all G-invariant closed subset of Z, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : \ h(gz) = g(hz), \ \forall g \in G\right\}$$

be the class of all G-equivariant mapping of Z. Finally, we call the set

$$O(z) := \{gz : \ g \in G\}$$

a G-orbit of z.

Definition 2.2. An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathbb{Z}_+ \cup \{+\infty\}$ (where $\mathbb{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \theta$;
(2) (Monotonicity) $A \subset B \Rightarrow i(A) \leq i(B)$;
(3) (Subadditivity) $i(A \cup B) \leq i(A) + i(B)$;
(4) (Supervariance) $i(A) \leq i(\overline{h(A)})$ for all $h \in \Gamma$;
(5) (Continuity) If A is compact and $A \cap \mathrm{Fix}(G) = 0$, then $i(A) < +\infty$ and there is a G-invariant neighbourhood N of A such that $i(\overline{N}) = i(A)$;
(6) (Normalization) If $x \notin \mathrm{Fix}(G)$, then $i(O(x)) = 1$.

Definition 2.3. An index theory is said to satisfy the d-dimension property if there is a positive integer d such that

$$i\left(V^{dk} \cap S_1(0)\right) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}(G) = \{0\}$, where $S_1(0)$ is the unit sphere in Z.

Suppose U and V are G-invariant closed subspaces of Z such that $Z = U \oplus V$, where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspaces of V, $j = 1, 2, \dots$, and $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$. Let

$$Z_j = U \oplus V_j$$

and for all $A \in \Sigma$, let

$$A_j = A \cap Z_j.$$

Definition 2.4. Let i be an index theory satisfying the d-dimension property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \sum \to \mathbb{Z} \cup \{-\infty; +\infty\}$$

given by $i^\infty(A) = \limsup_{j\to\infty}(i(A_j) - n_j)$.

Proposition 2.5. *Let $A, B \in \Sigma$. Then i^∞ satisfies*:

(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) (*Monotonicity*) $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) (*Subadditivity*) $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If $V \cap Fix(G) = \{0\}$, then $i^\infty(S_\rho(0) \cap V) = 0$, where $S_\rho(0) = \{z \in Z : \|z\| = \rho\}$*;
(5) *If Y_0 and $\widetilde{Y_0}$ are G-invariant closed subspaces of V such that $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ for some j_0 and $\dim \widetilde{Y_0} = dm$, then $i^\infty(S_\rho(0) \cap Y_0) \ge -m$.*

Definition 2.6. A functional $J \in C^1(Z, \mathbb{R})$ is said to satisfy the condition $(PS)_c^*$ with respect to (Z_n) if any sequence $\{z_{n_k}\} \subset Z$, $z_{n_k} \in Z_{n_k}$ such that

$$J_{n_k}(z_{n_k}) \to c, \quad J'_{n_k}(z_{n_k}) \to 0 \text{ as } k \to \infty,$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimension subspace of Z as in Definition 2.3 and $J_{n_k} = J|_{Z_{n_k}}$.

Proposition 2.7 (see [32]). *Assume that*

(B_1) *$J \in C^1(Z, \mathbb{R})$ is G-invariant*;
(B_2) *There exist G-invariant closed subspaces U and V such that V is infinite dimensional and $Z = U \oplus V$*;
(B_3) *There exists a sequence of G-invariant finite-dimensional subspaces $V_1 \subset V_2 \subset \cdots V_j \subset \cdots$, $\dim V_j = dn_j$, such that $V = \overline{\bigcup_{j=1}^\infty V_j}$*;
(B_4) *There exists an index theory i on Z satisfying the d-dimension property*;
(B_5) *There exist G-invariant subspaces $Y_0, \widetilde{Y_0}, Y_1$ of V such that $V = Y_0 \oplus \widetilde{Y_0}$, Y_1, $\widetilde{Y_0} \subset V_{j_0}$ for some j_0 and $\dim \widetilde{Y_0} = dm < dk = \dim Y_1$*;
(B_6) *There exist α and β, $\alpha < \beta$ such that J satisfies $(PS)_c^*$ for all $c \in [\alpha, \beta]$.*
(B_7) *It holds that*

$$\begin{cases} \text{(a) } \textit{either } Fix(G) \subset U \oplus Y_1 \textit{ or } Fix(G) \cap V = \{0\}, \\ \text{(b) } \textit{there is } \rho > 0 \textit{ such that for all } z \in Y_0 \cap S_\rho(0), \textit{ we have } J(z) \ge \alpha, \\ \text{(c) } \textit{for all } z \in U \oplus Y_1, \textit{ we have } J(z) \le \beta. \end{cases}$$

If i^∞ is the limit index corresponding to i, then the numbers

$$c_j := \inf_{i^\infty(A) \ge j} \sup_{z \in A} J(u), \quad -k+1 \le j \le -m,$$

are critical values of J, and $\alpha \le c_{-k+1} \le \cdots \le c_{-m} \le \beta$. Moreover, if $c = c_l = \cdots = c_{l+r}$, $r \ge 0$, then $i(\mathbb{K}_c) \ge r+1$, where $\mathbb{K}_c = \{z \in Z : J'(z) = 0, J(z) = c\}$.

3. Proof of the main result

In this section, we shall prove Theorem 1.2 using Proposition 2.7. Throughout this section, we denote by c_i general positive real number whose value may change from line to line. First, we recall the following useful result, the reader can consult its proof in [16, 29].

