J. Korean Math. Soc. **56** (2019), No. 5, pp. 1355–1370
https://doi.org/10.4134/JKMS.j180675
pISSN: 0304-9914 / eISSN: 2234-3008

CURVES WITH MAXIMAL RANK, BUT NOT ACM, WITH VERY HIGH GENERA IN PROJECTIVE SPACES

Edoardo Ballico

Abstract. A curve $X \subset \mathbb{P}^r$ has maximal rank if for each $t \in \mathbb{N}$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_X(t))$ is either injective or surjective. We show that for all integers $d \geq r+1$ there are maximal rank, but not arithmetically Cohen-Macaulay, smooth curves $X \subset \mathbb{P}^r$ with degree d and genus roughly $d^2/2r$, contrary to the case $r = 3$, where it was proved that their genus growths at most like $d^{3/2}$ (A. Dolcetti). Nevertheless there is a sector of large genera g, roughly between $d^2/(2r+2)$ and $d^2/2r$, where we prove the existence of smooth curves (even aCM ones) with degree d and genus g, but the only integral and non-degenerate maximal rank curves with degree d and arithmetic genus g are the aCM ones. For some (d, g, r) with high g we prove the existence of reducible non-degenerate maximal rank and non aCM curves $X \subset \mathbb{P}^r$ with degree d and arithmetic genus g, while (d, g, r) is not realized by non-degenerate maximal rank and non aCM integral curves.

1. Introduction

Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate curve. Set $d := \deg(X)$ and $g := p_a(X)$. We recall that X is said to be *arithmetically Cohen-Macaulay* (or *aCM* for short) if $h^1(\mathcal{I}_X(t)) = 0$ for all $t \in \mathbb{N}$ and that it is said to have *maximal rank* if for each $t \in \mathbb{N}$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$. Thus if X has maximal rank and for some $t \in \mathbb{N}$ we know the integer $h^0(\mathcal{O}_X(t))$, then we know the integers $h^0(\mathcal{I}_X(t)) = \max\{0, \binom{r+t}{t} - h^0(\mathcal{O}_X(t))\}$ and $h^1(\mathcal{I}_X(t)) = \max\{0, h^0(\mathcal{O}_X(t)) - \binom{r+t}{t}\}$. An aCM curve has maximal rank, but easy examples show that the converse does not hold. In the case $r = 3$ all pairs (d, g) realized by some integral (and then by some smooth, too) aCM curve are known ([9, 17]). For all integers $d \geq r$ set $\pi(d, r) := \binom{m}{2}(r-1) + m\epsilon$, where $m := \lfloor (d-1)/(r-1) \rfloor$ and $\epsilon := d - 1 - m(r-1)$. We recall that for any non-degenerate X Castelnuovo proved that $g \leq \pi(d, r)$ and classified the

Received October 8, 2018; Revised March 5, 2019; Accepted May 23, 2019.
2010 *Mathematics Subject Classification.* 14H50.
Key words and phrases. projective curve, Hilbert function, curve of maximal rank, aCM curve.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

curves with $g = \pi(d,r)$ ([11, Theorems 3.7 and 3.11]). Such curves exist for all $d \geq r$. Note that

$$\lim_{d \to +\infty} \pi(d,r)/d^2 = \frac{1}{2r-2}.$$

A. Dolcetti proved the existence of a real number $K > 0$ such that if the pair (d,g) is realized by a maximal rank, but not aCM, curve $X \subset \mathbb{P}^3$ (i.e., $d := \deg(X)$ and $g := p_a(X)$), then $g \leq Kd^{3/2}$. As far as we know the family of maximal rank space curves which asymptotically for large d have the largest ratio $d^{3/2}/g$ are the ones constructed by A. Hirschowitz and R. Hartshorne in [14, 5.4, 5.5 and 5.8] (see [8, Example 1.7] for a description of the pairs (d,g) obtained in this way). There is a smaller positive real number K_1 such that for all $d \gg 0$ and all $g \leq K_1 d^{3/2}$ there is a smooth maximal rank space curve $X \subset \mathbb{P}^3$ with degree d and genus g and these curves are not aCM, except for a few pairs (d,g) ([3]). The aim of this note is to prove that Dolcetti's result is peculiar to the case $r = 3$. We show this claim proving the following result.

Theorem 1.1. *Fix an integer $r \geq 4$.*

(1) *If (d,g) is realized by some non-degenerate integral maximal rank curve $X \subset \mathbb{P}^r$, which is not aCM, then $g \leq \pi(d, r+1)$.*
(2) *For each $r \geq 5$ and each integer $d \geq r+1$ there is a smooth, connected and non-degenerate maximal rank curve in $\mathbb{P}^r$ with degree d, genus $\pi(d, r+1)$ and not aCM.*
(3) *For each even integer $d \geq 6$ there is a smooth, connected and non-degenerate maximal rank curve in $\mathbb{P}^4$ with degree d, genus $\pi(d,5)$ and not aCM.*

For all integers $d \geq r$ there are smooth and non-degenerate aCM curves $X \subset \mathbb{P}^r$ with degree d and genus $\pi(d,r)$ (Remark 3.5) and so part (1) of Theorem 1.1 shows that to be of maximal rank, but not aCM, gives a (small) restriction on the growth of the genera. Parts (2) and (3) of Theorem 1.1 show that for $r > 3$ the growth is still quadratic in d. We stress that the restriction in part (1) does not arise for aCM curves (Remark 3.5). In Section 4 we prove the existence of maximal rank, but not aCM, curves with high genus $g < \pi(d, r+1)$. More precisely the maximal genus $< \pi(d,r)$ is the integer $\pi_1(d, r+1)$, which we define here following [11, Theorem 3.15 and Section 3.c]. For all integers $r \geq 4$ and $d \geq 2r+1$ set $\pi_1(d,r) := \binom{m_1}{2}r + m_1(\epsilon_1+1)+\mu_1$, where $m_1 := \lfloor (d-1)/r \rfloor$, $\varepsilon_1 := d - m_1 r - 1$, $\mu_1 := 1$ if $\varepsilon_1 = r-1$ and $\mu_1 := 0$ if $\varepsilon_1 \neq r-1$. Note that $\lim_{d\to+\infty} \pi_1(d,r)/d^2 = \frac{1}{2r}$.

We prove the following result.

Proposition 1.2. *Fix integers $r \geq 5$, $d \geq 2r+3$, and $g < \pi(d, r+1)$. If there is an integral, non-degenerate maximal rank, but not aCM, curve $X \subset \mathbb{P}^r$ with degree d and genus g, then $g \leq \pi_1(d, r+1)$.*

For $r \geq 9$ the only integers for which there is a smooth curve of degree d and genus $\pi_1(d, r+1)$ are $\equiv 0, 1 \pmod{r+2}$ (Remark 4.2). See Examples 4.3 and 4.4 for existence results in the set-up of Proposition 1.2 for $r = 7, 8$.

All the curves with maximal rank with very large genus appearing in Theorem 1.1 and potentially appearing in the set-up of Proposition 1.2 are contained in a quadric hypersurface. If we impose that the minimal degree of a hypersurface containing the curve X is at least 3 there are stronger upper bounds (but still quadratic in d) for the genus (Corollary 2.6).

In Section 5 we consider reducible, connected and non degenerate curves $W \subset \mathbb{P}^r$ with maximal rank, but they are not aCM and for which there is no integral, non-degenerate curve $X \subset \mathbb{P}^r$ with maximal rank and not aCM and with $(\deg(X), p_a(X)) = (\deg(W), p_a(W))$ (Remark 5.2).

We work over an algebraically closed field with characteristic 0.

We thank the referee for useful observations.

