J. Korean Math. Soc. **56** (2019), No. 5, pp. 1403–1418
https://doi.org/10.4134/JKMS.j180698
pISSN: 0304-9914 / eISSN: 2234-3008

ANNIHILATING CONTENT IN POLYNOMIAL AND POWER SERIES RINGS

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Abstract. Let R be a commutative ring with unity. If $f(x)$ is a zero-divisor polynomial such that $f(x) = c_f f_1(x)$ with $c_f \in R$ and $f_1(x)$ is not zero-divisor, then c_f is called an annihilating content for $f(x)$. In this case $Ann(f) = Ann(c_f)$. We defined EM-rings to be rings with every zero-divisor polynomial having annihilating content. We showed that the class of EM-rings includes integral domains, principal ideal rings, and PP-rings, while it is included in Armendariz rings, and rings having a.c. condition. Some properties of EM-rings are studied and the zero-divisor graphs $\Gamma(R)$ and $\Gamma(R[x])$ are related if R was an EM-ring. Some properties of annihilating contents for polynomials are extended to formal power series rings.

1. Introduction

In this article all rings R are assumed to be commutative rings with unity 1, $Z(R)$ the set of zero-divisors in R, and $reg(R) = R \setminus Z(R)$. Let $R[x]$ be the ring of polynomials defined on R and $R[[x]]$ the ring of formal power series.

Polynomial rings $R[x]$ are used in almost all branches of mathematics. Polynomials were used to construct splitting fields, and they were also generalized to power series, and Laurent series in complex analysis. The Hilbert basis theorem (if R is Noetherian, then so is $R[x]$) motivated mathematicians to study properties joint between R and $R[x]$. Zero-divisor polynomials were characterized in [17], and zero-divisor power series were characterized in [11], if R was Noetherian.

In this article, we study the concept of annihilating content of a zero-divisor polynomial, and define EM-rings to be rings in which all polynomials have annihilating contents. Annihilating content simplifies computing the annihilator

Received October 19, 2018; Accepted November 21, 2018.
2010 *Mathematics Subject Classification.* 13F20, 13F25, 13E05.
Key words and phrases. polynomial ring, power series ring, annihilating content, EM-ring, generalized morphic ring, zero-divisor graph.
The second author was supported by the “Scientific Research Deanship” at “The University of Jordan”.

of a polynomial, which is not always an easy task. Then we relate EM-rings to some famous rings.

It was shown in [17] that if $f(x) \in Z(R[x])$, then there exists a non-zero constant $c \in R \setminus \{0\}$ such that $cf(x) = 0$. Polynomials with such property are called McCoy polynomials, and rings with every polynomial having this property are called McCoy rings, and so, every commutative ring is a McCoy ring. A similar result was proved in [11] for power series in $Z(R[[x]])$, when R is a Noetherian ring, and a counterexample was given to show that the result may be false if R is not Noetherian. Thus one can conclude that if R is a Noetherian ring, $T = R[x]$ or $R[[x]]$, then $f = \sum a_i x^i \in Z(T)$ if and only if $Ann(a_0, a_1, \ldots) \neq \{0\}$. This motivated mathematicians to define rings with property A to be rings R, in which every finitely generated ideal contained in $Z(R)$ has a non-zero annihilator. It was proved that for any ring R, the ring $R[x]$ has property A. Kaplansky in [15, page 56], proved that every Noetherian ring has property A, and gave an example of a non-Noetherian ring that does not have property A. A ring R is said to have the a.c. condition if for each $a, b \in R$ there exists $c \in R$ such that $Ann(a, b) = Ann(c)$.

As a continuation of this, and trying to simplify computations with zero-divisors in $T = R[x]$ or $R[[x]]$, the authors in [1] proved that if R is a finite commutative principal ideal ring with unity, then for each $f \in Z^*(T)$, there exists $c_f \in R$, $f_1 \in reg(T)$ such that $f = c_f f_1$. They called the constant c_f an annihilating content for f. They proved the result first for finite local principal rings, and then used the fact that a finite ring is a product of finitely many local rings to generalize it. In fact their proof can be extended easily to Artinian principal ideal rings. Note that if a polynomial f has an annihilating content c_f, then $Ann_{R[x]}(f(x)) = Ann_{R[x]}(c_f)$, and $Ann_R(C(f)) = Ann_R(c_f)$, which relates annihilators of polynomials to annihilators of R, and simplifies computations.

In this article, we generalize the results of [1], and define the EM-rings to be the rings in which any zero-divisor polynomial has an annihilating content, and define strongly EM-rings to be EM-rings in which any zero-divisor power series has an annihilating content. We show that in Noetherian rings EM and strongly EM are equivalent, and gave an example of a non-Noetherian EM-ring that is not strongly EM. We show that the class of EM-rings includes integral domains, principal ideal rings, Bézout rings, Baer rings, von Neumann rings and PP-rings, while it is included in Armendariz rings and rings having a.c. condition. We prove that a Noetherian ring R is an EM-ring if and only if it is generalized morphic. Finally, we study some relations between the zero-divisor graphs of $R, R[x]$ and $R[[x]]$, if R was an EM-ring or strongly EM-ring.

2. Annihilating content

In [1], the authors gave the following definition:

Definition 2.1. Let R be a ring, $T = R[x]$ or $R[[x]]$, and let $f(x) \in Z(T)$ such that $f(x) = c_f f_1(x)$, where $c_f \in R$ and $f_1(x) \in reg(T)$. Then c_f is called an annihilating content for $f(x)$. In case of $R[x]$, it is clear that $\deg(f) \leq \deg(f_1)$.

It was proved in [1] that if R is a finite commutative principal ideal ring, then any zero-divisor in $R[x]$ or $R[[x]]$ has an annihilating content. We will generalize their result to any principal ideal ring, but first we will present the following lemma.

Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then the content of f is the ideal generated by the coefficients of f, in this case we write $C(f) = (a_0, a_1, \ldots, a_n)$.

The following lemma can be found in [2, Lemma 2.3].

Lemma 2.2. *Let $f = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $f(x) = ag(x)$ with $C(g) = R$ if and only if $C(f) = aR$, a principal ideal.*

It is clear that if $C(g) = R$ or even $C(g) \nsubseteq Z(R)$, then $Ann(b_0, b_1, \ldots, b_m) = \{0\}$, and a is an annihilating content for $f(x)$. So, the condition $C(f) = aR$ is sufficient to get an annihilating content for $f(x)$, but it is not necessary, since we will show later on that in $\mathbb{Z}[y] \times \mathbb{Z}[y]$ every polynomial has an annihilating content, but $f(x) = (y, 0) + (2, 0)x \in Z((\mathbb{Z}[y] \times \mathbb{Z}[y])[x])$, while $C(f)$ is not a principal ideal.

