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SELF-HOMOTOPY EQUIVALENCES OF MOORE SPACES DEPENDING ON COHOMOTOPY GROUPS

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Abstract. Given a topological space X and a non-negative integer k, $\mathcal{E}_k^\sharp(X)$ is the set of all self-homotopy equivalences of X that do not change maps from X to an t-sphere S^t homotopically by the composition for all $t \geq k$. This set is a subgroup of the self-homotopy equivalence group $\mathcal{E}(X)$. We find certain homotopic tools for computations of $\mathcal{E}_k^\sharp(X)$. Using these results, we determine $\mathcal{E}_k^\sharp(M(G,n))$ for $k \geq n$, where $M(G,n)$ is a Moore space type of (G,n) for a finitely generated abelian group G.

1. Introduction

For a topological space X, we denote $\mathcal{E}(X)$ as the set of all homotopy classes of self-homotopy equivalences of X. Then $\mathcal{E}(X)$ is a subset of $[X, X]$ and has a group structure given by the composition of homotopy classes. The subset $\mathcal{E}(X)$ has been studied extensively by various authors, including Arkowitz [2], Arkowitz and Maruyama [3], Lee [5,7], Rutter [9], Sawashita [10], and Sieradski [11]. Moreover several subgroups of $\mathcal{E}(X)$ have also been studied, notably the group $\mathcal{E}_\sharp^k(X)$, which consists of all elements of $\mathcal{E}(X)$ that induce the identity homomorphism on homotopy groups $\pi_t(X)$ for $t = 0, 1, 2, \ldots, k$. In our previous work [4], the first and second authors used homotopy techniques to calculate these subgroups for the wedge products of Moore spaces.

In [6], we introduced $\mathcal{E}_k^\sharp(X)$, which consists of the elements of $\mathcal{E}(X)$ that induce the identity homomorphism on cohomotopy groups $\pi^t(X)$ for $t \geq k$. Equivalently, it can be defined as follows: For a non-negative integer i, consider the self-map $f : X \to X$ such that $g \circ f$ is homotopic to g for each $g : X \to S^t$

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and for each $t \geq k$. The set of all homotopy classes of such self-maps of X is denoted by $[X, X]_k^\sharp$, that is,

$$[X, X]_k^\sharp = \left\{ f \in [X, X] \mid g \circ f \sim g \text{ for each } g : X \to S^t \text{ for all } t \geq k \right\}.$$

Then

$$\mathcal{E}_k^\sharp(X) = \mathcal{E}(X) \cap [X, X]_k^\sharp.$$

In [6], we proved that $\mathcal{E}_k^\sharp(X)$ is a subgroup of $\mathcal{E}(X)$ and, if X is a finite CW-complex, then $\mathcal{E}_k^\sharp(X)$ has a lower bound whereas $\mathcal{E}_\sharp^k(X)$ has an upper bound. Moreover, we calculated $\mathcal{E}_k^\sharp(X)$ for special Moore space $X = M(\mathbb{Z}_p, n)$ and co-Moore space $X = C(\mathbb{Z}_p, n)$.

In this paper, we determine $\mathcal{E}_k^\sharp(M(G, n))$ completely for $k \geq n$, where $M(G, n)$ is a Moore space type of (G, n) with G a finitely generated abelian group and $n \geq 3$. To solve this problem, we first study a subset $\mathcal{Z}_k^\sharp(Y, Z)$ of $[Y, Z]$ for spaces Y and Z. The subset $\mathcal{Z}_k^\sharp(Y, Z)$ is defined by the set of all $h \in [Y, Z]$ whose induced homomorphism $h^{\sharp t} : \pi^t(Z) \to \pi^t(Y)$ is the trivial homomorphism for $t \geq k$. Furthermore, we investigate the properties of $\mathcal{E}_k^\sharp(X)$ for given wedge product space X to prove the following theorem in Section 4:

Theorem 4.3. *Let $M(G, n)$ be a Moore space type of (G, n) and $G = F \oplus T$ be a finitely generated abelian group G with free part F and torsion part T. Then $\mathcal{E}_k^\sharp(M(G, n))$ is isomorphic to*

$$\mathcal{E}_k^\sharp(M(F, n)) \oplus \mathcal{Z}_k^\sharp(M(F, n), M(T, n)) \oplus \mathcal{Z}_k^\sharp(M(T, n), M(F, n)) \oplus \mathcal{E}_k^\sharp(M(T, n)).$$

From Theorem 4.3, the problem of computing $\mathcal{E}_k^\sharp(M(G, n))$ reduces to that of computing the $\mathcal{E}_k^\sharp$-groups and $\mathcal{Z}_k^\sharp$-groups for Moore spaces for (possibly infinite) cyclic groups. In Section 5, we compute explicitly $\mathcal{Z}_k^\sharp$-groups for Moore spaces for (possibly infinite) cyclic groups and combine the relevant $\mathcal{E}_k^\sharp$-groups computed in our previous paper and are recorded as Theorem 2.4 to obtain the following main result:

Corollary 5.13. *Let $G = \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) \bigoplus \left(\bigoplus_{j=1}^{s} \mathbb{Z}_{q_j}\right)$. Then, we have*

$$\mathcal{E}_k^\sharp(M(G, n)) \cong \begin{cases} GL(m, \mathbb{Z}) \bigoplus^{m \times t} \mathbb{Z}_2 \bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right) \bigoplus \mathcal{E}(M(T, n)) & \text{if } k \geq n+2, \\ GL(m, \mathbb{Z}) \bigoplus^{m \times t + t + \ell} \mathbb{Z}_2 \bigoplus \left(\bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right)\right) & \text{if } k = n+1, \\ GL(m, \mathbb{Z}) \bigoplus^{t+\ell} \mathbb{Z}_2 \bigoplus \left(\bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right)\right) & \text{if } k = n, \end{cases}$$

where $GL(m,\mathbb{Z})$ is the general linear group of degree m, t is the number of even q_j and ℓ is the number of pairs $\{i,j\} \subset \{1,\dots,s\}$ such that both q_i and q_j are even and $i \neq j$.

Throughout this paper, all topological spaces are based and have the based homotopy type of a CW-complex, and all maps and homotopies preserve base points. For the spaces X and Y, we denote by $[X,Y]$ the set of all homotopy classes of maps from X to Y. No distinction is made between the notation of a map $X \to Y$ and that of its homotopy class in $[X,Y]$. When a group G is generated by a set $\{a_1,\dots,a_n\}$, then we denote it by $G\{a_1,\dots,a_n\}$. Moreover, when $f : X \to Y$ is a map, $f^{\sharp k} : \pi^k(Y) \to \pi^k(X)$ denote the induced homomorphisms on the k-th cohomotopy group.