2. Preliminary results

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Let $s(X)$ denote the minimal positive integer x such that $h^0(\mathcal{I}_X(x)) > 0$. Since X is non-degenerate, we have $s(X) \geq 2$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Consider the exact sequence

$$0 \to \mathcal{I}_X(t-1) \to \mathcal{I}_X(t) \to \mathcal{I}_{X \cap H, H}(t) \to 0. \tag{2.1}$$

Let $\sigma(X)$ be the minimal integer x such that $h^0(H, \mathcal{I}_{X \cap H, H}(x)) \neq 0$. Obviously, $\sigma(X) \leq s(X)$. Since X is integral, we have $h^1(\mathcal{I}_X) = 0$. Thus the case $t = 1$ of (2.1) gives $h^0(H, \mathcal{I}_{X \cap H, H}(1)) = 0$, i.e., $X \cap H$ spans H. Thus $\sigma(X) \geq 2$.

Remark 2.1. Several times we will be in the following set-up. Let $W \subset \mathbb{P}^n$ be an integral and non-degenerate surface such that W spans $\mathbb{P}^n$, W is aCM and it is contained in at least one quadric surface; later we will take $r = n - 1$. Let $C \subset W$ be an integral and non-degenerate curve. Since $h^0(\mathcal{I}_W(2)) \neq 0$, we have $h^1(\mathcal{I}_C(2)) \neq 0$. Thus C has maximal rank if and only if $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq 2$, while C is aCM if and only if $h^1(\mathcal{I}_C(t)) = 0$ for all $t > 0$ (note that $h^1(\mathcal{I}_C) = 0$, because C is integral). Thus C is aCM if and only if it has maximal rank and it is linearly normal. Since W is aCM, C is aCM (resp. has maximal rank) if and only if for each integer $t > 0$ (resp. $t \geq 2$) the restriction map $H^0(\mathcal{O}_W(t)) \to H^0(\mathcal{O}_C(t))$ is surjective. We will always have $h^1(\mathcal{O}_W(t)) = 0$ for all $t > 0$. Thus the restriction map $H^0(\mathcal{O}_W(t)) \to H^0(\mathcal{O}_C(t))$ is surjective if and only if $h^1(\mathcal{O}_W(t)(-C)) = 0$. Now fix $o \in \mathbb{P}^n \setminus W$ such that the linear projection $\ell : \mathbb{P}^n \setminus \{o\} \to \mathbb{P}^{n-1}$ from o maps W isomorphically onto the surface $Y := \ell(W)$ (we need that either $n \geq 6$ or $n = 5$ and W is the Veronese surface). Hence $X := \ell(C)$ is isomorphic to C. Suppose that $h^1(\mathbb{P}^{n-1}, \mathcal{I}_Y(t)) = 0$ for all $t \geq 2$ and that C has maximal rank. Since X is not linearly normal, it is not aCM. We claim that X has maximal rank. Since X spans $\mathbb{P}^{n-1}$, it is sufficient to prove that $h^1(\mathbb{P}^{n-1}, \mathcal{I}_X(t)) = 0$ for all $t \geq 2$. Fix an integer $t \geq 2$.

Since ℓ induces an isomorphism between W and Y and between C and X and the restriction map $H^0(\mathcal{O}_W(t)) \to H^0(\mathcal{O}_C(t))$ is surjective, the restriction map $H^0(\mathcal{O}_Y(t)) \to H^0(\mathcal{O}_X(t))$ is surjective. Since $h^1(\mathbb{P}^{n-1}, \mathcal{I}_Y(t)) = 0$, we get $h^1(\mathbb{P}^{n-1}, \mathcal{I}_X(t)) = 0$.

We recall the following lemma proved for $r = 3$ in [8, Lemma 1.2]; the same proof works for any r.

Lemma 2.2. *Assume that X has maximal rank. X is aCM if and only if $s(X) = \sigma(X)$ and the restriction map $H^0(\mathcal{I}_X(s(X))) \to H^0(H, \mathcal{I}_{X\cap H,H}(s(X)))$ is surjective.*

Remark 2.3. Assume that the non-degenerate curve $X \subset \mathbb{P}^r$ is linearly normal, i.e., assume $h^1(\mathcal{I}_X(1)) = 0$. By (2.1) the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X\cap H,H}(2))$ is surjective. Assume $s(X) = 2$ and that X has maximal rank. Since X is integral, we have $h^1(\mathcal{I}_X) = 0$. Thus the case $t = 1$ of (2.1) gives $h^0(H, \mathcal{I}_{X\cap H,H}(1)) = 0$. Thus $\sigma(X) = 2$. Lemma 2.2 shows that X is aCM.

Lemma 2.4. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve such that $\sigma(X) > 2$. Set $d := \deg(X)$ and fix an integer σ such that $2 \le \sigma \le \sigma(X)$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Set $S := X \cap H$ and $\beta := h^0(H, \mathcal{I}_{S,H}(\sigma))$. Take $A, B \subset S$ and an integer $\alpha > 0$ such that $h^1(H, \mathcal{I}_{A,H}(\alpha)) = 0$ and $h^0(H, \mathcal{I}_{A,H}(\alpha)) > 0$. If $|B| \le \binom{r+\sigma-1}{r-1} - \beta$, then $h^1(H, \mathcal{I}_{A\cup B,H}(\alpha+\sigma)) = 0$ and $h^0(H, \mathcal{I}_{A\cup B,H}(\alpha+\sigma)) > 0$.*

Proof. Note that $\beta = 0$ if and only if $\sigma < \sigma(X)$. Since $h^0(H, \mathcal{I}_{S,H}(\sigma)) = \beta$, there is $D \subseteq S$ such that $|D| = \binom{r+\sigma-1}{r-1} - \beta$ and $h^1(H, \mathcal{I}_{D,H}(\sigma)) = 0$. Since S has the Uniform Position Property in the sense of [11, Ch. III], $h^1(H, \mathcal{I}_{F,H}(\sigma)) = 0$ for all $F \subseteq S$ such that $|F| \le \binom{r+\sigma-1}{r-1} - \beta$. Since $|A| < \binom{r+\alpha-1}{r-1}$, $|B| \le \binom{r+\sigma-1}{r-1}$ and $\binom{r+\alpha-1}{r-1} + \binom{r+\sigma-1}{r-1} \le \binom{r+\alpha+\sigma-1}{r-1}$, we have $h^0(H, \mathcal{I}_{A\cup B,H}(\alpha+\sigma)) > 0$. Taking $B \setminus B \cap A$ instead of B we reduce to the case $A \cap B = \emptyset$. Set $z := |B|$. We use induction on the integer z starting with the trivial case $z = 0$. Take a general $Q \in |\mathcal{I}_{A,H}(\alpha)|$. Since S is in uniform position, either $h^1(H, \mathcal{I}_{S,H}(\alpha)) = 0$ or $Q \cap (S \setminus A) = \emptyset$. Since in the former case the lemma is true, we may assume $Q \cap (S \setminus A) = \emptyset$ and in particular $Q \cap B = \emptyset$. We may assume $z > 0$. Take $p \in B$ and set $B' := B \setminus \{p\}$. By the inductive assumption we have $h^1(H, \mathcal{I}_{A\cup B',H}(\alpha+\sigma)) = 0$. Let Q' be a general element of $|\mathcal{I}_{B',H}(\sigma)|$. Since $h^1(H, \mathcal{I}_{D,H}(\sigma)) = 0$ and Q' is general, we have $p \notin Q'$. Thus $p \notin Q \cup Q'$. Hence $h^0(H, \mathcal{I}_{A\cup B,H}(\alpha+\sigma)) < h^0(H, \mathcal{I}_{A\cup B',H}(\alpha+\sigma))$. Thus $h^1(H, \mathcal{I}_{A\cup B,H}(\alpha+\sigma)) = 0$. □

We only use the case $\sigma = 2$ of Lemma 2.4 to prove Corollary 2.6, which will be used to prove part (1) of Theorem 1.1.