Recall that a ring R is called a Bézout ring if each finitely generated ideal in R is principal.

Corollary 2.3. *If R is a Bézout ring, then every polynomial in $Z(R[x])$ has an annihilating content.*

The following theorem shows the importance of the annihilating content of a polynomial or a power series.

Theorem 2.4. *Let R be a ring, $T = R[x]$ or $R[[x]]$. If $f, g \in Z(T)$, have annihilating contents, then $fg = 0$ if and only if $c_f c_g = 0$.*

One can ask about the uniqueness of an annihilating content of a polynomial or a power series. Unfortunately, the annihilating content is not unique, as seen in the following example, but they have the same annihilator.

Example 2.5. Let $R = \mathbb{Z} \times \mathbb{Z}$, and $f(x) = (6, 0) + (12, 0)x$. Then $f(x) = (2, 0)((3, 1) + (6, 1)x) = (3, 0)((2, 1) + (4, 1)x)$. Note that $Ann((2, 0)) = \{0\} \times \mathbb{Z} = Ann((3, 0))$, but $((2, 0))R = 2\mathbb{Z}\times\{0\} \neq 3\mathbb{Z}\times\{0\} = ((3, 0))R$.

Let R be a ring, and let $f(x) = c_f f_1(x) \in Z(R[[x]]) \setminus \{0\}$ such that $c_f \in R$ and $f_1(x) \in reg(R[[x]])$. It is clear that $g(x) \in Ann(c_f)$ if and only if $g(x)f(x) = 0$ if and only if $g(x) \in Ann(f(x))$. So, $Ann(c_f) = Ann(f(x))$. Thus, if $f(x) = af_1(x) = bf_2(x) \in Z(R[[x]]) \setminus \{0\}$ such that $a, b \in R$ and $f_1(x), f_2(x) \in reg(R[[x]])$, then $Ann(a) = Ann(f(x)) = Ann(b)$, and so, $aR \approx R/Ann_R(a) = R/Ann_R(b) \approx bR$ as R-modules. Thus if $(R, +)$ is a finite cyclic group, then $aR = bR$, since the two subgroups have the same cardinality.

Thus, one can easily compute the annihilator of a polynomial or a power series by computing the annihilator of the annihilating content, whenever it exists.

Do all polynomials in any ring R have annihilating content? The following example shows that this may be false, even if R was Artinian ring.

Example 2.6. Let $R = \mathbb{Z}_4[x]\big/(x^2) = \{a_0 + a_1 X : a_i \in \mathbb{Z}_4 \text{ and } X^2 = 0\}$. Then R is an Artinian local ring with maximal ideal $2R + XR$. If $f(y) = 2 + Xy \in R[y]$, then $f(y) \in Z^*(R[y])$, since $2Xf(y) = 0$. But $f(y)$ has no annihilating content, since if $f(y) = af_1(y), a \in R$ and $f_1(y) = \sum_{i=0}^{n} b_i y^i \in reg(R[y])$, then $2 = ab_0$ and $X = ab_1$, and so $2R + XR \subseteq aR \subset R$, which implies that $2R + XR = aR$, a contradiction.

3. EM-rings

If R is a principal ideal ring, then every polynomial in $Z(R[x])$ and every power series in $Z(R[[x]])$ has an annihilating content, but we found other classes of rings with this property, this motivated us to give the following definitions, in order to study the effect of annihilating contents.

Definition 3.1. Let R be a ring. Then R is called an EM-ring if every zero-divisor polynomial in $R[x]$ has an annihilating content. R is called strongly EM-ring if it is an EM-ring and every zero-divisor power series in $R[[x]]$ has an annihilating content.

Question. According to the above definition, one can ask whether if every power series in $R[[x]]$ has an annihilating content, then every polynomial in $R[x]$ has an annihilating content.

This is not a trivial question. The following example gives a polynomial that has an annihilating content as a power series but has no annihilating content as a polynomial.

Example 3.2. Let $K = \{y, t, s_0, s_1, \{x_i\}_{i=0}^{\infty}\}$ be a set of indeterminants, $S = \mathbb{Q}[K]$ and let the ideal $I = (yx_0 - s_0, yx_1 - s_1, ty, tx_0, tx_1, t^2, \{yx_i, x_i^2 : i \geq 2\})$. For each $a \in S$, let $\Lambda(a) = \{\alpha \in K : \alpha$ has non-zero coefficient in the expansion of $a\}$. It is clear that for each $a \in S \setminus \{0\}$, $|\Lambda(a)| < \infty$. Now consider the ring $R = S/I$. Then $Z(R) = (\overline{y}, \overline{t}, \overline{s}_0, \overline{s}_1, \{\overline{x}_i\}_{i=0}^{\infty})$ is an ideal in R. Note that $\sum_{i=2}^{\infty} \overline{x}_i T^i$ is a zero-divisor power series that has no annihilating content. Consider the polynomial $f(T) = \overline{s}_0 + \overline{s}_1 T \in R[T]$. Then $\overline{t}f = \overline{0}$, and f has no annihilating content in $R[T]$. But we have

$$f(T) = \overline{s}_0 + \overline{s}_1 T = \overline{y} \sum_{i=0}^{\infty} \overline{x}_i T^i.$$

We want to show that $g(T)=\sum_{i=0}^{\infty}\overline{x}_iT^i$ is a regular power series. Assume to the contrary that there exists $h(T)=\sum_{i=0}^{\infty}\overline{h}_iT^i$ such that

$$\left(\sum_{i=0}^{\infty}\overline{x}_iT^i\right)\left(\sum_{i=0}^{\infty}\overline{h}_iT^i\right)=\overline{0}.$$

Without loss of generality we can assume that $\overline{h}_0\neq\overline{0}$. Now we have:

$$\overline{x}_0\overline{h}_0=\overline{0},$$

and so, $\overline{h}_0\in Ann(\overline{x}_0)=\overline{t}R$, i.e., $\overline{t}|\overline{h}_0$. Also we have:

$$\overline{x}_0\overline{h}_1+\overline{x}_1\overline{h}_0=\overline{x}_0\overline{h}_1+\overline{0}=\overline{0},$$

and so, $\overline{h}_1\in Ann(\overline{x}_0)=\overline{t}R$, i.e., $\overline{t}|\overline{h}_1$. Assume now that $\overline{t}|\overline{h}_i$ for all $i<m$, then we have:

$$\overline{x}_0\overline{h}_m+\overline{x}_1\overline{h}_{m-1}+\cdots+\overline{x}_m\overline{h}_0=\overline{0}.$$

Multiply by $\overline{x}_0$ to get:

$$\overline{x}_0^2\overline{h}_m=\overline{0},$$

and so, $\overline{h}_m\in Ann(\overline{x}_0^2)=Ann(\overline{x}_0)=\overline{t}R$. Thus, $\overline{t}|\overline{h}_i$ for all i. Therefore we have:

$$\overline{0}=\left(\sum_{i=0}^{\infty}\overline{x}_iT^i\right)\left(\sum_{i=0}^{\infty}\overline{h}_iT^i\right)=\left(\sum_{i=2}^{\infty}\overline{x}_iT^i\right)\left(\sum_{i=0}^{\infty}\overline{h}_iT^i\right).$$

But $\overline{x}_i^2=\overline{0}$ for each $i\geq 2$, so we can follow the technique in [13] to write $\sum_{i=2}^{\infty}\overline{x}_iT^i=P_k+T^{k+1}g_{k+1}$, where $P_k=\sum_{i=2}^{k}\overline{x}_iT^i$ for $k=2,3,\ldots$.