2. Preliminaries

In this section, we review some results provided in [3,6], knowledge of which would be useful when reading this paper. First, we introduce the following proposition from [3] that is a basic concept in developing this paper.

Proposition 2.1 (Arkowitz and Maruyama [3])**.** *If X is $(k-1)$-connected, Y is $(\ell-1)$-connected and, further, if $k,\ell \geq 2$ and $\dim P \leq k+\ell-1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection*:

$$[P, X \vee Y] \to [P,X] \oplus [P,Y].$$

Consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1,n_1)$ and $M_2 = M(G_2,n_2)$. When $X = M_1 \vee M_2$, we denote by $i_j : M_j \to X$ the inclusion and by $p_j : X \to M_j$ the projection, where $j = 1,2$. If $f : X \to X$, then we define $f_{jk} = M_k \to M_j$ by $f_{jk} = p_j \circ f \circ i_k$ for $j,k = 1,2$. Then, by Proposition 2.1, we have

$$[X,X] \cong [M_1,M_1] \oplus [M_1,M_2] \oplus [M_2,M_1] \oplus [M_2,M_2]$$

and, from [3, Proposition 2.6], there exists a bijective function θ that assigns to each $f \in [X,X]$ a 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$. In addition, we have the following:

(1) $\theta(f+g) = \theta(f) + \theta(g)$, so θ is an isomorphism $[X,X] \to \bigoplus_{j,k=1,2} [M_k, M_j]$;

(2) $\theta(f \circ g) = \theta(f)\theta(g)$, where $f \circ g$ denotes composition in $[X,X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.

Moreover, for each $f \in [X,X]$, the induced homomorphism $f^{\sharp k}$ on the cohomotopy groups $\pi^k(X)$ is determined as in the following propositions:

Proposition 2.2 (Proposition 3.4 in [6])**.** *For any $f \in [M_1 \vee M_2, M_1 \vee M_2]$, we have*

$$f^{\sharp k}(\gamma_1,\gamma_2) = (f_{11}^{\sharp k}(\gamma_1) + f_{21}^{\sharp k}(\gamma_2), f_{12}^{\sharp k}(\gamma_1) + f_{22}^{\sharp k}(\gamma_2)),$$

where $\gamma_1 \in \pi^k(M_1)$ *and* $\gamma_2 \in \pi^k(M_2)$.

Proposition 2.3 (Proposition 3.5 in [6])**.** *If* $f \in \mathcal{E}_k^\sharp(M_1 \vee M_2)$, *then*

$$f^{\sharp k} = \begin{pmatrix} 1_{\pi^k(M_1)} & 0 \\ 0 & 1_{\pi^k(M_2)} \end{pmatrix}.$$

In [6], we computed $\mathcal{E}_k^\sharp(M(\mathbb{Z}_q, n))$ and obtained the following table:

Theorem 2.4 (Theorems 4.1, 4.2, 4.3, and 4.4 in [6])**.**

	q : odd	$q \equiv 0 \pmod 4$	$q \equiv 2 \pmod 4$
$\mathcal{E}_{n+1}^\sharp(M(\mathbb{Z}_q, n))$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathcal{E}_{n}^\sharp(M(\mathbb{Z}_q, n))$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathcal{E}_{n-1}^\sharp(M(\mathbb{Z}_q, n))$	1	1	1

3. Maps inducing a trivial homomorphism on cohomotopy groups

In [8], Maruyama studied the subset $\mathcal{Z}_\sharp^k(Y,Z)$ of $[Y,Z]$. $\mathcal{Z}_\sharp^k(Y,Z)$ is the subset of all homotopy classes from Y to Z that induce the trivial homomorphism $\pi_t(Y)$ to $\pi_t(Z)$ for $0 \leq t \leq k$. In this section, we introduce a subset $\mathcal{Z}_k^\sharp(Y,Z)$ of $[Y,Z]$ that is a dual concept of $\mathcal{Z}_\sharp^k(Y,Z)$ and, in particular, investigate some properties of $\mathcal{Z}_k^\sharp(Y,Z)$ for wedge spaces Y and Z.

Definition 3.1. Let Y and Z be topological spaces. Then the subset $\mathcal{Z}_k^\sharp(Y,Z)$ is defined by the set of all $h \in [Y,Z]$ whose induced homomorphism $h^{\sharp t} : \pi^t(Z) \to \pi^t(Y)$ is the trivial homomorphism for $t \geq k$. Equivalently,

$$\mathcal{Z}_k^\sharp(Y,Z) = \{f \in [Y,Z] \mid \alpha \circ f \simeq 0 \text{ for all } \alpha : Z \to S^t,\ t \geq k\}.$$

If $Z = Y$, then $\mathcal{Z}_k^\sharp(Y,Y)$ is simply denoted by $\mathcal{Z}_k^\sharp(Y)$.

It is well known that there is a bijective map $\tau : [Y \vee Z, W] \to [Y,W] \oplus [Z,W]$ defined by $\tau(f) = (f \circ i_Y, f \circ i_Z)$, where $i_I : I \to Y \vee W$ is an inclusion map for $I = Y, W$. The inverse of τ is defined by $\rho : [Y,W] \oplus [Z,W] \to [Y \vee Z, W]$ defined by $\rho(g,h) = \nabla \circ (g \vee h)$, where ∇ is the folding map.

Proposition 3.1. *Let* Y, Z, *and* W *be CW-complexes. Then there is a bijective map* $\tau : \mathcal{Z}_k^\sharp(Y \vee Z, W) \to \mathcal{Z}_k^\sharp(Y,W) \oplus \mathcal{Z}_k^\sharp(Z,W)$ *defined by* $\tau(f) = (f \circ i_Y, f \circ i_Z)$.

Proof. It is sufficient to show that $\tau(\mathcal{Z}_k^\sharp(Y \vee W)) \subset \mathcal{Z}_k^\sharp(Y,W) \oplus \mathcal{Z}_k^\sharp(Z,W)$ and $\rho(\mathcal{Z}_k^\sharp(Y,W) \oplus \mathcal{Z}_k^\sharp(Z,W)) \subset \mathcal{Z}_k^\sharp(Y \vee Z, W)$.

Let $f \in \mathcal{Z}_k^\sharp(Y \vee Z, W)$ and $t \geq k$. Since $f^{\sharp t} = 0$, $(f \circ i_I)^{\sharp t} = i_I^\sharp \circ f^{\sharp t} = 0$ for $I = Y, W$. Hence, $\tau(f) = (f \circ i_Y, f \circ i_Z) \in \mathcal{Z}_k^\sharp(Y,W) \oplus \mathcal{Z}_k^\sharp(Z,W)$.