Lemma 2.5. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve such that $\sigma(X) > 2$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Set $S := X \cap H$ and $d := \deg(X)$. Write $d = a\binom{r+1}{2} + b$ with $a \in \mathbb{N}$ and $-1 \le b \le \binom{r+1}{2} - 2$.*

(a) *We have* $h^1(H,\mathcal{I}_{S,H}(1)) = d-r$ *and* $h^1(H,\mathcal{I}_{S,H}(2)) = d-\binom{r+1}{2}$.
(b) *We have* $h^1(H,\mathcal{I}_{S,H}(t)) = 0$ *for all* $t \ge 2a+2$.
(c) *We have* $h^1(H,\mathcal{I}_{S,H}(2a+1)) \le \max\{0, b-r+1\}$.
(d) *If* $4 \le t \le 2a$ *and* $t = 2x$ *is even we have* $h^1(H,\mathcal{I}_{S,H}(t)) \le d-x\binom{r+1}{2}+1$.
(e) *If* $3 \le t \le 2a-1$ *and* $t = 2x+1$ *is odd, we have* $h^1(H,\mathcal{I}_{S,H}(t)) \le d-x\binom{r+1}{2}+1-r$.

Proof. Since $h^0(H,\mathcal{I}_{S,H}(2)) = 0$, we have $d \ge \binom{r+1}{2}$ and hence part (a) is trivial. Since H is general, the set S has cardinality d and it is in uniform position, i.e., $h^0(H,\mathcal{I}_{A,H}(t)) = h^0(H,\mathcal{I}_{B,H}(t))$ for all $t \in \mathbb{N}$ and any $A, B \subseteq S$ with $|A| = |B|$ ([11, page 85]). Since $h^0(H,\mathcal{I}_{S,H}(2)) = 0$, we have $h^1(H,\mathcal{I}_{A,H}(2)) = 0$ for all $A \subset S$ such that $|A| \le \binom{r+1}{2}$.

Now we prove part (d). Fix any $A \subset S$ with $|A| \le \binom{r+1}{2}-1$. Thus $h^0(H,\mathcal{I}_{A,H}(2)) = \binom{r+1}{2}-|A| > 0$. Take a general $Q \in |\mathcal{I}_{A,H}(2)|$. Since S is in uniform position and Q is general, we have $Q \cap S = A$. Write $S = A_1 \sqcup B_2 \sqcup \cdots \sqcup B_{a-1} \sqcup D$ with $|B_i| = \binom{r+1}{2}$ for all i and $|D| = b$. Part (d) is empty if $a = 3$. To get part (d) for $x = 2$ (hence $a \ge 4$) use Lemma 2.4 with $\alpha = \sigma = 2$, and $B := B_1$. Then use Lemma 2.4 for $\alpha = 2x-2$ and $\sigma = 2$ to prove $h^1(H,\mathcal{I}_{A\cup\cdots\cup B_x,H}(2x)) = 0$ by induction on x. This vanishing proves part (d).

To prove part (b) it is sufficient to prove that $h^1(H,\mathcal{I}_{S,H}(2a+2)) = 0$, which is proved from the case $t = 2a$ using that $|D| \le \binom{r+1}{2}$ and applying Lemma 2.4 for $\sigma = 2$ and $\alpha = 2a$.

Now we prove part (e). We may assume $a \ge 2$, because if $a = 1$ part (e) is empty. We fix $E \subset S$ with $|E| = r-1$. Since S is in uniform position and it spans H, E spans a hyperplane M of H such that $M \cap S = E$. Write $S = E \sqcup F_1 \sqcup \cdots \sqcup F_x \sqcup D'$ with $|F_i| = \binom{r+1}{2}$ for all i. Apply x times Lemma 2.4, always with $\sigma = 2$ and $\alpha = 1, 3, \dots, 2x-3$. We get $h^1(H,\mathcal{I}_{E\sqcup F_1\sqcup\cdots\sqcup F_x,H}(2x+1)) = 0$ and so $h^1(H,\mathcal{I}_{S,H}(2x+1)) \le |D'| = d-r+1-x\binom{r+1}{2}$.

Part (c) follows from part (e) for $t = 2a$ using Lemma 2.4 with $\alpha = 2a-1$ and $\sigma = 2$. □

Corollary 2.6. *Let* $X \subset \mathbb{P}^r$ *be an integral and non-degenerate curve such that* $\sigma(X) > 2$. *Set* $d := \deg(X)$ *and* $g := p_a(X)$. *Write* $d = a\binom{r+1}{2}+b$ *with* $a \in \mathbb{N}$ *and* $-1 \le b \le \binom{r+1}{2}-2$. *Then* $g \le 2ad - a^2\binom{r+1}{2} - ra + a - 2 + \max\{0, b-r+1\}$.

Proof. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Set $S := X \cap H$. By the Castelnuovo's method ([11, Corollary 3.2]) we have $g \le \sum_{t\ge1} h^1(H,\mathcal{I}_{S,H}(t))$. Parts (a), (b) and (c) of Lemma 2.5 give $h^1(H,\mathcal{I}_{S,H}(1)) = d-r$, $h^1(H,\mathcal{I}_{S,H}(2)) = d-\binom{r+1}{2}$, $h^1(H,\mathcal{I}_{S,H}(t)) = 0$ for all $t \ge 2a+2$ and $h^1(H,\mathcal{I}_{S,H}(2a+1)) \le \max\{0, b-r+1\}$. For all integers x such that $2 \le x \le a$ we have $h^1(H,\mathcal{I}_{S,H}(2x)) \le d-x\binom{r+1}{2}+1$ (part (d) of Lemma 2.5). For all integers x such that $1 \le x \le a-1$ we have $h^1(H,\mathcal{I}_{S,H}(2x+1)) \le d-x\binom{r+1}{2}+1-r$ (part (e) of Lemma 2.5).

Since $\sum_{x=2}^{a}(d - x\binom{r+1}{2} + 1) = (a-1)d + a - 1 - \binom{r+1}{2}(a+2)(a-1)/2$ and $\sum_{x=1}^{a-1}(d - x\binom{r+1}{2} + 1 - r) = (a-1)d - (a-1)(r-1) - \binom{r+1}{2}a(a-1)/2$, we get $g \le 2ad - a^2\binom{r+1}{2} - ra + a - 2 + \max\{0, b-r+1\}$. $\square$

Remark 2.7. Note that the upper bound on the arithmetic genus g in Corollary 2.6 is quadratic in d, but with a leading coefficient, $\frac{2}{r(r+1)}$, which is far smaller both of the one for the upper bound, $\pi(d,r)$, for degree d non-degenerate curves, $\frac{1}{2r-2}$ ([11, Theorem 3.7]), and the one for non-linearly normal non-degenerate curves (i.e., $\pi(d,r+1)$), $\frac{1}{2r}$ ([11, Theorem 3.15]) and $\pi_1(d,r+1)$, $\frac{1}{2r+2}$. Set $\gamma(d,r) := 2ad - a^2\binom{r+1}{2} - ra + a - 2 + \max\{0, b-r+1\}$. Set $m_1 := \lfloor (d-1)/(r+1) \rfloor$, $\epsilon_1 := d - m_1(r+1) - 1$, $\mu_1 = 0$ if $\epsilon_1 \ne r$ and $\mu_1 = 1$ if $\epsilon_1 = r$. Recall that $\pi_1(d,r+1) = \binom{m_1}{2}(r+1) + m_1(\epsilon_1 + 1) + \mu_1$.