Since P_k is nilpotent, there exists n_k such that $P_k^{n_k}=\overline{0}$. Now since $h(T)\sum_{i=2}^{\infty}\overline{x}_iT^i=\overline{0}$, we get $-h(T)P_k=h(T)T^{k+1}g_{k+1}$. Thus

$$\begin{aligned}\overline{0}&=h(T)\left(P_k+T^{k+1}g_{k+1}\right)^{n_k}=h(T)\sum_{i=0}^{n_k}\binom{n_k}{i}P_k^{n_k-i}(T^{k+1}g_{k+1})^i\\&=h(T)(T^{k+1}g_{k+1})^{n_k},\end{aligned}$$

and hence $\overline{h}_0\overline{x}_{k+1}\overline{x}_{k+2}\cdots\overline{x}_{k+n_k}=\overline{0}$.

If $\overline{h}_0\notin\overline{y}R$, then $\overline{h}_0=\overline{a}+\overline{yc}$, with $a\notin I$. But

$$\overline{x}_2\overline{h}_0=\overline{0}.$$

So, $\overline{x}_2\overline{h}=\overline{0}$, that is $x_2a\in I$, and hence we have $x_2\in\Lambda(a)$. Also for $k=2,3,4$ we have:

$$\overline{x}_k\overline{x}_{k+1}\cdots\overline{x}_{k+n_k}\overline{h}_0=\overline{0},$$

which implies that $x_kx_{k+1}\cdots x_{k+n_k}a\in I$, and hence we have $x_{k+i}\in\Lambda(a)$ for some i. Continue to get $|\Lambda(a)|=\infty$, that is $a=0$. Hence we have $\overline{t\alpha}=\overline{h}_0=\overline{yc}$, which implies that $\overline{c}\in Ann(\overline{y}^2)=Ann(\overline{y})$. Therefore, $\overline{h}_0=\overline{yc}=\overline{0}$, a contradiction. Therefore $g(T)=\sum_{i=0}^{\infty}\overline{x}_iT^i$ is a regular power series in $R[[T]]$, although every polynomial made up of $g(T)$ is zero-divisor.

In the following, we will give some positive partial answers, showing that the result is true in a wide class of rings. But first we note that if every power series in $R[[x]]$ has an annihilating content, then $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = 0$ if and only if $a_i b_j = 0$ for each i and j. In particular $\sum_{i=0}^{\infty} a_i x^i$ is a zero-divisor in $R[[x]]$ if and only if there exists $a \in R \setminus \{0\}$ such that $aa_i = 0$ for each i.

We here give a partial answer to the previous question. But we still don't know the answer for the general case.

Recall that a ring R is called a countably McCoy ring, according to [16], if each countably generated ideal contained in $Z(R)$ has a non-zero annihilator.

Theorem 3.3. *Assume R is a countably McCoy ring. Then R is a strongly EM-ring if and only if every power series in $R[[x]]$ has an annihilating content.*

Proof. Let $\sum_{i=0}^{n} f_i x^i$ be a zero-divisor polynomial in $R[x]$. Then there exists a regular power series $\sum_{i=0}^{\infty} g_i x^i \in R[[x]]$ such that $\sum_{i=0}^{n} f_i x^i = c\sum_{i=0}^{\infty} g_i x^i$. Then we have:

$$f_i = cg_i \text{ for each } i \in \{0, 1, \ldots, n\},$$
$$0 = cg_i \text{ for each } i > n.$$

But the countably generated ideal $(g_0, g_1, \ldots)$ is not contained in $Z(R)$, since $\sum_{i=0}^{\infty} g_i x^i$ is regular, and so there exists $r = \sum_{i=0}^{k} r_i g_i \in reg(R)$. Let $m = \mathrm{Max}\{n, k\}$. Then $\sum_{i=0}^{n} f_i x^i = c\sum_{i=0}^{m} g_i x^i$, and $\bigcap_{i=0}^{m} Ann(g_i) = \{0\}$. Thus R is an EM-ring. □

Corollary 3.4. *Assume R is a Noetherian ring. Then R is a strongly EM-ring if and only if every power series in $R[[x]]$ has an annihilating content.*

Theorem 3.5. *Let R be a ring such that $Z(R[[x]])$ is an ideal in $R[[x]]$. Then R is a strongly EM-ring if and only if every power series in $R[[x]]$ has an annihilating content.*

Proof. Let $\sum_{i=0}^{n} f_i x^i$ be a zero-divisor polynomial in $R[x]$. Then there exists a regular power series $\sum_{i=0}^{\infty} g_i x^i \in R[[x]]$ such that $\sum_{i=0}^{n} f_i x^i = c\sum_{i=0}^{\infty} g_i x^i$. Then we have:

$$f_i = cg_i \text{ for each } i \in \{0, 1, \ldots, n\},$$
$$0 = cg_i \text{ for each } i > n.$$

But $\sum_{i=0}^{\infty} g_i x^i = \sum_{i=0}^{n} g_i x^i + \sum_{i=n+1}^{\infty} g_i x^i$ is regular and $\sum_{i=n+1}^{\infty} g_i x^i$ is zero-divisor, so $\sum_{i=0}^{n} g_i x^i$ must be regular. Hence $\sum_{i=0}^{n} f_i x^i = c\sum_{i=0}^{n} g_i x^i$ with $\sum_{i=0}^{n} g_i x^i$ is regular. Thus R is an EM-ring. □

We have shown that any Bézout ring, or in particular any principal ideal ring is an EM-ring.

Clearly, any strongly EM-ring is an EM-ring, and we will give later on an example of an EM-ring that is not strongly EM.

We now study some basic properties of EM-rings.