Let $(g,h) \in \mathcal{Z}_k^\sharp(Y,W) \oplus \mathcal{Z}_k^\sharp(Z,W)$. Since $g^{\sharp t} = 0$ and $h^{\sharp t} = 0$ for $t \geq k$, $\rho(g,h)^{\sharp t} = (\nabla \circ (g \vee h))^{\sharp t} = (g \vee h)^{\sharp t} \circ \nabla^{\sharp t} = (g^{\sharp t} \vee h^{\sharp t}) \circ \nabla^{\sharp t} = (0 \vee 0) \circ \nabla^{\sharp t} = 0$. Hence, $\rho(g,h) = \nabla \circ (g \vee h) \in \mathcal{Z}_k^\sharp(Y \vee Z, W)$. $\square$

Define a map

$$\Phi : [Y, W \vee Z] \to [Y, W] \oplus [Y, Z]$$

by $\Phi(f) = (p_W \circ f, p_Z \circ f)$, where $p_I = W \vee Z \to I$ is the projection map for $I = W, Z$. If Y is a suspension of a space, that is $Y = \Sigma Y'$ for some Y', then there is a suspension co-multiplication C_Y. Using C_Y, we can obtain a map

$$\Psi : [Y, W] \oplus [Y, Z] \to [Y, W \vee Z]$$

given by $\Psi(g, h) = (g \vee h) \circ C_Y$. Then $\Phi \circ \Psi = id_{[Y,W]} \oplus id_{[Y,Z]}$. Hence Ψ is an injection and Φ is a surjection.

Proposition 3.2. *Let Y, Z, and W be CW-complexes. Suppose that $Y = \Sigma Y'$ with $\dim(Y) \leq k + \ell - 1$, Z is $(k-1)$-connected, and W is $(\ell - 1)$-connected for $k, \ell \geq 2$. Then there is a bijection map from $\mathcal{Z}_k^\sharp(Y, Z \vee W)$ to $\mathcal{Z}_k^\sharp(Y, Z) \oplus \mathcal{Z}_k^\sharp(Y, W)$.*

Proof. By Proposition 2.1, $\Phi : [X, Y \vee Z] \to [X, Y] \oplus [X, Z]$ is bijective. Since $Y = \Sigma Y'$, there is a suspension co-multiplication C_Y and the map $\Psi : [X, Y] \oplus [X, Z] \to [X, Y \vee Z]$ is bijective. By a method similar to Proposition 3.1, we can complete the proof. □

Let Y be a Moore space of type $(G_1 \oplus G_2, n)$ for $n \geq 3$ and let Y_1 and Y_2 be Moore spaces of type (G_1, n) and (G_2, n), respectively. Then we have $Y \simeq Y_1 \vee Y_2$.

Corollary 3.3. *Let Y be a Moore space type of $(G_1 \oplus G_2, n)$ for $n \geq 3$. Then we have*

$$\mathcal{Z}_k^\sharp(Y) \cong \mathcal{Z}_k^\sharp(Y_1) \oplus \mathcal{Z}_k^\sharp(Y_1, Y_2) \oplus \mathcal{Z}_k^\sharp(Y_2, Y_1) \oplus \mathcal{Z}_k^\sharp(Y_2),$$

where $Y_1 = M(G_1, n)$ and $Y_2 = M(G_2, n)$.

4. Properties of $\mathcal{E}_k^\sharp(M(G, n))$

In [6], we investigated some properties and determined $\mathcal{E}_k^\sharp(M(\mathbb{Z}_p, n))$. In this section, we extend and apply these results to a Moore space $M(G, n)$, where G is a finitely generated abelian group and $n \geq 3$. Since G is a finitely generated abelian group, $G = \bigoplus_{i=1}^s G_i$, where G_i is a cyclic group. Then, we have $M(G, n) \simeq \bigvee_{i=1}^s M(G_i, n)$. Therefore, we have

$$[M(G, n), M(G, n)] \cong \bigoplus_{i,j=1}^{s} [M(G_i, n), M(G_j, n)]$$

and

$$\pi^k(M(G, n)) \cong \bigoplus_{i=1}^{s} \pi^k(M(G_i, n))$$

by Proposition 2.1.

Let $i_i = M(G_i, n) \to M(G, n)$ be the inclusion and let $p_j : M(G, n) \to M(G_j, n)$ be the projection. For a self-map $f : M(G, n) \to M(G, n)$, we define

$f_{ji} = M_i \to M_j$ by $f_{ji} = p_j \circ f \circ i_i$. Then there is a bijection θ that assigns each $f \in [M(G,n), M(G,n)]$ the $s \times s$ matrix

$$\theta(f) = \begin{pmatrix} f_{11} & \cdots & f_{1s} \\ \vdots & \ddots & \vdots \\ f_{s1} & \cdots & f_{ss} \end{pmatrix}.$$

Thus, there is a bijection $\theta^\sharp$ that assigns each

$$f^{\sharp k} \in \mathrm{Hom}(\pi^k(M(G,n)), \pi^k(M(G,n)))$$

the $s \times s$ matrix

$$\theta^\sharp(f^{\sharp k}) = \begin{pmatrix} f_{11}^{\sharp k} & \cdots & f_{1s}^{\sharp k} \\ \vdots & \ddots & \vdots \\ f_{s1}^{\sharp k} & \cdots & f_{ss}^{\sharp k} \end{pmatrix}.$$

Throughout this paper, $\theta(f)$ and $\theta^\sharp(f^{\sharp k})$ are identified with f and $f^{\sharp k}$ and are called the *matrix representations* of f and $f^{\sharp k}$, respectively. Furthermore, each $\gamma \in \pi^k(M(G,n))$ can be represented as $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, where $\gamma_i = \gamma \circ i_i \in \pi^k(M(G_i,n))$.

Proposition 4.1. *For each $f \in [M(G,n), M(G,n)]$, we have*

$$f^{\sharp k}(\gamma) = (\sum_{i=1}^{s} f_{i1}^{\sharp k}(\gamma_i), \ldots, \sum_{i=1}^{s} f_{is}^{\sharp k}(\gamma_i))$$

for $\gamma \in \pi^k(M(G,n))$ and $\gamma_i = \gamma \circ i_i$.