Claim 1: We have $\gamma(d,r) < \pi(d,r+1)$ for all $r \ge 4$ and $d \ge \binom{r+1}{2}$.

Claim 2: We have $\gamma(d,r) < \pi_1(d,r+1)$ for all $r \ge 5$ and $d \ge \binom{r+1}{2}$.

Proofs of Claims 1 and 2: Since $\pi_1(d,r+1) \le \pi(d,r+1)$ when the former is defined, i.e., for $d \ge 2r+5$, for $r \ge 5$ it is sufficient to prove Claim 2 and then prove Claim 1 for $r = 4$ (note that $\binom{r+1}{2} \ge 2r+5$ if and only if $r \ge 5$).

(a) Take $d = \binom{r+1}{2}$. Thus $a = 1$, $b = 0$ and $\gamma(\binom{r+1}{2}, r) = \binom{r+1}{2} - r - 1 = \binom{r}{2} - 1$.

Now we compute $\pi_1(\binom{r+1}{2}, r+1)$. First assume r even. We get $m_1 = r/2 - 1$ and $\epsilon_1 = r$. Thus $\mu_1 = 1$ and $\pi_1(\binom{r+1}{2}, r+1) = (r-2)(r-4)(r+1)/8 + (r/2 - 1)(r+1) + 1 = (r-2)(r-4)(r+1)/8 + \binom{r}{2}$. Thus $\pi_1(\binom{r+1}{2}, r+1) > \gamma(\binom{r+1}{2}, r)$ for all even $r \ge 4$. Now assume r odd. We get $m_1 = (r-1)/2 = \epsilon_1$ and hence $\pi_1(\binom{r+1}{2}, r+1) = (r-1)(r-3)(r+1)/8 + (r^2-1)/4$. Thus $\pi_1(\binom{r+1}{2}, r+1) > \gamma(\binom{r+1}{2}, r)$ for all odd $r \ge 5$.

(b) Now we take $r \ge 5$ and any $d \ge \binom{r+1}{2}$, assume $\pi_1(d,r+1) > \gamma(d,r)$ and prove that $\pi_1(d+1,r+1) > \gamma(d+1,r)$. By step (a) this would conclude the proof of Claim 2. Set $z := \pi_1(d+1,r+1) - \pi_1(d,r+1)$ and $w := \gamma(d+1,r) - \gamma(d,r)$. It is sufficient to prove that $z \ge w$. We call $a, b, m_1, \epsilon_1, \mu_1$ the integers associated to d and compute the corresponding integers for $d+1$ (which we will write with a prime $'$, say a', b' and so on). If $b \ne \binom{r+1}{2} - 2$ we have $a' = a$ and $b' = b+1$. Thus $2a \le w \le 2a+1$ (the first inequality holding if and only if $b \le r-2$). If $b = \binom{r+1}{2} - 2$, i.e., $d = (a+1)\binom{r+1}{2} - 2$, then $a' = a+1$ and $b' = 0$. In this case we have $\gamma((a+1)\binom{r+1}{2} - 2, r) = (2a^2+2a)\binom{r+1}{2} - 4a - a^2\binom{r+1}{2} - ar + a - 2 + \binom{r+1}{2} - 1 - r$, $\gamma((a+1)\binom{r+1}{2} - 1, r) = (2a^2+4a+1)\binom{r+1}{2} - 2(a+1) - (a+1)^2\binom{r+1}{2} - ar - r + a - 1$ and hence $w = 2a - r$.

If $\epsilon_1 < r$ we have $m_1' = m_1$, $\epsilon_1' = \epsilon_1$. We have $\mu_1' = 1$ if and only if $\epsilon' = r-1$. Thus $m_1 \le z \le m_1 + 1$ in this case and $z = m_1 + 1$ if and only if $\epsilon_1 = r - 1$. Now assume $\epsilon_1 = r$ and so $\mu_1 = 1$. We have $m_1' = m_1 + 1$ and $\epsilon_1' = \mu_1' = 0$. Since $\binom{m_1+1}{2} - \binom{m_1}{2} = m_1$, we get $z = m_1$ in this case. Thus to prove that $z > w$ it is sufficient to prove that $m_1 \ge 2a+1$. We have $(r+1)m_1 = d - 1 - \epsilon_1 \ge d - r - 1$

and $d = a\binom{r+1}{2} + b$. For any $a \ge 1$, $b \ge -1$ is sufficient to assume either $r \ge 6$ or $r = 5$, $a \ge 2$ and $b \ge 1$.

Now assume $r = 5$. Hence $\binom{r+1}{2} = 15$. Thus $\gamma(15 + x, 5) = 9 + 2x$ for $0 \le x \le 3$, $\gamma(19 + x, 5) = 18 + 3x$ for $0 \le x \le 9$ and $\gamma(29 + x, 5) = 46 + 4x$ for $0 \le x \le 4$. To compute $\pi_1(15, 6)$ we use that $m_1 = 2$ and $\epsilon_1 = 1$. Thus $\pi_1(15 + x, 6) = 12 + 2x$ for $0 \le x \le 2$, $\pi_1(18 + x, 6) = 19 + 3x$ for $0 \le x \le 4$, $\pi_1(24, 6) = 37$, $\pi_1(25 + x, 6) = 40 + 4x$ for $0 \le x \le 4$, and so on.

(c) Take $r = 4$ and so $\binom{r+1}{2} = 10$. Thus $\gamma(10, 4) = 5$. Call m and ϵ the integers associated to the pair $(10, 5)$ for the computation of $\pi(10, 5)$. Since $m = 2$ and $\epsilon = 1$, we have $\pi(10, 5) = 6$. Now take an integer $d \ge 10$ and assume $\gamma(d, 4) \le \pi(d, 4)$. Take $a, b, w := \gamma(d+1, 4) - \gamma(d, 4)$ as in step (b). Set $z' := \pi(d+1, 5) - \pi(d, 5)$. Call m, ϵ (resp. m', ϵ') the integers needed to compute $\pi(d, 5)$ (resp. $\pi(d + 1, 5)$). If $\epsilon \le 3$ we have $m' = m$, $\epsilon' = \epsilon + 1$ and so $z' = m$. If $\epsilon = 4$ we have $m' = m + 1$, $\epsilon' = 0$ and so $z' = m$. Recall that $w \le 2a + 1$ with $w = 2a$ if $b \le 3$, that $(a, b) \ne (1 - 1)$ and that $d - 1 = 4m + \epsilon = 15a + b - 1$ with $-1 \le b \le 8$. To get $z' \ge w$ use that $10a + 7 \ge 4(2a + 1)$ and $10a - 2 \ge 8a$ for all $a \ge 1$.

3. Proof of Theorem 1.1

Proof of part (1) *of Theorem 1.1.* Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate maximal rank curve which is not aCM. Set $d := \deg(X)$ and $g := g(X)$ and assume $g > \pi(d, r + 1)$. By Castelnuovo's theory ([11, Theorem 3.7]), X is linearly normal and hence $h^1(\mathcal{I}_X(1)) = 0$. Assume for the moment $h^0(\mathcal{I}_X(2)) \ne 0$ and hence $h^0(\mathcal{I}_X(t)) \ne 0$ for all $t \ge 2$. Since X has maximal rank, $h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge 2$. Since X is integral, we have $h^1(\mathcal{I}_X) = 0$. Thus X is aCM, contradicting one of our assumptions. Thus $h^0(\mathcal{I}_X(2)) = 0$. Since $h^1(\mathcal{I}_X(1)) = 0$, we have $\sigma(X) > 2$. Taking a general hyperplane section and using the definition of $\sigma(X)$ we get $d \ge \binom{r+1}{2}$. To get a contradiction and conclude the proof of part (1) of Theorem 1.1 it is sufficient to quote Claim 1 of Remark 2.7. □

For the constructive proof of part (2) of Theorem 1.1 we recall the description of the Hirzebruch surfaces, i.e., the $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ ([12, §V.2]; to translate the notation below to the one used in [12] set $H := h + ef$).