Lemma 3.1. *Assume* $1 \leq \theta_1, \theta_2, \theta < +\infty$, $f \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *and*

$$f(x,s,t) \leq C\left(|s|^{\frac{\theta_1}{\theta}} + |t|^{\frac{\theta_1}{\theta}}\right), \quad \forall (x,s,t) \in \overline{\Omega} \times \mathbb{R}^2, \quad C > 0.$$

Then, for every $(u,v) \in L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$, *we have* $f(\cdot,u,v) \in L^{\theta}(\Omega)$ *and the operator* $T : (u,v) \mapsto f(x,u,v)$ *is a continuous map from* $L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$ *to* $L^{\theta}(\Omega)$.

Now, we turn to prove Theorem 1.2. In order to apply Proposition 2.7, let us denote $E = H^2(\Omega) \cap H_0^1(\Omega)$ and an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for E which are characterized by the relations $\langle e_i, e_j \rangle = \delta_{ij}$, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Moreover, we define

$$H = U \oplus V, \quad U = \{0\} \times E, \quad V = E \times \{0\},$$
$$Y_0 = E_1^{\perp} \times \{0\}, \quad V = Y_0 \oplus \widetilde{Y}_0,$$
$$Y_1 = E_{k_0} \times \{0\}, \quad E_{k_0} = span\{e_1, \ldots, e_{k_0}\},$$

then $\dim(\widetilde{Y}_0) = 1$, $\dim(Y_1) = k_0$.

Define a group action $G = \{1, \tau\} \cong \mathbb{Z}_2$ by setting $\tau(u,v) = (-u,-v)$, then $\mathrm{Fix}(G) = \{0\} \times \{0\}$ (also denote by $\{0\}$). It is clear that U and V are G-invariant closed subspaces of X, and Y_0, $\widetilde{Y}_0$ and Y_1 are G-invariant subspace of V. Set

$$\sum = \{A \subset H \backslash \{0\} : \ A \text{ is closed in } H \text{ and } (u,v) \in A \Rightarrow (-u,-v) \in A\}.$$

Define an index γ on $\sum$ by

$$\gamma(A) = \begin{cases} \min\left\{N \in \mathbb{Z} : \exists h \in C(A, \mathbb{R}^N \backslash \{0\}) \text{ such that } h(-u,-v) = h(u,v)\right\}, \\ 0, \quad \text{if } A = \emptyset, \\ +\infty, \quad \text{if such } h \text{ does not exist.} \end{cases}$$

From [16], we deduce that γ is an index satisfying the properties given in Definition 2.2. Moreover, γ satisfies the one-dimension property. According to Definition 2.4 we can obtain a limit index γ^{∞} with respect to (H_n) from γ.

As we stated at the beginning of the paper, in order to prove the main result, let us define the functional $J : H \to \mathbb{R}$ by

$$\begin{aligned} J(w) = & \frac{1}{2}\int_{\Omega} |\Delta u|^2 \, dx - \frac{1}{2}\int_{\Omega} |\Delta v|^2 \, dx \\ & + \frac{1}{2}\widehat{M}\left(\int_{\Omega} |\nabla u|^2 \, dx\right) - \frac{1}{2}\widehat{M}\left(\int_{\Omega} |\nabla v|^2 \, dx\right) \\ & - \frac{1}{2_*}\int_{\Omega} |u|^{2_*} \, dx - \frac{1}{2_*}\int_{\Omega} |v|^{2_*} \, dx - \int_{\Omega} F(x,u,v) \, dx, \ w = (u,v) \in H, \end{aligned}$$

we then obtain that $J \in C^1(H, \mathbb{R})$ using Lemma 3.1 and its derivative is given by

$$J'(u,v)(\varphi, \psi) = \int_{\Omega} \Delta u \Delta \varphi \, dx - \int_{\Omega} \Delta v \Delta \psi \, dx$$

$$+ M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u\cdot\nabla\varphi\,dx$$
$$- M\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega \nabla v\cdot\nabla\psi\,dx$$
$$- \int_\Omega |u|^{2_*-2}u\varphi\,dx - \int_\Omega |v|^{2_*-2}v\psi\,dx$$
$$- \int_\Omega \left(F_u(x,u,v)\varphi + F_v(x,u,v)\psi\right)\,dx$$

for all $(u,v),(\varphi,\psi)\in H$. Moreover, weak solutions of problem (1.1) are exactly the critical points of the functional J.

Lemma 3.2. *Let* $(\mathcal{M}_1)$-$(\mathcal{M}_2)$ *and* $(\mathcal{F}_1)$-$(\mathcal{F}_3)$ *hold. Then the functional* J *satisfies the local* $(PS)_c^*$ *with*

$$c\in\left(-\infty,\frac{1}{2}\left(\frac{\sigma}{2}-\frac{1}{2_*}\right)S^{\frac{N}{4}}-\widetilde{C}|\Omega|\right)$$

in the following sense: if $\{w_{n_k}\}\subset H$ *is a sequence such that* $w_{n_k}=(u_{n_k},v_{n_k})\in H_{n_k}$ *and*

$$J_{n_k}(u_{n_k},v_{n_k})\to c,\quad J'_{n_k}(u_{n_k},v_{n_k})\to 0\quad \text{as } k\to\infty, \tag{3.1}$$

where $J_{n_k}=J|_{H_{n_k}}$ *with* $H_{n_k}=E\times E_{n_k}$. *Then* $\{(u_{n_k},v_{n_k})\}$ *possesses a subsequence which converges strongly in* H *to a critical point of the functional* J.