Proof. This can be proved in a manner similar to Proposition 3.4 in [6]. $\square$

The matrix representations of the identity map $1_{M(G,n)}$ and the induced map $1_{M(G,n)}^{\sharp t}$ are

$$\theta(1_{M(G,n)}) = \begin{pmatrix} 1_{M_1} & 0 & \cdots & 0 \\ 0 & 1_{M_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1_{M_s} \end{pmatrix}$$

and

$$\theta^\sharp(1_{M(G,n)}^{\sharp t}) = \begin{pmatrix} 1_{M_1}^{\sharp t} & 0 & \cdots & 0 \\ 0 & 1_{M_2}^{\sharp t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1_{M_s}^{\sharp t} \end{pmatrix},$$

respectively, where 1_{M_i} is the identity map in $[M(G_i,n), M(G_i,n)]$ and $1_{M_i}^{\sharp t}$ is the induced map of 1_{M_i} on the t-th cohomotopy group. $\theta(1_{M(G,n)})$ and $\theta^\sharp(1_{M(G,n)}^{\sharp t})$ are simply denoted by I_s and $I_s^{\sharp t}$, respectively.

Proposition 4.2. *Let $G = \bigoplus_{j=1}^{s} G_j$ be a finitely generated abelian group. If $f \in \mathcal{E}_k^{\sharp}(M(G,n))$, then*

$$f^{\sharp t} = I_s^{\sharp t}$$

for $t \geq k$.

Proof. For any $f \in \mathcal{E}_k^{\sharp}(M(G,n))$, f induces the identity homomorphism on $\pi^t(M(G,n))$ for $t \geq k$. Thus $f^{\sharp t} = id_{\pi^t(M(G,n))} = 1_{M(G,n)}^{\sharp t}$. Hence, $\theta^{\sharp}(f^{\sharp t}) = \theta^{\sharp}(1_{M(G,n)})^{\sharp t}) = I_s^{\sharp t}$. Therefore,

$$f^{\sharp t} = I_s^{\sharp t}$$

for $t \geq k$. □

We determine the set of self-homotopy equivalences that induce the identity map on cohomotopy groups for Moore space of type (G,n), where G is a finitely generated abelian group and n is a positive integer. G can be represented as follows:

$$G \cong \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{s} \mathbb{Z}_{q_j}\right),$$

where $\mathbb{Z}_{q_j}$ is a primary cyclic group and q_j represents powers of prime numbers.

Let $G_1 = (\bigoplus_{i=1}^{m} \mathbb{Z}) \oplus \left(\bigoplus_{j=1}^{s} \mathbb{Z}_{q_j}\right)$ and $G_2 = (\bigoplus_{i=1}^{r} \mathbb{Z}) \oplus \left(\bigoplus_{j=1}^{t} \mathbb{Z}_{q'_j}\right)$. Then,

$$G_1 \oplus G_2 \cong \left(\bigoplus_{i=1}^{m+r} \mathbb{Z}\right) \bigoplus \left(\bigoplus_{j=1}^{s+t} \mathbb{Z}_{q_j}\right), \tag{4.1}$$

where $q_{j+s} = q'_j$ for $1 \leq j \leq t$. Let $X = M(G_1 \oplus G_2, n)$ be a Moore space. Since

$$M(G_1 \oplus G_2, n) \simeq M(G_1, n) \vee M(G_2, n),$$

we have

$$X \simeq (\vee_{i=1}^{m} M(\mathbb{Z}, n)) \vee \left(\vee_{j=1}^{s} M(\mathbb{Z}_{q_j}, n)\right). \tag{4.2}$$

From (4.2), we have

$$\begin{aligned}
&[X,X] \\
&\cong [\vee_{i=1}^{m} M(\mathbb{Z},n), \vee_{i=1}^{m} M(\mathbb{Z},n)] \bigoplus \left[\vee_{j=1}^{s} M(\mathbb{Z}_{q_j},n), \vee_{i=1}^{m} M(\mathbb{Z},n)\right] \\
&\quad \bigoplus \left[\vee_{i=1}^{m} M(\mathbb{Z},n), \vee_{j=1}^{s} M(\mathbb{Z}_{q_j},n)\right] \bigoplus \left[\vee_{j=1}^{s} M(\mathbb{Z}_{q_j},n), \vee_{j=1}^{s} M(\mathbb{Z}_{q_j},n)\right].
\end{aligned}$$

For each $f \in [X,X]$, the matrix representation of f is

$$\theta(f) = \begin{pmatrix} p_1 \circ f \circ i_1 & \cdots & p_{m+s} \circ f \circ i_1 \\ \vdots & \ddots & \vdots \\ p_1 \circ f \circ i_{m+s} & \cdots & p_{m+s} \circ f \circ i_{m+s} \end{pmatrix}$$

and we have

$$f^{\sharp k}(\gamma) = \left(\sum_{i=1}^{m+s} (p_i \circ f \circ i_1)^{\sharp k}(\gamma_i),\ \ldots\ , \sum_{i=1}^{m+s} (p_i \circ f \circ i_{m+s})^{\sharp k}(\gamma_i) \right),$$

where $\gamma_i = \gamma \circ i_i$.

Now, we divide the matrix representation of f into

$$\theta(f) = \left(\begin{array}{c|c} M_1(f) & M_2(f) \\ \hline M_3(f) & M_4(f) \end{array} \right),$$

where $M_1(f)$ is the square matrix of degree n whose components are the maps in $[M(\mathbb{Z}, n), M(\mathbb{Z}, n)]$, $M_2(f)$ is the $m \times s$ matrix whose components are the maps in $[M(\mathbb{Z}, n), M(\mathbb{Z}_{q_j}, n)]$, $M_3(f)$ is the $s \times m$ matrix whose components are the maps in $[M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}, n)]$, and $M_4(f)$ is the square matrix of degree s matrix whose components are the maps in $[M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q'_j}, n)]$.

From the above matrix, we have the following induced matrix:

$$\theta^\sharp(f^{\sharp t}) = \left(\begin{array}{c|c} M_1(f)^{\sharp t} & M_2(f)^{\sharp t} \\ \hline M_3(f)^{\sharp t} & M_4(f)^{\sharp t} \end{array} \right).$$

In [3], for given $X = M(F \oplus T, n)$,

$$\begin{aligned} \mathcal{E}(X) \cong \mathcal{E}(M(F, n)) \oplus [M(F, n), M(T, n)] \oplus [M(T, n), M(F, n)] \\ \oplus\, \mathcal{E}(M(T, n)), \end{aligned} \tag{4.3}$$

where F and T are finitely generated abelian groups.