Let F_e be the Hirzebruch surface with a section of the ruling with self-intersection $-e$. The embeddings of these surfaces, plus the cones over rational normal curves give the minimal degree surfaces ([11, Proposition 3.10]). We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^2$ and we take as a $\mathbb{Z}$-basis of $\mathrm{Pic}(F_e)$ a fiber f of one of its ruling (the only one if $e > 0$) and a section h of the ruling with $h^2 = -e$; h is unique if $e > 0$. We have $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h - (e + 2)f)$. All the smooth surfaces $Y \subset \mathbb{P}^r$, $r \ge 3$, with minimal degree $r - 1$ are obtained embedding some F_e with $e \equiv r - 1 \mod 2$ and $0 \le e \le r - 2$ by the complete linear system $|\mathcal{O}_{F_e}(h + \frac{r-1+e}{2} f)|$. From now on we often identify F_e and Y, so that a curve $X \subset Y$ belongs to a certain linear system $|\mathcal{O}_{F_e}(ah + bf)|$, $(a, b) \in \mathbb{N}^2$.

Fix an integral and non-degenerate curve $X \subset Y$ with $X \in |\mathcal{O}_{F_e}(ah+bf)|$. Since X is integral and non-degenerate, we have $a > 0$, $b > 0$, $b \ge ae$ and either $a \ge 2$ or $a = 1$ and $b > (r-1+e)/2$. We have $d := \deg(X) = a(r-1+e)/2 + b - ea = a(r-1-e)/2 + b$. Set $g := p_a(X)$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h-(e+2)f)$, the adjunction formula gives $\omega_X \cong \mathcal{O}_X((a-2)h + (b-e-2)f)$ and hence $2g-2 = -ea(a-2)+a(b-e-2)+b(a-2)$, i.e., $g = 1+ab-b+(ea-ea^2)/2-a$.

Now we check for which $(x,y) \in \mathbb{Z}^2$ we have $h^1(\mathcal{O}_{F_e}(xh+yf)) = 0$. Let $\pi : F_e \to \mathbb{P}^1$ denote the ruling induced by the complete linear system $|\mathcal{O}_{F_e}(f)|$. For any integer $c \ge 0$ we have $\pi_*(\mathcal{O}_{F_e}(ch+df)) \cong \oplus_{i=0}^{c}\mathcal{O}_{\mathbb{P}^1}(d-ie)$. We have $h^1(\mathcal{O}_{\mathbb{P}^1}(t)) = 0$ if an only if $t \ge -1$. Thus if $x \ge 0$ we have $h^1(\mathcal{O}_{F_e}(xh+yf)) = 0$ if and only if $y \ge ex-1$.

Now assume $x = -1$. Since $h^i(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, $i = 0,1$, and π is flat, the changing basis theorem gives $\pi_*(\mathcal{O}_{F_e}(ah+bf)) = R^1\pi_*(\mathcal{O}_{F_e}(ah+bf)) = 0$. Thus the Leray spectral sequence of π gives $h^1(\mathcal{O}_{F_e}(-h+yf)) = 0$.

Now assume $x \le -2$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h-(e+2)f)$, duality gives $h^1(\mathcal{O}_{F_e}(xh+yf)) = h^1(\mathcal{O}_{F_e}((-2-x)f+(-b-e-2)h))$. Since $-2-x \ge 0$, we just saw that $h^1(\mathcal{O}_{F_e}(xh+yf)) = 0$ if and only if $-y-e-2 \ge e(-2-x)-1$, i.e., if and only if $y \le (x+1)e-1$.

Fix an integer $x > 0$. We have $\mathcal{O}_{F_e}(x) \cong \mathcal{O}_{F_e}(xh + x\frac{r_1+e}{2}f)$. Since Y is projectively normal, we have $h^1(\mathcal{I}_X(x)) = 0$ if and only if $h^1(\mathcal{O}_{F_e}(x)(-X)) = 0$, i.e., if and only if $h^1(\mathcal{O}_{F_e}((x-a)h + (x\frac{r_1+e}{2} - b)f)) = 0$. We do not need to show all the possible solutions for arbitrary e for the following reason. In the particular cases (i.e., $e = 0,1$) we will do below we will get all integers $d \ge r+1$ and for each of these cases the genus is $\pi(d, r+1)$ by Remark 3.1 below. So the long discussion of the Hirzebruch surfaces, rational cones and the Veronese embedding would only give (by the Castelnuovo's theorem explained in [11, Ch. III]) all possible smooth curves in part (2) of Theorem 1.1.

Remark 3.1. Take any smooth curve $X \subset \mathbb{P}^r$ with maximal rank, but not aCM constructed as an isomorphic linear projection of an aCM and linearly normal curve $X' \subset \mathbb{P}^{r+1}$ contained in a minimal degree surface T, i.e., either a cone of a rational normal curve of $\mathbb{P}^{r+1}$ or the isomorphic image of an Hirzebruch surface F_e, $e \equiv r+1 \pmod 2$, $0 \le e \le r-2$. Set $d := \deg(X)$. We claim that $p_a(X) = \pi(d, r+1)$, i.e., that $p_a(X') = \pi(d, r+1)$. Indeed, since X' is aCM, we have $h^1(\mathcal{I}_{X'}(x)) = 0$ for all $x \ge 0$. Let $H \subset \mathbb{P}^{r+1}$ be a general hyperplane. Note that $H \cap X'$ are d points of the rational normal curve $T \cap H$. Use the proof of Castelnuovo's theorem given in [11, Ch. III].

3.1. $\mathbb{P}^r$, $r \ge 6$ and r even

Fix an integer $t \ge 3$. In this section we consider non-degenerate curves $X \subset \mathbb{P}^{2t}$, which are not linearly normal (and in particular they are not aCM), but which have maximal rank. Set $F_0 := \mathbb{P}^1 \times \mathbb{P}^1$. The line bundle $\mathcal{O}_{F_0}(1,t)$ is very ample and it gives a linearly normal embedding $\phi : F_0 \to \mathbb{P}^{2t+1}$ as a minimal degree surface (and in particular as an aCM surface). Let $Y \subset \mathbb{P}^{2t}$

be a general linear projection of $\phi(F_0)$. Since $2t \ge 6$, $Y \cong F_0$ and hence for each curve $X \subset F_0$ we get an embedding of X into $\mathbb{P}^{2t}$. Fix integers $a \ge 2$ and $b > 0$ and take any smooth $C_{a,b} \in |\mathcal{O}_{F_0}(a,b)|$. Let $X_{a,b} \subset Y$ denote the curve obtained by the linear projection of $\phi(C_{a,b})$. Since $a \ge 2$, $\phi(C_{a,b})$ spans $\mathbb{P}^{2t+1}$ and hence $X_{a,b}$ is non-degenerate and $h^1(\mathcal{I}_{X_{a,b}}(1)) > 0$. We have $\omega_{F_0} \cong \mathcal{O}_{F_0}(-2,-2)$. Hence the adjunction formula gives $\omega_{C_{a,b}} \cong \mathcal{O}_{X_{a,b}}(a-2,b-2)$. Thus $p_a(X_{a,b}) = ab-a-b-1$. Note that $\deg(X_{a,b}) = b+ta$. For any integer $x > 0$ let $\eta_{x,a,b} : H^0(\mathcal{O}_{F_0}(x,tx)) \to H^0(\mathcal{O}_{X_{a,b}}(x))$ denote the restriction map.