Proof. We first show that $\{w_{n_k}\}=\{(u_{n_k},v_{n_k})\}$ is bounded in H. Indeed, note that by relation (3.1), conditions $(\mathcal{M}_1)$ and $(\mathcal{F}_3)$, we have

$$\begin{aligned} o_k(1)\|v_{n_k}\|_E &\geq \left\langle -J'_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\right\rangle\\ &= \int_\Omega |\Delta v_{n_k}|^2\,dx + M\left(\int_\Omega |\nabla v_{n_k}|^2\,dx\right)\int_\Omega |\nabla v_{n_k}|^2\,dx\\ &\qquad + \int_\Omega |v_{n_k}|^{2_*}\,dx + \int_\Omega F_v(x,u_{n_k},v_{n_k})v_{n_k}\,dx\\ &\geq \min\{1,m_0\}\|v_{n_k}\|_E^2. \end{aligned} \tag{3.2}$$

From relation (3.2), it follows that $\|v_{n_k}\|_E$ is bounded. On the other hand, by relations (1.7), (3.1) and conditions $(\mathcal{M}_1)$-$(\mathcal{M}_2)$, we deduce that

$$\begin{aligned} &c + o_k(1)\|u_{n_k}\|_E\\ &\geq J_{n_k}(u_{n_k},0) - \frac{\sigma}{2}\left\langle J'_{n_k}(u_{n_k},v_{n_k}),(u_{n_k},0)\right\rangle\\ &= \frac{1}{2}\int_\Omega |\Delta u_{n_k}|^2\,dx + \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right) - \frac{1}{2_*}\int_\Omega |u_{n_k}|^{2_*}\,dx\\ &\quad - \int_\Omega F(x,u_{n_k},0)\,dx - \frac{\sigma}{2}\int_\Omega |\Delta u_{n_k}|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\sigma}{2} M \left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) \int_\Omega |\nabla u_{n_k}|^2 \, dx \\
&\quad + \frac{\sigma}{2} \int_\Omega |u_{n_k}|^{2_*} \, dx + \frac{\sigma}{2} \int_\Omega F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \, dx \\
&= \left(\frac{1}{2} - \frac{\sigma}{2} \right) \int_\Omega |\Delta u_{n_k}|^2 \, dx + \frac{1}{2} \widehat{M} \left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) \\
&\quad - \frac{\sigma}{2} M \left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) \int_\Omega |\nabla u_{n_k}|^2 \, dx + \left(\frac{\sigma}{2} - \frac{1}{2_*} \right) \int_\Omega |u_{n_k}|^{2_*} \, dx \\
&\quad - \int_\Omega \left(F(x, u_{n_k}, 0) - \frac{1}{2_*} F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \right) dx \\
&\geq \left(\frac{\sigma}{2} - \frac{1}{2_*} \right) \int_\Omega |u_{n_k}|^{2_*} \, dx - \int_\Omega \left(F(x, u_{n_k}, 0) - \frac{1}{2_*} F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \right) dx \\
&\geq \left(\frac{\sigma}{2} - \frac{1}{2_*} \right) \int_\Omega |u_{n_k}|^{2_*} \, dx - \int_\Omega \left(c(\epsilon) + \epsilon |u_{n_k}|^{2_*} \right) dx \\
&\leq \left[\frac{\sigma}{2} - \frac{1}{2_*} - \epsilon \right] \int_\Omega |u_{n_k}|^{2_*} \, dx - c(\epsilon) |\Omega|,
\end{aligned}
$$

which yields

$$
\left[\frac{\sigma}{2} - \frac{1}{2_*} - \epsilon \right] \int_\Omega |u_{n_k}|^{2_*} \, dx \leq c(\epsilon)|\Omega| + c + o_k(1) \|u_{n_k}\|_E, \tag{3.3}
$$

where $|\cdot|$ denote by Lebesgue measure. Setting $\epsilon = \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right)$, we get from (3.3) that

$$
\int_\Omega |u_{n_k}|^{2_*} \, dx \leq c_1 + o_k(1) \|u_{n_k}\|_E, \tag{3.4}
$$

where $o_k(1) \to 0$ and $c_1 > 0$. On the other hand, by (1.6), (3.1) conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ we have

$$
\begin{aligned}
c + o_k(1) \|u_{n_k}\| &= J(u_{n_k}, 0) \\
&= \frac{1}{2} \int_\Omega |\Delta u_{n_k}|^2 \, dx + \frac{1}{2} \widehat{M} \left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) - \frac{1}{2_*} \int_\Omega |u_{n_k}|^{2_*} \, dx \\
&\qquad - \int_\Omega F(x, u_{n_k}, 0) \, dx \\
&\geq \frac{1}{2} \int_\Omega |\Delta u_{n_k}|^2 \, dx + \frac{m_0 \sigma}{2} \int_\Omega |\nabla u_{n_k}|^2 \, dx - \frac{1}{2_*} \int_\Omega |u_{n_k}|^{2_*} \, dx \\
&\qquad - \int_\Omega F(x, u_{n_k}, 0) \, dx \\
&\geq \frac{1}{2} \min\{1, m_0 \sigma\} \|u_{n_k}\|_E^2 - \left(\frac{1}{2_*} + \epsilon \right) \int_\Omega |u_{n_k}|^{2_*} \, dx - a(\epsilon) |\Omega|. \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5), it implies that $\{u_{n_k}\}$ is bounded in E. Hence, $\|w_{n_k}\|_H = \|u_{n_k}\|_E + \|v_{n_k}\|_E$ is bounded.