Theorem 4.3. *Let $X = M(F \oplus T, n)$ be a Moore space with finitely generated abelian groups F and T and positive integer n. Then*

$$\begin{aligned} \mathcal{E}^\sharp_k(X) \cong\ & \mathcal{E}^\sharp_k(M(F, n)) \oplus \mathcal{Z}^\sharp_k(M(F, n), M(T, n)) \\ & \oplus \mathcal{Z}^\sharp_k(M(T, n), M(F, n)) \oplus \mathcal{E}^\sharp_k(M(T, n)). \end{aligned}$$

Proof. From (4.3), for $f \in \mathcal{E}(X)$, f can be represented by $f = (f_1, f_2, f_3, f_4)$ for some $f_1 \in \mathcal{E}(M(F, n))$, $f_2 \in [M(F, n), M(T, n)]$, $f_3 \in [M(T, n), M(F, n)]$, and $f_4 \in \mathcal{E}(M(T, n))$. Then,

$$M_1(f) = \theta(f_1), M_2(f) = \theta(f_2), M_3(f) = \theta(f_3), \text{ and } M_4(f) = \theta(f_4).$$

Since, for $t \geq k$,

$$\theta^\sharp(f^{\sharp t}) = \left(\begin{array}{c|c} M_1(f)^{\sharp t} & M_2(f)^{\sharp t} \\ \hline M_3(f)^{\sharp t} & M_4(f)^{\sharp t} \end{array} \right) = I^{\sharp t}_{m \times s}$$

by Proposition 4.2,

$$M_1(f)^{\sharp t} = I^{\sharp t}_m, M_2(f)^{\sharp t} = O^{\sharp t}_{m \times s}, M_3(f)^{\sharp t} = O^{\sharp t}_{s \times m}, M_4(f)^{\sharp t} = I^{\sharp t}_s$$

for $t \geq k$, where $O^{\sharp t}_{m \times s}$ and $O^{\sharp t}_{s \times m}$ are an $m \times s$ zero matrix and an $s \times m$ zero matrix, respectively. Thus, if $f = (f_1, f_2, f_3, f_4) \in \mathcal{E}^\sharp_k(X)$, then $f_1 \in$

$\mathcal{E}_k^\sharp(M(F,n)), f_2 \in \mathcal{Z}_k^\sharp(M(F,n)), f_3 \in \mathcal{Z}_k^\sharp(M(F,n))$, and $f_4 \in \mathcal{E}_k^\sharp(M(T,n))$. Therefore, $\mathcal{E}_k^\sharp(X)$ is contained in

$$\mathcal{E}_k^\sharp(M(F,n)) \oplus \mathcal{Z}_k^\sharp(M(F,n), M(T,n)) \oplus \mathcal{Z}_k^\sharp(M(T,n), M(F,n)) \oplus \mathcal{E}_k^\sharp(M(T,n)).$$

Conversely, let $f = (f_1, f_2, f_3, f_4)$ belong to

$$\mathcal{E}_k^\sharp(M(F,n)) \oplus \mathcal{Z}_k^\sharp(M(F,n), M(T,n)) \oplus \mathcal{Z}_k^\sharp(M(T,n), M(F,n)) \oplus \mathcal{E}_k^\sharp(M(T,n)).$$

Then $f \in \mathcal{E}(X)$ from (4.3). Since $\theta^\sharp(f_1^{\sharp t}) = I_m^{\sharp t}, \theta^\sharp(f_2^{\sharp t}) = O_{s\times m}^{\sharp t}, \theta^\sharp(f_3^{\sharp t}) = O_{m\times s}^{\sharp t}$, and $\theta^\sharp(f_4^{\sharp t}) = I_s^{\sharp t}$, $\theta^\sharp(f^{\sharp t}) = I_{s\times m}^{\sharp t}$ for each $t \geq k$. Therefore, $f \in \mathcal{E}_k^\sharp(X)$. □

5. Computations of $\mathcal{E}_k^\sharp(M(G,n))$

Let $X_1 = M(\bigoplus_{i=1}^m \mathbb{Z}, n)$ and $X_2 = M(\bigoplus_{j=1}^s \mathbb{Z}_{q_j}, n)$. For the wedge product space $X = X_1 \vee X_2$, to determine $\mathcal{E}_k^\sharp(X)$ by Theorem 4.3, we need to calculate each $\mathcal{E}_k^\sharp(X_1)$, $\mathcal{Z}_k^\sharp(X_2, X_1)$, $\mathcal{Z}_k^\sharp(X_1, X_2)$, and $\mathcal{E}_k^\sharp(X_2)$.

We first compute $\mathcal{E}_k^\sharp(X_1)$. Since $X_1 \simeq \vee_{i=1}^m M(\mathbb{Z}, n) \simeq \vee_{i=1}^m S_i^n$, where S_i^n is a copy of S^n, we have

$$[X_1, X_1] \cong \bigoplus_{i,j=1}^{m} [S_i^n, S_j^n] \cong \bigoplus_{i=1}^{m\times m} \mathbb{Z}$$

by Proposition 2.1 in [9]. Thus, $\mathcal{E}(X_1) = GL(m, \mathbb{Z})$, where $GL(m,\mathbb{Z})$ is the general linear group of degree m.

Let $i_i : S_i^n \to X_1$ be the inclusion and $p_j : X_1 \to S_j^n$ be the projection. For a self-map $f : X_1 \to X_1$, we define $f_{ji} : S_i^n \to S_j^n$ by $f_{ji} = p_j \circ f \circ i_i$. Since S_i^n and S_j^n are the copies of the n-dimensional sphere, we see that $[S_i^n, S_j^n] = [S^n, S^n]$. Let ι_n be the identity map on $[S^n, S^n]$; in particular, let $(\iota_n)_{ji}$ be the identity map on $[S_i^n, S_j^n]$. Then the matrix representation of f is given by

$$\theta(f) = \begin{pmatrix} t_{11}(\iota_n)_{11} & \cdots & t_{1m}(\iota_n)_{1m} \\ \vdots & \ddots & \vdots \\ t_{m1}(\iota_n)_{m1} & \cdots & t_{mm}(\iota_n)_{mm} \end{pmatrix},$$

where t_{ji} is the degree of f_{ji} for $j, i = 1, 2, \ldots, m$.