Remark 3.2. Fix $(u,v) \in \mathbb{Z}^2$. By the Künneth formula we have $h^1(\mathcal{O}_{F_0}(u,v)) = 0$ if and only if either $u \ge 0$ and $v \ge -1$ or $u = -1$ or $u \le -2$ and $v \ge 0$.

Remark 3.3. By [1, Corollary 3.3] or [2, Theorem 2] we have $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \ge 2$. Thus for every integer $x \ge 2$ the restriction map $\rho_x : H^0(\mathcal{O}_{\mathbb{P}^{2t}}(x)) \to H^0(\mathcal{O}_Y(x))$ is surjective. Note that $\mathcal{O}_Y(x) \cong \mathcal{O}_{F_0}(1,xt)$ and that $h^0(\mathcal{O}_{F_0}(2,2t)) = 6t+3$. Since ρ_2 is surjective, we have $h^0(\mathcal{I}_Y(2)) = \binom{2t+2}{2} - 6t - 3 > 0$. Thus every curve contained in Y is contained in a quadric hypersurface. Note that $\eta_{x,a,b}$ is surjective (a condition equivalent to $h^1(\mathcal{I}_{C_{a,b}}(x)) = 0$ if $x \ge 2$) if and only if $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$. By the Künneth's formula we have $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$ if and only if either $x \ge a-1$ or $b \ge tx-1$. Since $X_{a,b}$ is an isomorphic linear projection of $\phi(C_{a,b})$ and $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \ge 2$, we have $h^1(\mathcal{I}_{C_{a,b}}(x)) = 0$ if and only if $h^1(\mathcal{O}_{F_0}(x-a,xt-b)) = 0$, i.e., if and only if either $x \ge a$ and $xt-b \ge -1$ or $x-a = -1$ or $x-a \le -2$ and $tx-b < 0$.

Take positive integers a,b. The restriction maps $\eta_{x,a,b}$ are surjective for all $x \ge 2$ if and only if $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$ for all $x \ge 2$. Recall that $t \ge 3$ and $\mathcal{O}_{F_0}(1) = \mathcal{O}_{F_0}(1,t)$. First assume $x \ge a$. In this case $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$ if and only if $tx-b \ge x-a-1$ (Remark 3.2) and this is the case if and only if $b \le ta+1$. If $x = a-1$, then $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$ for any b. Now assume $x \le a-2$. Since $x \ge 2$, in this part we are assuming $a \ge 4$). By Remark 3.2 we have $h^1(\mathcal{O}_{F_0}(x-a,tx-b)) = 0$ if and only if $b \ge tx$. Thus $\eta_{x,a,b}$ is surjective for all $x \ge 2$ if and only if

$$2ta - 2t \le b \le 2ta + 1. \tag{3.1}$$

So for each fixed $t \ge 3$ (i.e., for each fixed even $r \ge 6$) and each fixed $a \ge 2$ we have $2t+1$ possible degrees (ranging from $2ta-2t$ and $2ta+1$), each of them with a different genus (which increases from $ta^2-2at-ta+2t-a+1 = ta^2-3at-a+1$ to $ta^2+a-ta-a+1 = ta^2-ta+1$). The maximal degree $2ta+1$, for the integer a is higher than the minimal degree, $2ta$, for the integer $a+1$. Thus increasing a we get as degrees all integers which are at least the minimal degree which occurs when $a = 4$, i.e., all $d \ge 6t$. Then we add the non-degenerate examples coming for the integers $a = 1,2,3$ (i.e., for $a = 1$ we assume $b > t$ and hence $b = t+1$). We get all examples with $d \ge 2t+1 = r+1$.

3.2. $\mathbb{P}^r$, $r \geq 5$ and r odd

Fix an odd integer $r \geq 5$. Thus $r = 2t - 1$ for some integer $t \geq 3$. The linear system $|\mathcal{O}_{F_1}(h + tf)|$ induces an embedding $\phi : F_1 \to \mathbb{P}^{2t}$. Let $Y \subset \mathbb{P}^{2t-1} = \mathbb{P}^r$ be a general linear projection of $\phi(F_1)$. We have $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \geq 2$ by either [2, Theorem 2] or [1, Corollary 3.3]. Fix integer $b \geq a \geq 2$. Fix a smooth curve $X \in |\mathcal{O}_{F_1}(ah + bf)|$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$, the adjunction formula gives $\omega_X \cong \mathcal{O}_X((a-2)h + (b-3)f)$ and so $2p_a(X) - 2 = a(b-3) + (a-2)b - a(a-2)$, i.e., X has genus $1 + ab - a(a+3)/2$. Let $X_{a,b} \subset Y$ be the image of X by the linear projection sending $\phi(F_1)$ isomorphically onto Y. Note that $\deg(X_{a,b}) = b + ta - a$. For any $x \in \mathbb{N}$ let $\eta_{x,a,b} : H^0(\mathcal{O}_{F_1}(xh + xtf)) \to H^0(\mathcal{O}_X(x))$ denote the restriction map. We only consider the case $b = ta$.

Lemma 3.4. *We have $h^1(\mathcal{O}_{F_1}(uh + vf)) = 0$ if and only if either $u \geq 0$ and $v \geq u - 1$ or $u = -1$ or $u \leq -2$ and $u \geq v$.*

Proof. First assume $u \geq 0$. Let $\pi : F_1 \to \mathbb{P}^1$ be the ruling of F_1. We have $\pi_*(\mathcal{O}_{F_1}(uh + vf)) \cong \oplus_{i=0}^{u} \mathcal{O}_{F_1}(v - i)$. Since $h^1(\mathcal{O}_{\mathbb{P}^1}(x)) = 0$ if and only if $x \geq -1$, the Leray spectral sequence of π gives $h^1(\mathcal{O}_{F_1}(uh + vf)) = 0$ if and only if $v \geq u - 1$.

Now assume $u = -1$. Since $\pi_*(\mathcal{O}_{F_1}(-h + vf)) = 0$ and $R^1\pi_*(\mathcal{O}_{F_1}(-h + vf)) = 0$, the Leray spectral sequence of π gives $h^1(\mathcal{O}_{F_1}(-h + vf)) = 0$.

Now assume $u \leq -2$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$, duality gives $h^1(\mathcal{O}_{F_1}(uh + vf)) = h^1(\mathcal{O}_{F_1}((-2-u)h + (-3-v)f))$. Thus $h^1(\mathcal{O}_{F_1}(uh + vf)) = 0$ if and only if $-3 - v \geq -2 - u - 1$, i.e., $u \geq v$. □

Take integers $b \geq a > 0$ and a smooth $X \in |\mathcal{O}_{F_1}(ah + bf)|$. Note that X is connected.

Fix positive integers a and b. The restriction maps $\eta_{x,a,b}$ are surjective for all $x \geq 2$ if and only if $h^1(\mathcal{O}_{F_1}((x-a)h + (tx-b)f)) = 0$ for all $x \geq 2$. Recall that $t \geq 3$ and $\mathcal{O}_{F_1}(1) = \mathcal{O}_{F_1}(h + tf)$. First assume $x \geq a$. In this case $h^1(\mathcal{O}_{F_1}((x-a)h + (tx-b)f)) = 0$ if and only if $tx - b \geq x - a - 1$ (Remark 3.4) and this is the case if and only if $b \leq ta + 1$. If $x = a - 1$, then $h^1(\mathcal{O}_{F_1}((x-a)h + (tx-b)f)) = 0$ for any b (Remark 3.4). Now assume $x \leq a - 2$. Since $x \geq 2$ we are assuming $a \geq 4$. Thus $\eta_{x,a,b}$ is surjective for all $x \geq 2$ if and only if $ta - 2t \leq b \leq ta + 1$. Note that an element of $|\mathcal{O}_{F_1}(ah + bf)|$ has degree $ta + b - a$. Thus for a fixed a we get all integers between $2ta - 2t - a$ and $2ta + 1 - a$. Note that the maximum of this set of integers for the integer a is smaller than the minimum one arising for the integer $a + 1$. We add the non-degenerate maximal rank curves coming from the integers $a = 1, 2, 3$. The non-degeneracy gives no restriction if $a > 1$, while if $a = 1$ we need $b > t$ and hence $b = t + 1$. Thus as in the case with even $r \geq 6$ we get as degrees of maximal rank, but not aCM, curves of all degrees $d \geq r + 1$.