Next, we prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in H. We note that $\{v_{n_k}\}$ is bounded in E. Hence, up to a subsequence, $v_{n_k} \rightharpoonup v$ weakly in E and $v_n(x) \to v(x)$ a.e. in Ω. We claim that $v_{n_k} \to v$ strongly in E. In fact, using relation (3.1) and conditions $(\mathcal{M}_1)$, $(\mathcal{F}_3)$, we have

$$\begin{aligned} o_k(1) = &\ \left\langle -J'_{n_k}(u_{n_k}, v_{n_k} - v), (0, v_{n_k} - v)\right\rangle \\ = &\ \int_\Omega |\Delta(v_{n_k} - v)|^2\,dx + M\left(\int_\Omega |\nabla(v_{n_k} - v)|^2\,dx\right)\int_\Omega |\nabla(v_{n_k} - v)|^2\,dx \\ &+ \int_\Omega |v_{n_k} - v|^{2_*}\,dx + \int_\Omega F_v(x, u_{n_k}, v_{n_k} - v)(v_{n_k} - v)\,dx \\ \geq &\ m_0\|v_{n_k} - v\|_E^2, \end{aligned}$$

which implies that $v_{n_k} \to v$ strongly in E. In the following, we shall prove that there exists $u \in E$ such that $u_{n_k} \to u$ strongly in E.

We know that $\{u_{n_k}\}$ is also bounded in E. Hence, up to a subsequence, we may assume that $u_{n_k} \rightharpoonup u$ in E, $u_{n_k} \to u$ strongly in $L^s(\Omega)$ for $1 \leq s < 2_*$ and $u_{n_k}(x) \to u(x)$ a.e. $x \in \Omega$. Using the Concentration Compactness Principle due to Lions [21, 22], there exist bounded nonnegative measures ν, μ and γ on $\mathbb{R}^N$ and some at most countable index set Λ, sequences $(x_j)_{j\in\Lambda} \subset \overline{\Omega}$, $(\nu_j)_{j\in\Lambda}$, $(\mu_j)_{j\in\Lambda}$ and $(\gamma_j)_{j\in\Lambda}$ in $[0, +\infty)$ such that

$$|u_{n_k}|^{2_*} \rightharpoonup \nu = |u|^{2_*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j}, \tag{3.6}$$

$$|\Delta u_{n_k}|^2 \rightharpoonup \mu \geq |\Delta u|^2 + \sum_{j\in\Lambda} \mu_j \delta_{x_j}, \quad |\nabla u_{n_k}|^2 \rightharpoonup \gamma \geq |\nabla u|^2 + \sum_{j\in\Lambda} \gamma_j \delta_{x_j}, \tag{3.7}$$

$$\nu_j^{\frac{2}{2_*}} \leq \frac{\mu_j}{S} \tag{3.8}$$

for all $j \in \Lambda$, where δ_{x_j} is the Dirac mass at $x_j \in \overline{\Omega}$, where S is given by (1.9).

Consider $\phi \in C_0^\infty(\Omega, [0, 1])$ such that $\phi \equiv 1$ on $B_1(0)$, $\phi \equiv 0$ on $\Omega\backslash B_2(0)$, $|\nabla\phi|_\infty \leq 2$ and $|\Delta\phi|_\infty \leq 2$. For each $j \in \Lambda$ and $\epsilon > 0$, let us define $\phi_{j,\epsilon} = \phi\left(\frac{x - x_j}{\epsilon}\right)$, we have that $\{u_{n_k}\phi_{j,\epsilon}\}$ is bounded in the space E, it then follows from (3.1) that $J'_{n_k}(u_{n_k}, v_{n_k})(u_{n_k}\phi_{j,\epsilon}, 0) \to 0$ as $k \to \infty$, that is,

$$\begin{aligned} J'_{n_k}(u_{n_k}, v_{n_k})(u_{n_k}\phi_{j,\epsilon}, 0) = &\ \int_\Omega \Delta u_{n_k}\Delta(u_{n_k}\phi_{j,\epsilon})\,dx \\ &+ M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega \nabla u_{n_k}\cdot\nabla(u_{n_k}\phi_{j,\epsilon})\,dx \\ &- \int_\Omega |u_{n_k}|^{2_*-2}u_{n_k}(u_{n_k}\phi_{j,\epsilon})\,dx \\ &- \int_\Omega F_u(x, u_{n_k}, v_{n_k})(u_{n_k}\phi_{j,\epsilon})\,dx \to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.9}$$

It is noticed that

$$
\begin{aligned}
\Delta(u_{n_k}\phi_{j,\epsilon}) &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}(u_{n_k}\phi_{j,\epsilon}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u_{n_k}}{\partial x_i}\phi_{j,\epsilon} + u_{n_k}\frac{\partial \phi_{j,\epsilon}}{\partial x_i}\right) \\
&= \sum_{i=1}^{n}\left[\frac{\partial^2 u_{n_k}}{\partial^2 x_i}\phi_{j,\epsilon} + \frac{\partial u_{n_k}}{\partial x_i}.\frac{\partial \phi_{j,\epsilon}}{\partial x_i} + \frac{\partial u_{n_k}}{\partial x_i}.\frac{\partial \phi_{j,\epsilon}}{\partial x_i} + u_{n_k}.\frac{\partial^2 \phi_{j,\epsilon}}{\partial^2 x_i}\right] \\
&= \sum_{i=1}^{n}\left[\frac{\partial^2 u_{n_k}}{\partial^2 x_i}\phi_{j,\epsilon} + 2\frac{\partial u_{n_k}}{\partial x_i}.\frac{\partial \phi_{j,\epsilon}}{\partial x_i} + u_{n_k}.\frac{\partial^2 \phi_{j,\epsilon}}{\partial^2 x_i}\right] \\
&= \Delta u_{n_k}\phi_{j,\epsilon} + 2\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon} + u_{n_k}\Delta\phi_{j,\epsilon}.
\end{aligned}
$$