Lemma 5.1. *For $n \geq 1$,*

$$\mathcal{Z}_k^\sharp(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k > n, \\ 0 & \text{if } k \leq n. \end{cases}$$

Proof. If $k > n$, then $[S^n, S^k] = 0$. Thus, each $f \in [S^n, S^n]$ induces the trivial homomorphism on $\pi^t(S^n)$ for $t \geq k$. Hence, $\mathcal{Z}_k^\sharp(S^n) \cong \mathbb{Z}$. If $k = n$, then $[S^n, S^n] \cong \mathbb{Z}\{\iota_n\}$. Since $f = (\deg f)\iota_n$ for each $f \in [S^n, S^n]$, $f^{\sharp n}(\iota_n) = ((\deg f)\iota_n)^{\sharp n}(\iota_n) = (\deg f)\iota_n$. Thus, if $f \in \mathcal{Z}_k^\sharp(S^n)$, then $\deg f$ must be 0. Hence, $\mathcal{Z}_n^\sharp(S^n) = 0$. From the definition, we have $\mathcal{Z}_k^\sharp(S^n) = 0$ for $k < n$. □

Theorem 5.2. *For $n \geq 2$,*

$$\mathcal{E}_k^\sharp(X_1) \cong \begin{cases} GL(m,\mathbb{Z}) & k > n, \\ 1 & k \leq n. \end{cases}$$

Proof. Since $\pi^k(X_1) = 0$ for $k > n$, $\mathcal{E}_k^\sharp(X_1) = \mathcal{E}(X_1)$ for $k > n$. Suppose that $k \leq n$. Then, for each $f \in \mathcal{E}_k^\sharp(X_1)$, $\theta^\sharp(f^{\sharp t}) = I_s^{\sharp t}$ for $t \geq k$. Thus

$$f_{ji}^{\sharp t} = \begin{cases} (\iota_n)_{ji}^{\sharp t} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, if $i = j$, then $f_{ji} \in \mathcal{E}_k^\sharp(S^n)$ and if $i \neq j$, then $f_{ji} \in \mathcal{Z}_k^\sharp(S^n)$. By Lemma 5.1, $\theta^\sharp(f^{\sharp t}) = id_{X_1}^{\sharp t}$ for $t \geq k$. □

Now, we investigate $\mathcal{Z}_k^\sharp(X_2, X_1)$ and $\mathcal{Z}_k^\sharp(X_1, X_2)$. We review briefly the following lemmas in [1] and [4].

Lemma 5.3. *Let $M(\mathbb{Z}_q, n)$ be a Moore space type of $(\mathbb{Z}_q, n)$. Then the k-th cohomotopy groups $\pi^k(M(\mathbb{Z}_q, n))$ are isomorphic to*

	$k \geq n+2$	$k = n+1$	$k = n$
$q \equiv 1 \pmod 2$	0	$\mathbb{Z}_q$	0
$q \equiv 0 \pmod 2$	0	$\mathbb{Z}_q$	$\mathbb{Z}_2$
Generator	-	$\iota_{n+1} \circ \pi_q$	$\eta_n \circ \pi_q$

Lemma 5.4. *Let $M(\mathbb{Z}_q, n)$ be a Moore space type of $(\mathbb{Z}_q, n)$. Then the k-th homotopy groups $\pi_k(M(\mathbb{Z}_q, n))$ are isomorphic to*

	$k = n+2$	$k = n+1$	$k = n$	$k \leq n-1$
$q \equiv 1 \pmod 2$	0	0	$\mathbb{Z}_q$	0
$q \equiv 0 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_q$	0
$q \equiv 2 \pmod 4$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_q$	0
Generator	-	$i_q \circ \eta_n$	$i_q \circ \iota_n$	-

Proposition 5.5. *For $k > 0$,*

$$\mathcal{Z}_k^\sharp(M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)) \cong \begin{cases} 0 & \text{if } q \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_n \circ \pi_q\} & \text{if } k \geq n+1 \text{ and } q \equiv 0 \pmod 2, \\ 0 & \text{if } k \leq n \text{ and } q \equiv 0 \pmod 2. \end{cases}$$

Proof. By Lemma 5.3,

$$[M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)] = \pi^n(M(\mathbb{Z}_q, n)) \cong \begin{cases} 0 & \text{if } q \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_n \circ \pi_q\} & \text{if } q \equiv 0 \pmod 2. \end{cases}$$

If q is odd, then $\mathcal{Z}_k^\sharp(M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)) = 0$. Let q be even. If $k \geq n+1$, then $\pi^k(M(\mathbb{Z}, n)) = 0$. Thus $\mathcal{Z}_k^\sharp(M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)) = [M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)] =$

$\pi^n(M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_2$. If $k = n$, then $\pi^n(M(\mathbb{Z}, n)) \cong \mathbb{Z}\{\iota_n\}$ and $\pi^n(M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_2\{\eta_n \circ \pi_q\}$ and so

$$(\eta_n \circ \pi_q)^{\sharp n}(\iota_n) = \iota_n \circ \eta_n \circ \pi_q = \eta_n \circ \pi_q \neq 0.$$

Therefore, $\mathcal{Z}_n^\sharp(M(\mathbb{Z}_q, n), M(\mathbb{Z}, n)) = 0$. $\square$

Proposition 5.6. *For $k \geq n$,*

$$\mathcal{Z}_k^\sharp(M(\mathbb{Z}, n), M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_q\{i_q \circ \iota_n\}.$$

Proof. By Lemma 5.4, $[M(\mathbb{Z}, n), M(\mathbb{Z}_q, n)] = \pi_n(M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_q\{i_q \circ \iota_n\}$. Since $\pi^k(S^n) = 0$ for $k \geq n+1$, $\mathcal{Z}_k^\sharp(M(\mathbb{Z}, n), M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_q$. If $k = n$, then $\pi^n(S^n) \cong \mathbb{Z}\{\iota_n\}$ and

$$\pi^n(M(\mathbb{Z}_q, n)) \cong \begin{cases} 0 & \text{if } q \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_n \circ \pi_q\} & \text{if } q \equiv 0 \pmod 2. \end{cases}$$

If $q \equiv 1 \pmod 2$, then $\mathcal{Z}_n^\sharp(M(\mathbb{Z}, n), M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_q$. If $q \equiv 0 \pmod 2$, then $(i_q \circ \iota_n)^{\sharp n}(\eta_n \circ \pi_q) = \eta_n \circ \pi_q \circ i_q \circ \iota_n = 0$ because $\pi_q \circ i_q \simeq 0$. Thus, $\mathcal{Z}_n^\sharp(M(\mathbb{Z}, n), M(\mathbb{Z}_q, n)) \cong \mathbb{Z}_q$. $\square$

Theorem 5.7. *For $k > 0$,*

$$\mathcal{Z}_k^\sharp(X_2, X_1) \cong \begin{cases} \bigoplus^{m\times t} \mathbb{Z}_2 & \text{if } k \geq n+1, \\ 0 & \text{if } k \leq n, \end{cases}$$

where t is the number of even q_j.