3.3. Case $r = 4$

Call $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ the order 2 Veronese embedding. Set $W := \phi(\mathbb{P}^2)$. The Veronese surface is aCM and a general linear projection Y of it in $\mathbb{P}^4$ is isomorphic to it. Thus for each integer $a \geq 3$ and any smooth degree a plane curve $A \subset \mathbb{P}^2$ we get a smooth curve $\phi(A)$ with degree $2a$ and genus $(a-1)(a-2)/2 = \pi(2a, 5)$. The curve $\phi(A)$ is aCM, because W is aCM and $h^1(\mathcal{O}_{\mathbb{P}^2}(a-t)) = 0$ for all $t \in \mathbb{N}$, i.e., for all $t \in \mathbb{N}$ the restriction map $H^0(\mathcal{O}_W(t)) \to H^0(\mathcal{O}_{\phi(A)}(t))$ is surjective. Let $X \subset Y$ be the image of $\phi(A)$ by the isomorphic linear projection $W \to Y$. It is well-known that $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \geq 2$ (this also follows from the case $(d, k) = (2, 4)$ of [2, Theorem 3]). Since all restriction maps $H^0(\mathcal{O}_W(t)) \to H^0(\mathcal{O}_{\phi(A)}(t))$, $t \geq 2$, are surjective, X has maximal rank. Since X is not linearly normal, X is not aCM. Since $a \geq 3$, X is non-degenerate. Note that we really need to exclude the case $a = 2$, because the only non-degenerate degree 4 curves of $\mathbb{P}^4$ are the rational normal curves, which are aCM.

Proof of part (2) *of Theorem 1.1.* For r even and $r \geq 6$ we use the linear projection of the embeddings of F_0. For all odd $r \geq 5$ we use the linear projections of the embeddings of F_1. Remark 3.1 gives that if $X \subset \mathbb{P}^r$ arises in that way we have $p_a(X) = \pi(\deg(X), r+1)$. □

Remark 3.5. We explain the existence for all integers $d \geq r$, $r \geq 4$, of smooth, aCM and non-degenerate curves $X \subset \mathbb{P}^r$, $r \geq 4$, with $p_a(X) = \pi(d, r)$. If r is odd use the images of the smooth curves $X \in |\mathcal{O}_{F_0}(ah + a\frac{r-1}{2}f)|$ by the embedding of F_0 induced by the complete linear system $|\mathcal{O}_{F_0}(h + \frac{r-1}{2}f)|$. If r is even use the images of the smooth curves $X \in |\mathcal{O}_{F_1}(ah + a\frac{r}{2}f)|$ by the embedding of F_1 induced by the complete linear system $|\mathcal{O}_{F_1}(h + \frac{r}{2}f)|$; note that these linearly normal examples work even when $r = 4$, because $\mathcal{O}_{F_1}(h+2f)$ is very ample and $h^0(\mathcal{O}_{F_1}(h+2f)) = 5$.

4. Genus $g < \pi(d, r+1)$

Remark 4.1. Let $W \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate surface of degree n, i.e., the next degree after the minimal one for non-degenerate surfaces. T. Fujita studied these surfaces (and their higher dimensional generalization) in the set-up of polarized varieties with Δ-genera 1 (more precisely, either they have Δ-genus 1 or they are the isomorphic linear projection of a polarized variety of Δ-genus 0). By [4, Theorem 1.2] they are either normal, aCM and anti-canonically embedded (often called normal del Pezzo surfaces) or exterior linear projection of minimal degree surfaces of $\mathbb{P}^{n+1}$. In the latter case if W is smooth we run in the case which we have handled in Section 3 and which only gives degree d maximal rank curves of genus $\pi(d, n+1)$; if W is singular, still no new smooth curve may arise in this way and so the degree d maximal rank curves contained in them have genus $\pi(d, n+1)$. We recall that the classification of so-called normal del Pezzo surfaces is quite complicated if we allow non-Gorestein singularities or we allow that ω_W is not ample ([5, 10]), but for us it is sufficient to look at normal Gorenstein surfaces for which ω_W is a line bundle and ω_W^* is ample (we even only need them when ω_X^* is very ample). In this case there is a complete and easy classification ([15, 16]). The main point is that in this case the surface $Y \subset \mathbb{P}^n$ is anticanonically embedded and it is either the cone over a linearly normal elliptic curve of $\mathbb{P}^{n-1}$ or it is the one described in [7] as the image of a blowing up of $\mathbb{P}^2$ by a system of plane cubics (when W is not smooth it corresponds to the sequences of blowing ups in almost general position in the terminology of [7]). In the latter cases we have $n \le 9$ and hence they give examples in the set-up of Proposition 1.2 only for $r := n-1 \le 8$. We only do the cases which gives examples for $r = 7, 8$.

Remark 4.2. The effective Weil divisors of cones are described in [12, Ex. II.6.3 and Ex. V.2.9]. Fix a hyperplane $H \subset \mathbb{P}^n$ and $o \in \mathbb{P}^n \setminus \{o\}$. Fix a smooth and non-degenerate curve $C \subset H$ and let $W \subset \mathbb{P}^n$ denote the cone with vertex o and base C. Let $\ell_1 : W \setminus \{o\} \to C$ denote the morphism induced by the linear projection from o. Let $X \subset W$ be a smooth and non-degenerate curve. If $o \notin X$, then ℓ_1 induces a morphism $\ell : X \to C$. If $o \in X$, then $\ell_{1|X\setminus\{o\}}$ induces a morphism $\ell : X \to C$, because C and X are assumed to be smooth. Set $a := \deg(\ell)$. Since X is non-degenerate, we have $a \ge 2$. If $o \notin X$, we have $\deg(X) = a\deg(C)$, because $\mathcal{O}_X(1) \cong \ell^*(\mathcal{O}_C(1))$. If $o \in X$, we have $\deg(X) = a\deg(C) + 1$, because $\mathcal{O}_X(1) \cong \ell^*(\mathcal{O}_C(1))(o)$ (here we use that o is a smooth point of X). Thus for all smooth and non-degenerate curves $X \subset W$ there is an integer $a \ge 2$ such that either $\deg(X) = a\deg(W)$ or $\deg(X) = a\deg(W) + 1$. For this statement the smoothness of X is essential. Now take as $W \subset \mathbb{P}^{r+1}$ a cone over a linearly normal elliptic curve of $\mathbb{P}^r$. We have $\deg(W) = r + 2$. Thus for a fixed r the degrees d of the smooth curves contained in such cones are very restrictive: $d \equiv 0, 1 \pmod{r+2}$. By Remark 4.1 we get that for $r \ge 9$ the only possible integers d appearing as smooth curves with genus $\pi_1(d, r+1)$ are $\equiv 0, 1 \pmod{r+2}$, contrary to the case of Theorem 1.1.