Hence, relation (3.9) gives us

$$
\begin{aligned}
&\int_\Omega (u_{n_k}\Delta u_{n_k}\Delta\phi_{j,\epsilon} + 2\Delta u_{n_k}(\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon}))\,dx \\
&+ M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega u_{n_k}\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon}\,dx \\
= &-\int_\Omega |\Delta u_{n_k}|^2\phi_{j,\epsilon}\,dx - M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega |\nabla u_{n_k}|^2\phi_{j,\epsilon}\,dx \\
&+ \int_\Omega |u_{n_k}|^{2_*}\phi_{j,\epsilon}\,dx + \int_\Omega F_u(x,u_{n_k},v_{n_k})u_{n_k}\phi_{j,\epsilon}\,dx + o_k(1).
\end{aligned}
\tag{3.10}
$$

First, using the Hölder inequality and the boundedness of the sequence $\{u_{n_k}\}$ in E, we deduce that

$$
\begin{aligned}
&\left|\int_\Omega u_{n_k}\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon}\,dx\right| \\
\leq\ &\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}||u_{n_k}||\nabla\phi_{j,\epsilon}|\,dx \\
\leq\ &\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^2|\nabla\phi_{j,\epsilon}|^2\,dx\right)^{\frac{1}{2}} \\
\leq\ &c_2\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^2|\nabla\phi_{j,\epsilon}|^2\,dx\right)^{\frac{1}{2}} \\
\leq\ &c_2\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{2N}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla\phi_{j,\epsilon}|^N\,dx\right)^{\frac{1}{N}} \\
\leq\ &c_3\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{2N}} \to 0 \quad \text{as } k\to\infty, \quad \epsilon\to 0.
\end{aligned}
\tag{3.11}
$$

Since $\{u_{n_k}\}$ is bounded in E, we may assume that $\int_\Omega |\nabla u_{n_k}|^2\,dx \to t_1 \geq 0$ as $n \to \infty$. Observing that $M(t)$ is continuous, we then have

$$M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right) \to M(t_1) \geq m_0 > 0 \quad \text{as } k \to \infty.$$

Hence, by (3.11),

$$M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega u_{n_k}\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon}\,dx \to 0 \quad \text{as } k\to\infty,\ \epsilon\to 0. \tag{3.12}$$

Similarly, we also have

$$\begin{aligned}
&\left|\int_\Omega u_{n_k}\Delta u_{n_k}\Delta\phi_{j,\epsilon}\,dx\right| \\
&= \left|\int_{B_{2\epsilon}(x_j)\cap\Omega} \Delta u_{n_k}(u_{n_k}\Delta\phi_{j,\epsilon})\,dx\right| \\
&\leq \int_{B_{2\epsilon}(x_j)\cap\Omega} |\Delta u_{n_k}||u_{n_k}||\Delta\phi_{j,\epsilon}|\,dx \\
&\leq \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\Delta u_{n_k}|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^2|\Delta\phi_{j,\epsilon}|^2\,dx\right)^{\frac{1}{2}} \\
&\leq c_4\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^2|\Delta\phi_{j,\epsilon}|^2\,dx\right)^{\frac{1}{2}} \\
&\leq c_4\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^{2_*}\,dx\right)^{\frac{1}{2_*}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\Delta\phi_{j,\epsilon}|^{\frac{N}{2}}\,dx\right)^{\frac{2}{N}} \\
&\leq c_5\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |u_{n_k}|^{2_*}\,dx\right)^{\frac{1}{2_*}} \to 0 \quad \text{as } k\to\infty,\ \epsilon\to 0
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\left|\int_\Omega \Delta u_{n_k}(\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon})\,dx\right| \\
&= \left|\int_{B_{2\epsilon}(x_j)\cap\Omega} \Delta u_{n_k}(\nabla u_{n_k}\cdot\nabla\phi_{j,\epsilon})\,dx\right| \\
&\leq \int_{B_{2\epsilon}(x_j)\cap\Omega} |\Delta u_{n_k}||\nabla u_{n_k}||\nabla\phi_{j,\epsilon}|\,dx \\
&\leq \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\Delta u_{n_k}|^p\,dx\right)^{\frac{1}{2}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}|^2|\nabla\phi_{j,\epsilon}|^2\,dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_6 \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}|^2 |\nabla \phi_{j,\epsilon}|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq c_6 \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla \phi_{j,\epsilon}|^N \, dx \right)^{\frac{1}{N}} \\
&\leq c_7 \left(\int_{B_{2\epsilon}(x_j)\cap\Omega} |\nabla u_{n_k}|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \to 0 \quad \text{as } k\to\infty,\ \epsilon\to 0. \qquad (3.14)
\end{aligned}
$$