Theorem 3.6. *Assume R is an EM-ring, and S is a multiplicatively closed subset of R. Then $S^{-1}R$ is an EM-ring.*

Proof. Assume that $h(x) \in S^{-1}R[x]$. Then there exist $t \in S$, $f(x) \in R[x]$ such that $h(x) = \frac{f(x)}{t}$. Since R is an EM-ring there exist $c \in R$ and a regular polynomial $g(x) = \sum_{i=0}^{n} g_i x^i \in R[x]$ such that $f(x) = cg(x)$, and $g(x)$ is regular. So, $h(x) = c\,\frac{g(x)}{t}$. If there exists $\frac{d}{k} \in S^{-1}R$ such that $\frac{d}{k}\frac{g_i}{1} = 0$ for each i, then there exists $u_i \in S$ such that $u_i d g_i = 0$ for each i. Let $u = \prod_{i=0}^{n} u_i$. Then $dug_i = 0$ for each i and so, $ud = 0$, since $g(x)$ is a regular polynomial in $R[x]$. Thus $\frac{d}{k} = 0$, since $u \in S$. Therefore $\frac{g(x)}{t}$ is a regular polynomial in $S^{-1}R[x]$, and $S^{-1}R$ is an EM-ring. □

Recall that if R is a commutative ring with unity, the total quotient ring of R is the localization $T(R) = (reg(R))^{-1}R$.

Corollary 3.7. *If R is an EM-ring, then $S^{-1}R$ is an EM-ring for any multiplicatively closed subset $S \subseteq reg(R)$. In particular $T(R)$ is an EM-ring, if R is.*

Theorem 3.8. *If $T(R)$ is an EM-ring, then for every $f \in Z(R[x])$, there exists $k \in R$ such that $Ann_{R[x]}(f) = Ann_{R[x]}(k)$.*

Proof. Assume that $f(x) \in Z(R[x]) \subseteq Z(T(R)[x])$. Then there exist $\frac{k}{s} \in T(R)$ and a regular polynomial $g(x) \in T(R)[x]$ such that $f(x) = k(\frac{1}{s}g(x))$. Now, since $\frac{1}{s}g(x)$ is regular we have $Ann_{R[x]}(f) = Ann_{R[x]}(k)$. Moreover, $Ann_R(C(f)) = Ann_R(k)$. □

Definition 3.9. A ring R is called locally EM-ring if $T(R)$ is an EM-ring.

For an example of a locally EM-ring which is not an EM-ring, see Example 3.11 below.

Theorem 3.10. *If R is an EM-ring with property A, then $T(R)$ is a Bézout ring.*

Proof. Let $I = \sum_{i=0}^{n} a_i T(R)$ be a finitely generated ideal in $T(R)$. If $(a_0, \ldots, a_n) \nsubseteq Z(R)$, then I contains a unit and so, $I = T(R)$. If $(a_0, \ldots, a_n) \subseteq Z(R)$, and since R has property A, we must have $Ann(a_0, \ldots, a_n) \neq \{0\}$, and so, $f(x) = \sum_{i=0}^{n} a_i x^i \in Z(R[x]) \setminus \{0\}$. Hence, we have $f(x) = a\sum_{i=0}^{m} u_i x^i$, with $Ann(u_0, \ldots, u_m) = \{0\}$, and there exists $r = \sum_{i=0}^{m} r_i u_i \in R \backslash Z(R)$. So we have $I = \sum_{i=0}^{n} a_i T(R) = a\sum_{i=0}^{m} u_i T(R) = aT(R)$, since $\sum_{i=0}^{m} u_i T(R)$ contains a unit. Thus, $T(R)$ is a Bézout ring. □

The converse of these theorems needs not be true as shown in the following example.

Example 3.11. Let $R = \mathbb{Z}_6[x, y]\,/(xy)$. Then R is a reduced Noetherian ring, and so $T(R)$ is a von Neumann regular ring, and hence it is a Bézout EM-ring. But R is not an EM-ring, since $Ann(\overline{2}\overline{y}) = \overline{3}R + \overline{x}R$, which is not principal, and we will see in Theorem 3.26 that this prevents R to be an EM-ring. Hence $\mathbb{Z}_6[x,y]/(xy)$ is a locally EM-ring which is not an EM-ring.

Theorem 3.12. *Let $R=\prod_{\alpha\in I}R_\alpha$. Then R is an EM-ring if and only if R_α is an EM-ring for each α.*

Proof. Assume R is an EM-ring, and let $f(x)=\sum_{i=0}^n a_ix^i\in Z^*(\ R_j[x])$ for some $j\in I$. For each i construct the sequence $(b_{\alpha,i})$ such that $b_{\alpha,i}=\begin{cases} a_i, & \alpha=j\\ 0, & \text{otherwise.}\end{cases}$ Then $g(x)=\sum_{i=0}^n(b_{\alpha,i})x^i\in Z^*(R[x])$, and so there exist $c_g=(c_\alpha)\in R$ and $g_1(x)=\sum_{i=0}^m(d_{\alpha,i})x^i\in reg(R[x])$ such that $g(x)=c_gg_1(x)$, $m\geq n$. For each i, let $w_i=d_{j,i}$. Then $a_i=c_jw_i$ for each i. If $yw_i=0$ for each i, then let $z_\alpha=\begin{cases} y, & \alpha=j\\ 0, & \text{otherwise,}\end{cases}$ and so $(z_\alpha)(d_{\alpha,i})=(0)$ for each i. Hence, $y=0$ and $f(x)=c_j\sum_{i=0}^m w_ix^i$.

To see the converse, assume that R_α is an EM-ring for each α, and let $f(x)=\sum_{i=0}^n(a_{\alpha,i})x^i\in Z^*(R[x])$. Then there exists $(b_\alpha)\neq(0)$ such that $(b_\alpha)(a_{\alpha,i})=(0)$ for each i. Note that if $\bigcap_{i=0}^n Ann\{a_{\alpha,i}\}=\{0\}$, then $b_\alpha=0$. Since $(b_\alpha)\neq(0)$, one can construct for each $j\in I$ such that $\bigcap_{i=0}^n Ann\{a_{j,i}\}\neq\{0\}$, the polynomial $f_j(x)=\sum_{i=0}^n a_{j,i}x^i\in Z(R_j[x])$, and so we have $f_j(x)=c_{f_j}\sum_{i=0}^m d_{j,i}x^i$ with $\bigcap_{i=0}^n Ann\{d_{j,i}\}=\{0\}$, $m\geq n$. Now let

$$c_\alpha=\begin{cases} c_{f_\alpha}, & \bigcap_{i=0}^m Ann\{a_{\alpha,i}\}\neq\{0\}\\ 1, & \text{otherwise,}\end{cases}\quad\text{and}$$
$$k_{\alpha,i}=\begin{cases} d_{\alpha,i}, & \bigcap_{i=0}^m Ann\{a_{\alpha,i}\}\neq\{0\}\\ a_{\alpha,i}, & \text{otherwise}\end{cases}\quad\text{for each } i.$$

Then $f(x)=(c_\alpha)\sum_{i=0}^m(k_{\alpha,i})x^i$ with

$$\bigcap_{i=0}^m Ann\{(k_{\alpha,i})\}=\bigcap_{i=0}^m(\prod_{\alpha\in I}Ann\{k_{\alpha,i}\})=\prod_{\alpha\in I}(\bigcap_{i=0}^m Ann\{k_{\alpha,i}\})=\{(0)\}.$$
□

We now give examples of EM-rings that are not principal ideal rings.