Proof. By Corollary 3.3,

$$\mathcal{Z}_k^\sharp(X_2, X_1) \cong \bigoplus^m \left(\bigoplus_{i=1}^s \mathcal{Z}_k^\sharp(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}, n)) \right).$$

By Proposition 5.5, we have $\mathcal{Z}_k^\sharp(X_2, X_1) \cong \bigoplus^{m\times t} \mathbb{Z}_2$ for $k \geq n+1$, where t is the number of even q_j. If $k \leq n$, then $\mathcal{Z}_k^\sharp(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}, n)) = 0$. Therefore, $\mathcal{Z}_k^\sharp(X_2, X_1) = 0$. $\square$

Theorem 5.8. *For $k \geq n$,*

$$\mathcal{Z}_k^\sharp(X_1, X_2) \cong \bigoplus_{j=1}^s \left(\bigoplus^m \mathbb{Z}_{q_j} \right).$$

Proof. This follows immediately from Corollary 3.3 and Proposition 5.6. $\square$

Finally, we determine $\mathcal{E}_k^\sharp(X_2)$. Since $X_2 \cong \vee_{i=1}^s M(\mathbb{Z}_{q_i}, n)$, we have

$$[X_2, X_2] \cong \bigoplus_{j=1}^s \left(\bigoplus_{i=1}^s [M(\mathbb{Z}_{q_i}, n), M(\mathbb{Z}_{q_j}, n)] \right)$$

by Proposition 2.1. In [3], it was shown that

$$\mathcal{E}(X_2) \cong \left(\bigoplus_{i=1}^{s} \mathcal{E}(M(\mathbb{Z}_{q_i}, n)) \right) \bigoplus \left(\bigoplus_{i \neq j} [M(\mathbb{Z}_{q_i}, n), M(\mathbb{Z}_{q_j}, n)] \right).$$

Theorem 5.9. *For $k > 0$,*

$$\mathcal{E}_k^{\sharp}(X_2) \cong \left(\bigoplus_{i=1}^{s} \mathcal{E}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n)) \right) \bigoplus \left(\bigoplus_{i \neq j} \mathcal{Z}_k^{\sharp}(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q_i}, n)) \right).$$

Proof. Let $f \in \mathcal{E}_k^{\sharp}(X_2)$. Then $\theta^{\sharp}(f^{\sharp t}) = I_s^{\sharp t}$ for $t \geq k$. This means that $f_{ii}^{\sharp t} = 1_{\pi^t(M(\mathbb{Z}_{q_i}, n))}$ for all $1 \leq i \leq s$ and $\theta^{\sharp}(f_{ji}^{\sharp t}) = 0$ for $i \neq j$. Thus, $f_{ii} \in \mathcal{E}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n))$ and $f_{ji} \in \mathcal{Z}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n), M(\mathbb{Z}_{q_j}, n))$. Therefore,

$$f \in \left(\mathcal{E}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n)) \right) \bigoplus \left(\bigoplus_{i \neq j} \mathcal{Z}_k^{\sharp}(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q_i}, n)) \right).$$

Conversely, let $f \in \left(\mathcal{E}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n)) \right) \oplus \left(\bigoplus_{i \neq j} \mathcal{Z}_k^{\sharp}(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q_i}, n)) \right)$. Then,

$$f_{ji}^{\sharp t} = \begin{cases} 1_{\pi^t(M(\mathbb{Z}_{q_i}, n))} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $t \geq k$. By Proposition 4.2 and the definition of $\mathcal{E}_k^{\sharp}(M(\mathbb{Z}_{q_i}, n))$, the matrix representation of $f^{\sharp t}$ is equal to $I_s^{\sharp t}$ for all $t \geq k$. Hence, $f \in \mathcal{E}_k^{\sharp}(X_2)$. □

From [3], we have the following lemma:

Lemma 5.10. *Let $M_j = M(\mathbb{Z}_{q_j}, n)$ and $M_i = M(\mathbb{Z}_{q_i}, n)$. Then we have*

	either q_j or q_i : odd	$q_j \equiv q_i \equiv 2 \pmod 4$	$q_j \equiv q_i \equiv 0 \pmod 4$
$[M_j, M_i]$	$\mathbb{Z}_d$	$\mathbb{Z}_{2d}$	$\mathbb{Z}_d \oplus \mathbb{Z}_2$
Generator	α_j	α_j	$\alpha_j, i_q \circ \eta_n \circ \pi_{q_j}$

where $\pi_{q_i} \circ \alpha_j = \bar{j} \iota_{n+1} \circ \pi_{q_j}$, $\bar{j}$ is an integer such that $q_j = \bar{j} d$, and $d = (q_i, q_j)$ is the greatest common divisor.

Proposition 5.11. *Let $M_i = M(\mathbb{Z}_{q_i}, n)$ and $M_j = M(\mathbb{Z}_{q_j}, n)$. Then we have*

$$\mathcal{Z}_k(M_j, M_i) \cong \begin{cases} [M_j, M_i] & \text{if } k \geq n+2, \\ 0 & \text{if } q_j \text{ or } q_i : \text{odd and } k = n \text{ or } n+1, \\ \mathbb{Z}_2 & \text{if } q_j \equiv q_i \equiv 0 \pmod 2 \text{ and } k = n \text{ or } n+1. \end{cases}$$

Proof. If $k \geq n+2$, then $\pi^k(M_j) = \pi^k(M_i) = 0$. Thus, $\mathcal{Z}_k^{\sharp}(M_j, M_i) = [M_j, M_i]$ for $k \geq n+2$. By Lemma 5.3, we have $\pi^{n+1}(M_\ell) \cong \mathbb{Z}_{q_\ell}\{\iota_{n+1} \circ \pi_{q_\ell}\}$ and

$$\pi^n(M_\ell) \cong \begin{cases} 0 & \text{if } q_\ell \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_n \circ \pi_{q_\ell}\} & \text{if } q_\ell \equiv 0 \pmod 2, \end{cases}$$

where $\ell = i, j$. By Lemma 5.10, we have

$$[M_j, M_i] \cong \begin{cases} \mathbb{Z}_d\{\alpha_j\} & \text{if } q_j \text{ or } q_i : \text{odd}, \\ \mathbb{Z}_{2d}\{\alpha_j\} & \text{if } q_j \equiv q_i \equiv 2 \pmod 4, \\ \mathbb{Z}_d \oplus \mathbb{Z}_2\{\alpha_j, i_{q_i} \circ \eta_n \circ \pi_{q_j}\} & \text{if } q_j \equiv q_i \equiv 0 \pmod 4, \end{cases}$$

where $d = (q_j, q_i)$.

Case 1. Let q_j or q_i be odd.