On the other hand, by the compactness lemma of Strauss, the boundedness of $\{u_n\}$ in E and Sobolev embedding, it follows that

$$
\int_\Omega F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \phi_{j,\epsilon} \, dx = 0 \quad \text{as } k\to\infty,\ \epsilon\to 0. \qquad (3.15)
$$

By relations (3.12)-(3.15), letting $k\to\infty$ in (3.10), we deduce that

$$
\int_\Omega \phi_{j,\epsilon} \, d\mu \leq \int_\Omega \phi_{j,\epsilon} \, d\mu + m_0 \int_\Omega \phi_{j,\epsilon} \, d\gamma \leq \int_\Omega \phi_{j,\epsilon} \, d\nu + o_\epsilon(1).
$$

Letting $\epsilon\to 0$ and using the standard theory of Radon measures, we conclude that $\nu_j \geq \mu_j$. Using (3.8) we have

$$
S\nu_j^{\frac{2}{2_*}} \leq \mu_j \leq \nu_j,
$$

which implies that

$$
\nu_j = 0 \text{ or } \nu_j \geq S^{\frac{N}{4}} \text{ for all } j\in\Lambda. \qquad (3.16)
$$

From the conditions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$ and relations (1.7), (3.1), we get

$$
\begin{aligned}
o_k(1) + c &= J_{n_k}(u_{n_k}, 0) - \frac{\sigma}{2} J'_{n_k}(u_{n_k}, v_{n_k})(u_{n_k}, 0) \\
&= \frac{1}{2}\int_\Omega |\Delta u_{n_k}|^2 \, dx + \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) \\
&\quad - \frac{1}{2_*}\int_\Omega |u_{n_k}|^{2_*} \, dx - \int_\Omega F(x, u_{n_k}, 0) \, dx \\
&\quad - \frac{\sigma}{2}\int_\Omega |\Delta u_{n_k}|^2 \, dx - \frac{\sigma}{2} M\left(\int_\Omega |\nabla u_{n_k}|^2 \, dx \right) \int_\Omega |\nabla u_{n_k}|^2 \, dx \\
&\quad + \frac{\sigma}{2}\int_\Omega |u_{n_k}|^{2_*} \, dx + \frac{\sigma}{2}\int_\Omega F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \, dx \\
&\geq \left(\frac{\sigma}{2} - \frac{1}{2_*} \right) \int_\Omega |u_{n_k}|^{2_*} \, dx \\
&\quad - \int_\Omega \left(F(x, u_{n_k}, 0) - \frac{\sigma}{2} F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \right) dx \\
&\geq \left(\frac{\sigma}{2} - \frac{1}{2_*} \right) \int_\Omega |u_{n_k}|^{2_*} \, dx - \int_\Omega \left(c(\epsilon) + \epsilon |u_{n_k}|^{2_*} \right) dx
\end{aligned}
$$

$$\geq \left[\frac{\sigma}{2} - \frac{1}{2_*} - \epsilon\right] \int_\Omega |u_{n_k}|^{2_*}\, dx - c(\epsilon)|\Omega|$$

$$\geq \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right) \int_\Omega |u_{n_k}|^{2_*}\, dx - \widetilde{C}|\Omega|, \tag{3.17}$$

where $\epsilon = \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right)$ and $\widetilde{C} > 0$ is given by the hypothesis $(\mathcal{F}_4)$. Letting $k \to \infty$ in (3.17), we get

$$c \geq \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right) \lim_{k\to\infty} \int_\Omega |u_{n_k}|^{2_*}\, dx - \widetilde{C}|\Omega|. \tag{3.18}$$

Using (3.6), it implies that

$$\lim_{k\to\infty} \int_\Omega |u_{n_k}|^{2_*}\, dx = \int_\Omega |u|^{2_*}\, dx + \sum_{j\in\Lambda} \nu_j \geq \nu_j, \quad \forall j \in \Lambda,$$

if $\nu_s > 0$ for some $s \in \Lambda$, we deduce from relations (3.16) and (3.18) that

$$c \geq \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right) S^{\frac{N}{4}} - \widetilde{C}|\Omega|,$$

which is an absurd. This leads to the fact that $\nu_j = 0$ for any $j \in \Lambda$ and

$$\lim_{k\to\infty} \int_\Omega |u_{n_k}|^{2_*}\, dx = \int_\Omega |u|^{2_*}\, dx \tag{3.19}$$

and by the Brezis-Lieb lemma [4], the sequence $\{u_{n_k}\}_k$ converges strongly to u in $L^{2_*}(\Omega)$. For this reason, by the Hölder inequality we deduce that

$$\left| \int_\Omega \left(|u_{n_k}|^{2_*-2} u_{n_k} - |u|^{2_*-2} u\right)(u_{n_k} - u)\, dx \right|$$

$$\leq \int_\Omega \left(|u_{n_k}|^{2_*-1} + |u|^{2_*-1}\right) |u_{n_k} - u|\, dx$$

$$\leq \left(|u_{n_k}|_{2_*}^{2_*-1} + |u|_{2_*}^{2_*-1}\right) |u_{n_k} - u|_{2_*} \to 0 \quad \text{as } k \to \infty \tag{3.20}$$

and

$$\left| \int_\Omega \left(F_u(x, u_{n_k}, v_{n_k}) - F_u(x, u, 0)\right)(u_{n_k} - u)\, dx \right|$$

$$\leq \int_\Omega \left(|F_u(x, u_{n_k}, v_{n_k})| + |F_u(x, u, 0)|\right) |u_{n_k} - u|\, dx$$

$$\leq c_8 \int_\Omega (1 + |u_{n_k}|^{2_*-1} + |u|^{2_*-1}) |u_{n_k} - u|\, dx$$

$$\leq c_8 \left(|\Omega|^{\frac{2_*-1}{2_*}} + |u_{n_k}|_{2_*}^{2_*-1} + |u|_{2_*}^{2_*-1}\right) |u_{n_k} - u|_{2_*} \to 0 \quad \text{as } k \to \infty. \tag{3.21}$$