Theorem 3.13. *If R is an EM-ring, then $R[x]$ is also an EM-ring.*

Proof. Let $f(x,y)=\sum_{i=0}^n f_i(x)y^i$ be a zero-divisor in $R[x,y]=(R[x])[y]$. Then there exists nonzero $h(x)$ such that $hf_i=0$ for all i. Define

$$g(x)=f_0+f_1x^{\deg(f_0)+1}+f_2x^{\deg(f_0)+\deg(f_1)+2}+\cdots+f_nx^{\ \sum_{i=1}^{n-1}\deg(f_i)+n}.$$

Since $hg=0$, there exists $c_g\in Z(R)$ and nonzero-divisor $g_1=\sum_{i=1}^m b_ix^i$ such that $g=c_gg_1$. So, $\cap Ann(b_i)=\{0\}$, and $f_0=c_g\sum_{i=0}^{\deg(f_0)}b_ix^i=c_gh_0(x)$, $f_1=c_g\sum_{i=0}^{\deg(f_1)}b_{i+\deg(f_0)+1}x^i=c_gh_1$, and so on. Hence, $f(x,y)=c_g\sum_{i=0}^n h_i(x)y^i$. Note that g_1 belongs to the $R[x]$-content of the polynomial $\sum_{i=0}^n h_i(x)y^i$, thus $Ann_{R[x]}(h_0,h_1,\ldots,h_n)\subseteq Ann_{R[x]}(g_1)=\{0\}$, and we have $R[x]$ is an EM-ring. □

Corollary 3.14. *Let R be an EM-ring. Then $R[x_1, x_2, \dots, x_n]$ is an EM-ring.*

Lemma 3.15. *Let R_0 be ring, and let $R_i = R_0[x_1, \dots, x_i]$. If $r \in reg(R_j)$, then $r \in reg(R_i)$ for every $i \geq j$.*

Proof. Note that, for any ring S, $Z(S[x]) \cap S = Z(S)$, otherwise there exists $t \in Z(S[x]) \cap reg(S)$, and so $tf(x) = 0$ for some $f(x) = \sum_{i=1}^{n} a_i x^i \in Z^*(S[x])$. Then $a_i \neq 0$ for some i and $ta_i = 0$, a contradiction. □

Corollary 3.16. *Let R be an EM-ring, and let $f \in Z(R_n)$, where $R_i = R[x_1, \dots, x_i]$. Then there exist $c_f \in R$ and $g \in reg(R_n)$ such that $f = c_f g$.*

Proof. $f = c_n f_n$, where $c_n \in Z(R_{n-1})$ and $f_n \in reg(R_n)$. Repeat the work to get $f = c_{n-1} f_{n-1} f_n$, where $c_{n-1} \in Z(R_{n-2})$ and $f_{n-1} \in reg(R_{n-1})$. Continue to obtain $f = c_1 f_1 f_2 \cdots f_n$, where $c_1 \in R$ and $f_i \in reg(R_i)$ for $i = 1, 2, \dots, n$. Let $g = f_1 f_2 \cdots f_n$. Then by Lemma 3.15, $g \in reg(R_n)$, and the result holds. □

We now give an example of a non-Noetherian EM-ring.

Theorem 3.17. *Let S be an EM-ring, and let $R = S[x_1, x_2, \dots]$. Then R is an EM-ring.*

Proof. The ring R is not Noetherian, since the ideal $\sum_{i=1}^{\infty} x_i R$ is not finitely generated. Let $f \in Z(R[x_0])$. Then $f = \sum_{i=0}^{\infty} a_i \prod_{j=0}^{\infty} x_j^{n_{i,j}}$, such that $a_i = 0$ for all but finitely many i and $n_{i,j} = 0$ for all but finitely many i and j. Thus $f \in S[x_0, x_1, \dots, x_m]$ for some m. There exists $g \in S[x_0, x_1, \dots, x_n]$ such that $fg = 0$. Let $k = \text{Max}\{m, n\}$. Then $f, g \in Z(S[x_0, x_1, \dots, x_k])$, and it follows by Lemma 3.15 and Corollary 3.16 that $f = c_f f_1$ such that $c_f \in S \subset R$ and $f_1 \in reg(R[x_0])$. □

If R is an EM-ring and I is an ideal of R, then R/I needs not be EM-ring, since $\mathbb{Z}_6[x, y]$ is an EM-ring, but $\mathbb{Z}_6[x, y]/(xy)$ is not, as shown in Example 3.11.

Now, we relate EM-rings with some famous rings.

A ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. Note that the ring $\mathbb{Z}_4[x]/(x^2)$ in Example 2.6 is not an Armendariz ring, since if $f(y) = 2 + Xy = g(y)$, then $f(y)g(y) = 0$, but $2X \neq 0$.

Theorem 3.18. *If R is an EM-ring, then it is an Armendariz ring.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in Z(R[x])$, with $f(x)g(x) = 0$. Then $f(x) = c_f \sum_{i=0}^{n_1} \alpha_i x^i$, and $g(x) = c_g \sum_{j=0}^{m_1} \beta_j x^j$. So, $a_i b_j = c_f \alpha_i c_g \beta_j = 0$, for each i, j, since $c_f c_g = 0$. □

Theorem 3.19. *If a ring R is an integral domain, then $R[x]/(x^2) = \{a_0 + a_1 X : a_i \in R, X^2 = 0\} = S$ is an EM-ring.*

Proof. Note first that $a_0 + a_1X$ is a zero-divisor in S if and only if $a_0 = 0$. Thus if $f(y) = \sum_{i=0}^{n} (a_{i,0} + a_{i,1}X)y^i \in Z^*(S[y])$, then $f(y) = \sum_{i=0}^{n} a_{i,1}Xy^i = X\sum_{i=0}^{n} a_{i,1}y^i$. Thus $R[x]\big/(x^2)$ is an EM-ring. $\square$

The converse of this theorem is not true, since $\mathbb{Z}_6[x]\big/(x^2)$ is an EM-ring, while $\mathbb{Z}_6$ is not an integral domain.