Let $g \in [M_j, M_i]$. Then, $g = s\alpha_j$ for some $0 \le s \le d$. Thus, we have

$$\begin{aligned} g^{\sharp n+1}(\iota_{n+1} \circ \pi_{q_i}) &= \iota_{n+1} \circ \pi_{q_i} \circ s\alpha_j \\ &= s\iota_{n+1} \circ \pi_{q_i} \circ \alpha_j \\ &= s\iota_{n+1} \circ \bar{j}\iota_n \circ \pi_{q_j} \\ &= s\bar{j}\iota_n \circ \pi_{q_j}. \end{aligned}$$

Since $0 \le s\bar{j} \le q_j$, $g^{\sharp n+1}$ is trivial if and only if $s = 0$. Hence, $\mathcal{Z}^\sharp_{n+1}(M_j, M_i) = 0$. Moreover, $\mathcal{Z}^\sharp_n(M_j, M_i) = 0$ by the definition.

Case 2. Let $q_j \equiv q_i \equiv 2 \pmod 4$.

Let $g \in [M_j, M_i]$. Then $g = s\alpha_j$ for some $0 \le s \le 2d$. Thus, if $s = d$, then

$$g^{\sharp n+1}(\iota_{n+1} \circ \pi_{q_i}) = \iota_{n+1} \circ \pi_{q_i} \circ i_{q_i} \circ \eta_n \circ \pi_{q_j} = 0$$

because $\pi_{q_i} \circ i_{q_i} \simeq 0$. If $s \ne d$, then

$$g^{\sharp n+1}(\iota_{n+1}\pi_{q_i}) = s\iota_{n+1}\pi_{q_i} \circ \alpha_j = s\bar{j}\iota_{n+1} \circ \pi_{q_j}.$$

Hence, $s = 0$ or d. Therefore $\mathcal{Z}^\sharp_{n+1}(M_j, M_i) \cong \mathbb{Z}_2\{d\alpha_j\}$. Then, for $g \in \mathcal{Z}^\sharp_{n+1}(M_j, M_i)$,

$$g^{\sharp n+1}(\eta_n \circ \pi_{q_i}) = d\eta_n \circ \pi_{q_i} \circ \alpha_j = d\bar{j}\eta_{n+1} \circ \pi_{q_j} = q_j\eta_{n+1} \circ \pi_{q_j} = 0$$

because q_j is even. Therefore, $\mathcal{Z}^\sharp_n(M_j, M_i) \cong \mathbb{Z}_2\{d\alpha_j\}$.

Case 3. Let $q_j \equiv q_i \equiv 0 \pmod 4$.

Let $g \in [M_j, M_i]$. Then, $g = s\alpha_j \oplus t i_{q_i} \circ \eta_n \circ \pi_{q_j}$ for some $0 \le s < d$ and $t = 0, 1$. Then,

$$g^{\sharp n+1}(\iota_{n+1} \circ \pi_{q_i}) = s\bar{j}\iota_{n+1} \circ \pi_{q_j} \oplus t\iota_{n+1} \circ \pi_{q_i} \circ i_{q_i} \circ \eta_n \circ \pi_{q_j} = s\bar{j}\iota_{n+1} \circ \pi_{q_j} \oplus 0$$

because $\pi_{q_i} \circ i_{q_i} \simeq 0$. Thus, $s = 0$ and $t = 0, 1$. Hence, $\mathcal{Z}^\sharp_{n+1}(M_j, M_i) \cong \mathbb{Z}_2\{0 \oplus i_{q_i} \circ \eta_n \circ \pi_{q_j}\}$. Then, for $g \in \mathcal{Z}^\sharp_{n+1}(M_j, M_i)$,

$$g^{\sharp n}(\eta_n \circ \pi_{q_i}) = 0 \oplus \eta_n \circ \pi_{q_i} \circ i_{q_i} \circ \eta_n \circ \pi_{q_j} = 0.$$

Hence, $\mathcal{Z}^\sharp_n(M_j, M_i) \cong \mathbb{Z}_2\{0 \oplus i_{q_i} \circ \eta_n \circ \pi_{q_j}\}$. □

Theorem 5.12. *For $n \ge 3$,*

$$\mathcal{E}^\sharp_k(X_2) \cong \begin{cases} \left(\bigoplus_{i=1}^{s} \mathcal{E}(M(\mathbb{Z}_{q_i}, n))\right) \bigoplus \left(\bigoplus_{i \ne j} [M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q_i}, n)]\right) & \text{if } k \ge n+2, \\ \bigoplus^{t+\ell} \mathbb{Z}_2 & \text{if } k = n \text{ or } n+1, \end{cases}$$

where t is the number of even q_j and ℓ is the number of pairs $\{i,j\} \subset \{1,\dots,s\}$ such that both q_i and q_j are even and $i \neq j$.

Proof. If $k \geq n+2$, then $\pi^k(X_2) = 0$. Thus, $\mathcal{E}_k^\sharp(X_2) = \mathcal{E}(X_2)$. If $k = n$ or $n+1$, then by [6, Theorems 4.1 and 4.2], $\mathcal{E}_k^\sharp(M(\mathbb{Z}_{q_i}, n)) \cong \bigoplus^{t} \mathbb{Z}_2$, where t is the number of even q_i. By Proposition 5.11, $\bigoplus_{i\neq j} \mathcal{Z}_k^\sharp(M(\mathbb{Z}_{q_j}, n), M(\mathbb{Z}_{q_i}, n)) \cong \bigoplus^{\ell} \mathbb{Z}_2$, where ℓ is the number of pairs $\{i,j\} \subset \{1,\dots,s\}$ such that both q_i and q_j are even and $i \neq j$. Therefore,

$$\mathcal{E}_k^\sharp(X_2) \cong \bigoplus^{t+\ell} \mathbb{Z}_2. \qquad \square$$

If we combine Theorems 5.2, 5.7, 5.8, and 5.12, we obtain the following main result:

Corollary 5.13. *Let $G = \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) \bigoplus \left(\bigoplus_{j=1}^{s} \mathbb{Z}_{q_j}\right)$. Then, we have*

$$\mathcal{E}_k^\sharp(M(G,n)) \cong \begin{cases} GL(m,\mathbb{Z}) \bigoplus^{m\times t} \mathbb{Z}_2 \bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right) \bigoplus \mathcal{E}(M(T,n)) & \text{if } k \geq n+2, \\ GL(m,\mathbb{Z}) \bigoplus^{m\times t+t+\ell} \mathbb{Z}_2 \bigoplus \left(\bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right)\right) & \text{if } k = n+1, \\ GL(m,\mathbb{Z}) \bigoplus^{t+\ell} \mathbb{Z}_2 \bigoplus \left(\bigoplus_{j=1}^{s} \left(\bigoplus^{m} \mathbb{Z}_{q_j}\right)\right) & \text{if } k = n, \end{cases}$$

where $GL(m,\mathbb{Z})$ is the general linear group of degree m, t is the number of even q_j and ℓ is the number of pairs $\{i,j\} \subset \{1,\dots,s\}$ such that both q_i and q_j are even and $i \neq j$.

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