Since the sequence $\{u_{n_k}\}$ converges weakly to u in E, the sequence $\{u_{n_k} - u\}$ is bounded in E and $\langle J'_{n_k}(u_{n_k}, v_{n_k}) - J'_{n_k}(u, 0), (u_{n_k} - u, 0)\rangle \to 0$ as $k \to \infty$, that is,

$$o_k(1) = \langle J'_{n_k}(u_{n_k}, v_{n_k}) - J'_{n_k}(u, 0), (u_{n_k} - u, 0)\rangle$$

$$= \int_\Omega |\Delta(u_{n_k}-u)|^2\,dx + M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega |\nabla(u_{n_k}-u)|^2\,dx$$
$$+ \left[M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right) - M\left(\int_\Omega |\nabla u|^2\,dx\right)\right]\int_\Omega \nabla u\cdot\nabla(u_{n_k}-u)\,dx$$
$$- \int_\Omega \left(|u_{n_k}|^{2_*-2}u_{n_k} - |u|^{2_*-2}u\right)(u_{n_k}-u)\,dx$$
$$- \int_\Omega \left[F_u(x,u_{n_k},v_{n_k}) - F_u(x,u,0)\right](u_{n_k}-u)\,dx. \tag{3.22}$$

From relations (3.19)-(3.22), we have

$$\lim_{k\to\infty}\left[\int_\Omega |\Delta(u_{n_k}-u)|^2\,dx + M\left(\int_\Omega |\nabla u_{n_k}|^2\,dx\right)\int_\Omega |\nabla(u_{n_k}-u)|^2\,dx\right] = 0,$$

and by $(\mathcal{M}_1)$ it follows that

$$\lim_{n\to\infty}\int_\Omega \left(|\Delta(u_{n_k}-u)|^2 + |\nabla(u_{n_k}-u)|^2\right)\,dx = 0. \tag{3.23}$$

From relation (3.3), the sequence $\{u_{n_k}\}$ converges strongly to u in E and thus, J satisfies the $(PS)_c^*$ condition for $c \in \left(0, \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right)S^{\frac{N}{4}} - \widetilde{C}|\Omega|\right)$. □

Proof of Theorem 1.2. Now, we are in the position to verify the conditions of Proposition 2.7. Obviously, conditions (B_1), (B_2), (B_4) in Proposition 2.7 are satisfied. Set $V_j = E_j = span\{e_1, e_2, \dots, e_j\}$, then condition (B_3) is also satisfied. Since $1 = \dim(\widetilde{Y}_0) < k_0 = \dim(Y_1)$, (B_5) is satisfied. In the following we verify the conditions in (B_7). Because $\mathrm{Fix}(G)\cap V = \{0\}$, we deduce that (a) of (B_7) holds. It remains to verify (b), (c) of (B_7). Let us choose a real number α such that

$$\alpha < \min\left\{0, \left(\frac{1}{2} - \frac{1}{2_*}\right)\frac{(\min\{1,\sigma m_0\}S)^{\frac{N}{4}}}{(1+\epsilon)^{\frac{N-4}{4}}} - a\left(\frac{\epsilon}{2_*}\right)|\Omega|\right\}. \tag{3.24}$$

(i) If $(u,0) \in Y_0 \cap S_\rho$ (where ρ is to be determined), then by (1.6) and $(\mathcal{M}_1)$-$(\mathcal{M}_2)$, we obtain

$$J(u,0) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\,dx\right)$$
$$- \frac{1}{2_*}\int_\Omega |u|^{2_*}\,dx - \int_\Omega F(x,u,0)\,dx$$
$$\geq \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{\sigma}{2}M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega |\nabla u|^2\,dx - \frac{1}{2_*}\int_\Omega |u|^{2_*}\,dx$$
$$- \int_\Omega \left[a\left(\frac{\epsilon}{2_*}\right) + \frac{\epsilon}{2_*}|u|^{2_*}\right]\,dx$$
$$\geq \frac{1}{2}\min\{1,\sigma m_0\}\|u\|_E^p - \frac{1}{2_* S^{\frac{2_*}{2}}}(1+\epsilon)\|u\|_E^{2_*} - a\left(\frac{\epsilon}{2_*}\right)|\Omega|.$$