Theorem 3.20. *If $S = R[x]\big/(x^2)$ is an EM-ring, then R is a reduced EM-ring.*

Proof. S is an Armendariz ring, and so R is a reduced ring, see [3]. To show that R is an EM-ring, let $f(y) = \sum_{i=0}^{n} a_iy^i \in Z(R[y])$, and so, $f(y) \in Z(S[y])$. Hence $f(y) = (c_0 + c_1X)\sum_{i=0}^{m} (b_{i,0} + b_{i,1}X)y^i$, with $\bigcap_{i=0}^{m} Ann(b_{i,0} + b_{i,1}X) = \{0\}$, which yields that $\bigcap_{i=0}^{m} Ann(b_{i,0}) = \{0\}$. But $a_i = (c_0+c_1X)(b_{i,0}+b_{i,1}X) = c_ob_{i,0}$ for each i. Hence $f(y) = c_0\sum_{i=0}^{m} b_{i,0}y^i$, and R is an EM-ring. $\square$

Recall that a ring R is called a PP-ring if every principal ideal in R is a projective R-module, which is equivalent to annihilator of any element is generated by an idempotent.

Theorem 3.21. *If R is a PP-ring, then it is an EM-ring.*

Proof. Note first that if e_1 and e_2 are idempotents, then $e = e_1 + e_2 - e_1e_2$ is also an idempotent. Moreover, $e_ie = e_i$ for $i = 1, 2$. So, $eR = e_1R + e_2R$, and thus $Ann(e_1, e_2) = Ann(e) = (1-e)R$, also note that $1 = (e_1+1-e)+(e_2+1-e)-(e_1+1-e)(e_2+1-e)$, which means that $Ann((e_1+1-e), (e_2+1-e)) = \{0\}$. Using induction one can generalize this for any finite family of idempotents, with $1 - e = \prod_{i=1}^{n}(1 - e_i)$. Now, let $f(x) = \sum_{i=0}^{n} a_ix^i \in Z^*(R[x])$. Then $a_i = u_ie_i$, where $u_i \in reg(R)$ and e_i is an idempotent for each i, see [10, Lemma 2]. There exists $a \in R \setminus \{0\}$ such that $af(x) = 0$, and so $ae_i = 0$. Let e be as above. Then $\{0\} \neq Ann(e_1, e_2, \ldots, e_n) = Ann(e) = (1 - e)R$. Thus $f(x) = e\sum_{i=0}^{n} u_i(e_i + 1 - e)x^i$, and $\bigcap_{i=0}^{n} Ann(u_i(e_i + 1 - e)) = \{0\}$. $\square$

The converse of this theorem needs not be true, since $\mathbb{Z}_8$ is an EM-ring which is not a PP-ring, being nonreduced. Also this theorem shows that the EM-rings include a large class of rings such as Baer rings, von Neumann rings, and PP-rings.

If for every $a, b \in R$, there is $c \in R$ such that $Ann(a) \cap Ann(b) = Ann(c)$, then R is said to satisfy the annihilator condition or to be an a.c. ring, see [14].

Theorem 3.22. *If R is an EM-ring, then it is an a.c. ring.*

Proof. Assume that $d \in Ann(a_0, a_1)$. Then $f(x) = a_0 + a_1x \in Z(R[x])$, and so $a_0 + a_1x = c_f\sum_{i=0}^{n} b_ix^i$, with $Ann(b_0, b_1, \ldots, b_n) = \{0\}$, $n \geq 1$. Thus $0 = da_i = dc_fb_i$ for $i = 0, 1$ and $0 = c_fb_i = dc_fb_i$ for $i > 1$, and hence, $dc_f \in Ann(b_0, b_1, \ldots, b_n) = \{0\}$, which implies that $Ann(a_0, a_1) \subseteq Ann(c_f)$. If $k \in Ann(c_f)$, then $ka_i = kc_fb_i = 0$ for $i = 0, 1$. Hence, $k \in Ann(a_0, a_1)$, and so, $Ann(c_f) \subseteq Ann(a_0, a_1)$. Thus R is an a.c. ring. $\square$

To get deeper results, we will assume that R is a Noetherian ring.

A ring R is called a morphic ring if for each $a \in R$, there exists $b \in R$ such that $Ann(a) = bR$ and $Ann(b) = aR$. The ring R is called generalized morphic if for each $a \in R$, there exists $b \in R$ such that $Ann(a) = bR$. Clearly every morphic ring is generalized morphic, but the ring $\mathbb{Z}$ is a generalized morphic which is not morphic, see [19] and [9].

We first start with the following proposition.

Proposition 3.23. *Let R be an EM-ring, and assume that $b \in R$ such that $Ann(b)$ is finitely generated, then $Ann(b)$ is principal.*

Proof. Assume that $Ann(b) = \sum_{j=1}^{n} a_j R$ for some $a_1, \ldots, a_n \in R$. Let $f(x) = \sum_{j=1}^{n} a_j x^j$. Then $f(x) \in Z^*(R[x])$, since $bf(x) = 0$. But $f(x) = c_f f_1(x) = c_f \sum_{j=1}^{m} \alpha_j x^j$, and so $a_j = c_f \alpha_j \in c_f R$ for each j. Thus we have $Ann(b) = \sum_{j=1}^{n} a_j R \subseteq c_f R$. Also we have $0 = ba_j = bc_f \alpha_j$ for each j. Then $bc_f = 0$, since $Ann(\alpha_1, \alpha_2, \ldots, \alpha_m) = \{0\}$, because $f_1(x)$ is not a zero-divisor. Hence $c_f \in Ann(b) = \sum_{j=1}^{n} a_j R$, and so we have $Ann(b) = \sum_{j=1}^{n} a_j R = c_f R$ is principal. □

We now use a definition of irreducible elements in commutative ring with unity and contains zero-divisors, see [12].

Definition 3.24. A non-unit element $a \in R$ is called irreducible element if whenever $a = bc$, we have $aR = bR$ or $aR = cR$.

It was shown in [5, Theorem 3.2] that if R has ascending chain condition on principal ideals (in particular if R is Noetherian), then every non-zero non-unit element in R is a finite product of irreducibles.

Lemma 3.25. *Assume that R is a Noetherian ring, and bR is a prime principal ideal with $b \in Z(R)$. If $a \in bR \setminus \{0\}$, then $a = b^n s$ for some $n \in \mathbb{N}$ and $s \in R \setminus bR$.*

Proof. Case 1: $b^2 R = bR$. So $bR = eR$, where $e = e^2$. With a change of notation $R = R_1 \times R_2$, $e = (0, 1)$, $b = (0, u)$ where u is a unit in R_2. Now, $a \in bR \setminus \{0\}$ implies that $a = (0, r)$, with $r \in R_2 \setminus \{0\}$. Then $a = b(1, ru^{-1})$, and clearly $(1, ru^{-1}) \notin bR$.

Case 2: $b^2 R \subsetneq bR$. Note that if $s \in bR$ is irreducible, then $sR = bR$, since otherwise $sR \subsetneq bR$ implies that $b \notin Z(R)$, see [4, Theorem 1]. Now $sR = bR$ implies that $s = br$, with $r \notin bR$, since otherwise, we would have $bR = sR \subseteq b^2 R \subsetneq bR$.

Write $a = s_1 s_2 \cdots s_m$ where s_i is irreducible. Now either $s_i \notin bR$ or $s_i = bs_i^{'}$ with $s_i^{'} \notin bR$. If $s_i \notin bR$, put $s_i = s_i^{'}$. Let $s = s_1^{'} s_2^{2} \cdots s_m^{'}$. Then $a = b^n s$ where $s \notin bR$ and $n = |\{i : s_i R = bR\}|$. □

Theorem 3.26. *Let R be a Noetherian ring. Then the following are equivalent*:

(1) *R is a strongly EM-ring.*

(2) *R is an EM-ring.*
(3) *R is a generalized morphic ring.*
(4) *Every ideal in $Z(R)$ is contained in a principal ideal in $Z(R)$.*
(5) *The maximal prime ideals of zero divisors are principal.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Follows by Proposition 3.23.

(3) $\Rightarrow$ (4) Let I be an ideal contained in $Z(R)$. Then I is finitely generated, and since R has property A, being Noetherian, there exists $a \neq 0$ such that $aI = 0$. Thus $I \subseteq Ann(a) = bR$, a principal ideal.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (1) Let $T = R[x]$ or $R[[x]]$, and let $f = \sum a_i x^i \in Z^*(T)$, and let $C(f)$ the content of f, i.e., the ideal generated by the coefficients of f. Then $C(f) \subseteq Z(T)$ and so, $C(f) \subseteq M_1 = c_1 R \subseteq Z(R)$, where M_1 is a maximal ideal consisting of zero divisors, and so it is prime, see [15, Theorem 6]. Hence, using Lemma 3.25 if $a_i \neq 0$, $a_i = \alpha_i c_1^{k_i}$ with $\alpha_i \notin c_1 R$, and $k_i \geq 1$ for each i. Let $k_{11} = \mathrm{Min}\{k_i\}$, $b_i = \alpha_i c_1^{k_i - k_{11}}$ and let $f_1 = \sum b_i x^i$. Then $f = c_1^{k_{11}} f_1$ and $C(f_1) \nsubseteq c_1 R$, and thus $fT \subset f_1 T$. If $f_1 \in Z(T) \setminus \{0\}$, then repeat the work to write $f_1 = c_2^{k_{22}} f_2$ and $fT \subset f_1 T \subset f_2 T$. Continue to get $fT \subset f_1 T \subset f_2 T \subset f_3 T \subset \cdots$, and since T is Noetherian ring, this ascending chain terminates. Thus there exits $f_n \notin Z(T)$ and $f(x) = c_1^{k_{11}} c_2^{k_{22}} c_3^{k_{33}} \cdots c_n^{k_{nn}} f_n = c_f f_n(x)$. $\square$

It follows immediately from this theorem that any principal ideal ring is a strongly EM-ring.

Corollary 3.27. *Let R be a reduced Noetherian ring. Then the following statements are equivalent*:
(1) *R is an EM-ring.*
(2) *R is a generalized morphic ring.*
(3) *The maximal prime ideals of zero divisors are principal.*
(4) *Any minimal prime ideal of R is principal.*

Proof. (1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (3) Follows from Theorem 3.26.

(2) $\Rightarrow$ (4) Assume that R is a generalized morphic ring, and let P be a minimal prime ideal in R. Then it follows by [15, Theorem 86] that $P = Ann(a)$ for some $a \in R$. Hence P is a principal ideal.

(4) $\Rightarrow$ (3) Assume that every minimal prime ideal of R is principal. Since R is Noetherian, R has a finite number of minimal prime ideals, $P_1, \ldots, P_n$, see [15, Theorem 88]. Also since R is reduced, we have $Z(R) = P_1 \cup \cdots \cup P_n$. Let M be a maximal ideal of $Z(R)$. Then $M \subseteq P_1 \cup \cdots \cup P_n$, and so $M \subseteq P_i$ for some i. Thus, $M = P_i$ which is principal. $\square$

Although EM-rings and generalized morphic rings are equivalent in the class of Noetherian rings, but they are not equivalent in general. In the following, we will find an EM-ring that is not generalized morphic.

A ring R is called a PF-ring if every principal ideal of R is a flat R-module. It is well known that R is a PF-ring if and only if for each $a \in R, Ann(a)$ is pure (for each $b \in Ann(a)$ there exists $c \in Ann(a)$ such that $b = bc$). It is easy to see that a ring R is a PP-ring if and only if it is a generalized morphic PF-ring. Thus any PF-ring that is not a PP-ring is not generalized morphic. But the ring $C(X)$ of continuous real valued functions defined on a space X is a PF-ring if and only if it is Bézout. Thus if $C(X)$ is a PF-ring that is not a PP-ring, then it is an EM-ring that is not generalized morphic.

4. Application to zero-divisor graphs

The zero-divisor graph for a ring R is a simple graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$, and two distinct elements $a, b \in Z^*(R)$ are adjacent if $ab = 0$. This graph is usually denoted by $\Gamma(R)$. The idea of zero-divisor graphs was first introduced in [8], then modified in its current status in [6]. The zero-divisor graph is a tool to study $Z(R)$, and it relates the algebraic properties of R with the graph properties of $\Gamma(R)$. The zero-divisor graphs for $R[x]$ and $R[[x]]$, were studied in [7] and [16]. The authors tried to relate $\Gamma(R)$ to $\Gamma(R[x])$ and $\Gamma(R[[x]])$.

For any undefined terms concerning the zero-divisor graph, the reader may contact [6].

We will write $x \sim y$ if $Ann(y) = Ann(x)$. It is clear that $\sim$ is an equivalence relation on R with equivalence classes $[x] = \{y \in R : x \sim y\}$.

If $x \sim y$ and $xz = 0$, then $z \in Ann(x) = Ann(y)$, and so $zy = 0$. Thus multiplication on the equivalence classes of $\sim$ is well-defined. Hence the multiplication $[x].[y] = [xy]$ make sense.

The graph of equivalence classes of zero-divisors of a ring R, denoted by $\Gamma_E(R)$, is a graph associated to R, with vertex set $\{[x] : x \in Z^*(R)\}$ and two vertices $[x]$ and $[y]$ are adjacent if $[x].[y] = [xy] = [0]$. In many cases $\Gamma_E(R)$ is finite when $\Gamma(R)$ is infinite, for example in the ring $R = \mathbb{Z}[x, y] \big/ (x^3, xy)$, $\Gamma(R)$ is infinite, while $\Gamma_E(R)$ has only 4 vertices, see [18].

Now, we will study some relations between $\Gamma(R), \Gamma(R[x]), \Gamma(R[[x]])$ and $\Gamma_E(R)$, when R is strongly EM-ring.