Let us consider the function $h:(0,+\infty)\to\mathbb{R}$ given by

$$h(t)=\frac{1}{2}\min\{1,\sigma m_0\}t^2-\frac{1}{2_*S^{\frac{2_*}{2}}}(1+\epsilon)\,t^{2_*}-a\left(\frac{\epsilon}{2_*}\right)|\Omega|,$$

we have $\lim_{t\to 0^+}h(t)=-a(\epsilon)|\Omega|$, $\lim_{t\to+\infty}h(t)=-\infty$ and

$$h'(t)=\min\{1,\sigma m_0\}t-\frac{1}{S^{\frac{2_*}{2}}}(1+\epsilon)\,t^{2_*-1}=0$$

when

$$t=t_0=\left(\frac{\min\{1,\sigma m_0\}S^{\frac{2_*}{2}}}{1+\epsilon}\right)^{\frac{1}{2_*-2}}$$

and

$$h(t_0)=\left(\frac{1}{2}-\frac{1}{2_*}\right)\frac{(\min\{1,\sigma m_0\}S)^{\frac{N}{4}}}{(1+\epsilon)^{\frac{N-4}{4}}}-a\left(\frac{\epsilon}{2_*}\right)|\Omega|,$$

which gives

$$\max_{t\in(0,+\infty)}h(t)=\left(\frac{1}{2}-\frac{1}{2_*}\right)\frac{(\min\{1,\sigma m_0\}S)^{\frac{N}{4}}}{(1+\epsilon)^{\frac{N-4}{4}}}-a\left(\frac{\epsilon}{2_*}\right)|\Omega|,$$

so that there exists $\rho>0$ such that $J(u,0)\geq\alpha$ for every $\|u\|_E=\rho$ with α as stated in (3.24), that is (b) of (B_7) holds.

(ii) First of all, by condition $(\mathcal{M}_2)$, we obtain that

$$\widehat{M}(t)\leq\frac{\widehat{M}(t_2)}{t_0^{\frac{1}{\sigma}}}t^{\frac{1}{\sigma}}=c_9t^{\frac{1}{\sigma}},\quad\forall t\geq t_2>0. \tag{3.25}$$

From (3.25), condition $(\mathcal{F}_4)$ and the definition of the functional J, it follows that for each $(u,v)\in U\oplus Y_1$,

$$\begin{aligned}
J(u,v)&=\frac{1}{2}\int_\Omega|\Delta u|^2\,dx-\int_\Omega|\nabla v|^2\,dx\\
&\quad+\frac{1}{2}\widehat{M}\left(\int_\Omega|\nabla u|^2\,dx\right)-\frac{1}{p}\widehat{M}\left(\int_\Omega|\nabla v|^2\,dx\right)\\
&\quad-\frac{1}{2_*}\int_\Omega|u|^{2_*}\,dx-\frac{1}{2_*}\int_\Omega|v|^{2_*}\,dx-\int_\Omega F(x,u,v)\,dx\\
&\leq\frac{1}{2}\int_\Omega|\Delta u|^2\,dx+\frac{c_9}{2}\left(\int_\Omega|\nabla u|^2\,dx\right)^{\frac{2}{\sigma}}-\int_\Omega\left(L|u|^\mu-\xi\right)\,dx\\
&\leq\frac{1}{2}\|u\|_E^2+\frac{c_9}{2}\|u\|_E^{\frac{2}{\sigma}}-L|u|_\mu^\mu+\xi|\Omega|.
\end{aligned}$$

Since all norms are equivalent on the finite-dimensional space Y_1, there exists a constant $c_{10}>0$ such that $\|u\|_E\leq c_{10}|u|_\mu$ and thus,

$$J(u,v)\leq\frac{c_{10}^2}{2}|u|_\mu^2+\frac{c_9c_{10}^{\frac{2}{\sigma}}}{2}|u|_\mu^{\frac{2}{\sigma}}-L|u|_\mu^\mu+\xi|\Omega|$$

$$= \left(\frac{c_{10}^2}{2} - \frac{L}{2}|u|_\mu^{\mu-2}\right)|u|_\mu^2 + \left(\frac{c_9 c_{10}^{\frac{2}{\sigma}}}{2} - \frac{L}{2}|u|_\mu^{\mu-\frac{2}{\sigma}}\right)|u|_\mu^{\frac{2}{\sigma}} + \xi|\Omega|. \tag{3.26}$$

Put $r = \min\left\{\int_\Omega |u|^\mu \, dx : \ u \in E_{k_0}\right\}$, where $E_{k_0} = span\{e_1, e_2, \dots, e_{k_0}\}$. By taking

$$L \geq \max\left\{\frac{c_{10}^2}{r^{\frac{2-\mu}{\mu}}}, \frac{c_9 c_{10}^{\frac{2}{\sigma}}}{r^{\frac{\mu-2/\sigma}{\mu}}}\right\},$$

we deduce that

$$\left(\frac{c_{10}^2}{2} - \frac{L}{2}|u|_\mu^{\mu-2}\right)|u|_\mu^2 + \left(\frac{c_9 c_{10}^{\frac{2}{\sigma}}}{2} - \frac{L}{2}|u|_\mu^{\mu-\frac{2}{\sigma}}\right)|u|_\mu^{\frac{2}{\sigma}} \leq 0. \tag{3.27}$$

It follows from relations (3.26)-(3.27) and $(\mathcal{F}_4)$ that

$$J(u,v) \leq \xi|\Omega| < \min\left\{0, \frac{1}{2}\left(\frac{\sigma}{2} - \frac{1}{2_*}\right) S^{\frac{N}{4}} - \widetilde{C}|\Omega|\right\}.$$

Let $\beta = \xi|\Omega|$, so we get (c) in (B_7). By Lemma 3.2, for any $c \in [\alpha, \beta]$, the functional J satisfies the condition of $(PS)_c^*$, then condition (B_6) in Proposition 2.7 holds. Finally, according to Proposition 2.7, we conclude that

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{w=(u,v)\in A} J(w), \quad -k_0 + 1 \leq j \leq -1,$$

are critical values of J, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta < 0$ and J has at least $k_0 - 1$ pairs critical points. The proof of Theorem 1.2 is now complete. □

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