Theorem 4.1. *If R is a strongly EM-ring, then the zero-divisor graph of equivalence classes $\Gamma_E(R)$ is isomorphic to $\Gamma_E(R[x])$ and $\Gamma_E(R[[x]])$ with the correspondence $f \leftrightarrow c_f$.*

A graph is called bipartite if its vertex set can be partitioned into two parts with no adjacency between the vertices in each part. If R is a non-reduced ring, then clearly, $\Gamma(R[x])$ and $\Gamma(R[[x]])$ cannot be bipartite, while $\Gamma(R)$ could be bipartite or not.

Theorem 4.2. *If R is a reduced strongly EM-ring, then the following are equivalent*:

(1) $\Gamma(R[[x]])$ *is bipartite.*

(2) $\Gamma(R[x])$ *is bipartite.*
(3) $\Gamma(R)$ *is bipartite.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) If $f(x)-g(x)-h(x)-f(x)$ is a triangle in $\Gamma(R[[x]])$, then since R is a reduced ring, $c_f-c_g-c_h-c_f$ is indeed a triangle in $\Gamma(R)$, a contradiction. $\square$

A graph is called complete if all of its vertices are adjacent. It was shown in [7] that if $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ is complete if and only if $\Gamma(R[x])$ is complete if and only if $\Gamma(R[[x]])$ is complete, and so the diameter of the graph $diam(\Gamma(R)) = diam(\Gamma(R[x])) = diam(\Gamma(R[[x]])) = 1$, when $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, where the diameter of a graph is the supremum of the distances between its vertices.

Theorem 4.3. *Let R be a strongly EM-ring. Then the following are equivalent*:
(1) $diam(\Gamma(R)) = 2$.
(2) $diam(\Gamma(R[x])) = 2$.
(3) $diam(\Gamma(R[[x]])) = 2$.

Proof. (1) $\Rightarrow$ (2) Assume that $f, g \in Z^*(R[x])$ such that $fg \neq 0$, then $c_f c_g \neq 0$ where c_f, c_g are annihilating contents of f and g in $R[x]$, respectively. Since $diam(\Gamma(R)) = 2$, there exists $r \in Z(R) - \{c_f, c_g\}$ such that $c_f - r - c_g$ is a path in $\Gamma(R)$, even if $c_f = c_g$ or not. Therefore $f - r - g$ is a path in $\Gamma(R[x])$ and hence $d(f, g) = 2$.
(2) $\Rightarrow$ (3) Let $f, g \in Z^*(R[[x]])$ such that $fg \neq 0$, then $c_f c_g \neq 0$ where c_f, c_g are an annihilating contents of f and g in $R[[x]]$ respectively. Since $diam(\Gamma(R[x])) = 2$, there exists $h \in Z(R[x])$ such that $c_h \in Z(R) - \{c_f, c_g\}$ and $c_f - c_h - c_g$ is a path in $\Gamma(R[x])$, even if $c_f = c_g$ or not. Hence $f - h - g$ is a path in $\Gamma(R[[x]])$.
(3) $\Longrightarrow$ (1) Since $diam(\Gamma(R)) \leq diam(\Gamma(R[[x]])) = 2$ and $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, $diam(\Gamma(R)) = 2$. $\square$

Since $diam(\Gamma(S)) \leq 3$ for any ring S, we have the following result.

Corollary 4.4. *Let R be a strongly EM-ring, such that R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Then $diam(\Gamma(R[[x]])) = diam(\Gamma(R[x])) = diam(\Gamma(R))$.*

If R is an EM-ring, then we will get similar relations between $\Gamma(R)$ and $\Gamma(R[x])$.

Unfortunately, the converse of the above corollary is not true, since there are rings R such that $diam(\Gamma(R[[x]])) = diam(\Gamma(R[x])) = diam(\Gamma(R))$, but R is not a strongly EM-ring, as shown in the following example. But we recall first that if R is a ring, and M is an R-module, then the idealization $R(+)M$ is the set of all ordered pairs $(r, m) \in R \times M$, equipped with addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sn)$. It is well-known that $R(+)R \simeq R[x] \,/ (x^2)$.

Example 4.5. Let p be a prime integer, and consider the idealization ring $R = \mathbb{Z}(+)\mathbb{Z}_p$. It was shown in [16] that $diam(\Gamma(R[[x]])) = diam(\Gamma(R[x])) = diam(\Gamma(R)) = 2$, and $Z(R) = p\mathbb{Z}(+)\mathbb{Z}_p = Ann((0,1))$. But R is not an EM-ring, since the polynomial $f(x) = (p,0)+(0,1)x \in Z^*(R[x])$ has no annihilating content.

It is clear that if R is a strongly EM-ring, then it is an EM-ring, and we showed in Theorem 3.26 that these conditions are equivalent in Noetherian rings. Now, we will use the results obtained in Corollary 4.4 concerning zero-divisor graphs to give an example of an EM-ring that is not a strongly EM-ring.

Example 4.6. Let p be a prime integer, and consider the idealization ring $R = \mathbb{Z}(+)\mathbb{Z}(p^\infty)$. It was shown in [16] that $diam(\Gamma(R)) = diam(\Gamma(R[x])) = 2$, while $diam(\Gamma(R[[x]])) = 3$, and $Z(R) = p\mathbb{Z}(+)\mathbb{Z}(p^\infty) = Ann((0, \frac{1}{p}))$. So, it follows by Corollary 4.4 that R is not a strongly EM-ring. Now to show that R is an EM-ring, let $f(x) = \sum_{i=0}^{n} a_i x^i \in Z^*(R[x])$. We have two cases:

Case I: $a_i = (0, \frac{m_i}{p^{k_i}})$, $\gcd(m_i, p) = 1$ for each i. Let $k = \mathrm{Max}\{k_1, \dots, k_n\}$, $f_1(x) = \sum_{i=0}^{n} (m_i p^{k-k_i}, \frac{1}{p^{k_i}}) x^i$. Then $f_1(x) \in reg(R[x])$, and $f(x) = (0, \frac{1}{p^k}) f_1(x)$.

Case II: $a_i = (n_i p^{l_i}, \frac{m_i}{p^{k_i}})$, $\gcd(m_i, p) = 1 = \gcd(n_i, p)$, $l_i \geq 1$ and $n_i \neq 0$ for some i. Let $I = \{i : n_i \neq 0\}$ and let $l = \mathrm{Min}\{l_i : i \in I\}$, $f_1(x) = \sum_{i=0}^{n} (n_i p^{l_i - l}, \frac{m_i}{p^{k_i + l}}) x^i$. Then $f_1(x) \in reg(R[x])$, and $f(x) = (p^l, 0) f_1(x)$. Thus R is an EM-ring. Moreover, R is a generalized morphic ring, since $Ann((0, \frac{n}{p^k})) = (p^k, 0)R$ and $Ann((mp^l, \frac{n}{p^k})) = (0, \frac{1}{p^l})R$.

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