4.1. Y rational

In this case W is the image by the anticanonical linear system of a sequence of $c := 9 - r$ blowing-ups starting with $\mathbb{P}^2$ and the sequence is called *in almost general position*. Since $c \ge 0$, this implies $r \le 9$ and if $r = 9$ we just have the order 3 Veronese embedding of $\mathbb{P}^2$. We do only the cases $r = 7, 8$, because the lower r are messy (just to give a sample, look at [13] in which the only problem is to find that all pairs (d, g) in certain ranges are realized (and here the 5 points are assumed to be in general position)).

Example 4.3. Assume $r = 8$. In this case Y is an isomorphic linear projection of the order 3 Veronese embedding W of $\mathbb{P}^2$. Let $X' \subset W$ be the image of a degree m integral curve of $\mathbb{P}^2$. We have $g := p_a(X) = (m-1)(m-2)/2$ and $d := \deg(X') = 3m$. Thus $g = (d-3)(d-6)/18$. The curve X' is aCM and its isomorphic linear projection in $\mathbb{P}^8$ has maximal rank, but it is not aCM, because it is not linearly normal.

Example 4.4. Assume $r = 7$. In this case the surface $W \subset \mathbb{P}^7$ whose isomorphic linear projection gives the example is smooth and it is the embedding of F_1 by its anticanonical linear system $|\mathcal{O}_{F_1}(2h + 3f)|$. We use the notation of section 3. Suppose that $C \subset Y$ is the image of an integral curve $X \in |\mathcal{O}_{F_1}(ah + bf)|$ with $a > 0$, $b > 0$ and $b \geq a$. To have X and C non degenerate we need either $b \geq 4$ or $(a, b) = (3, 3)$. We have $d := \deg(X) = 2b + 3a - 2a = 2b + a$ and $g := p_a(X) = 1 + ab - (a^2 + a)/2$. We are in the set-up of Remark 2.1. Fix an integer $t > 0$. Let $Y \subset \mathbb{P}^7$ be a general linear projection from $\mathbb{P}^8$ of the anticanonical embedding of F_1. By [2] we have $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \geq 2$. By Remark 2.1 we have $h^1(\mathcal{I}_X(t)) = 0$ if and only if $h^1(\mathcal{O}_{F_1}((2t-a)h + (3t-b)f)) = 0$. We quote the case $e = 1$ of Section 3. If $a = 2t + 1$ we have $h^1(\mathcal{O}_{F_1}((2t-a)h + (3t-b)f)) = 0$ for any b. If $a \leq 2t$ we have $h^1(\mathcal{O}_{F_1}((2t-a)h + (3t-b)f)) = 0$ if and only if $3t - b \geq 2t - a - 1$, i.e., $b \leq a + t + 1$. If $a \geq 2t + 2$ we have $h^1(\mathcal{O}_{F_1}((2t-a)h + (3t-b)f)) = 0$ if and only if $3t - b \leq 2t - a$, i.e., $b \leq a + t$.

Proof of Proposition 1.2. Assume for the moment $\sigma(X) > 2$. Taking a general hyperplane section and using the definition of $\sigma(X)$ we get $d \geq \binom{r+1}{2}$. Claim 2 of Remark 2.7 gives a contradiction. Thus $\sigma(X) = 2$. First assume $h^1(\mathcal{I}_X(1)) = 0$. Thus (2.1) gives $s(X) = 2$ and that the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(H, \mathcal{I}_{X \cap H, H}(2))$ is surjective, Lemma 2.2 gives that X is aCM, a contradiction. Thus X is an isomorphic projection of $X' \subset \mathbb{P}^{r+1}$. Apply [11, Corollary 3.17] to X'. □

5. Reducible curves with maximal rank

We only consider reduced curves and so our curves in this section are irreducible if and only if they are integral. In the case $r = 3$ there is a complete description of all (d, g) for with there is a reducible aCM space curve, but no irreducible aCM space curve ([9, 17]) and this description is exploited in [6] from the geometric point of view. No such description is known for $r > 3$ (and it is not expected, since we not even know the triples (d, g, r) for which aCM curves exists). Much less is expected for maximal rank curves. However for $r \geq 5$ we construct in this section several examples of reducible maximal rank curves, but not aCM, with degree d and arithmetic genus g with (d, g) not covered by any integral maximal rank curve. Take a reduced maximal rank curve $X \subset \mathbb{P}^r$. X is degenerate if and only if $h^0(\mathcal{I}_X(1)) \neq 0$, i.e., $h^0(\mathcal{I}_X(t)) \neq 0$ for all $t > 0$. Since X has maximal rank by assumption, it is aCM and X is just an aCM curve in some proper linear subspace of $\mathbb{P}^r$. Thus it is not restrictive to focus our attention on the case of maximal rank non-degenerate curves. We claim that for $r \geq 5$ the examples given in part (2) of Theorem 1.1 and in Examples 4.3, 4.4 or in Remark 4.2 have $h^0(\mathcal{I}_X(2)) \geq 2$. Note that in all these examples we have $h^0(\mathcal{I}_X(1)) = 0$ and

$h^1(\mathcal{I}_X(1)) = 1$. The long cohomology exact sequence of the exact sequence (2.1) gives $h^0(\mathcal{I}_X(2)) \geq h^0(H, \mathcal{I}_{X\cap H,H}(2)) - 1$. Since X has maximal rank, but it is not aCM, Lemma 2.2 shows that if $h^0(H, \mathcal{I}_{X\cap H,H}(2)) - 1 > 0$, then $h^0(\mathcal{I}_X(2)) = h^0(H, \mathcal{I}_{X\cap H,H}(2)) - 1$. Thus for $r \geq 5$ the examples given in part (2) of Theorem 1.1 have $h^0(\mathcal{I}_X(2)) = \binom{r+1}{2} - 2(r-1) - 2$, while the examples used in Proposition 1.2 have $h^0(\mathcal{I}_X(2)) = \binom{r+1}{2} - 2r - 1$. Thus there are many examples with large $h^0(\mathcal{I}_X(2))$.

In this section we prove the following result.

Proposition 5.1. *Let $X \subset \mathbb{P}^r$, $r \geq 5$, be a reduced and connected maximal rank curve having an irreducible component T spanning $\mathbb{P}^r$. Assume $h^0(\mathcal{I}_X(2)) \geq 2$ and set $a := \lfloor h^0(\mathcal{I}_X(2))/2 \rfloor$. Fix an integer b such that $1 \leq b \leq a$. Let $Y \subset \mathbb{P}^r$ be the union of X and b general lines $L_1, \dots, L_b$, each of them intersecting quasi-transversally T and at a unique point. Then $\deg(Y) = \deg(X) + b$, $p_a(Y) = p_a(X)$, $h^0(\mathcal{I}_Y(2)) = h^0(\mathcal{I}_X(2)) - 2b$ and Y has maximal rank, but it is not aCM.*

Proof. The word "quasi-transversally" means that for each L_i we have $L_i \cap \mathrm{Sing}(T) = \emptyset$ and that at each $q \in T \cap L_i$ the line L_i is not the tangent line to T at q. Note that the set $A(T)$ of all lines $L \subset \mathbb{P}^r$ intersecting T at a unique point and quasi-transversally is a non-empty and irreducible quasi-projective variety of dimension r (use that $\dim T = 1$ and that for each $p \in \mathbb{P}^r$ the set of all lines of $\mathbb{P}^r$ containing p is a projective space of dimension $r-1$). Thus it makes sense to speak about the general point of $A(T)$, i.e., of general L_i's. With our formulation of the proposition the case $b = 1$ gives the general case (if $b > 1$ use induction on b and apply the case $b = 1$ to the same T and the maximal rank curve $X' := X \cup L_1 \cup \cdots \cup L_{b-1}$ which have $h^0(\mathcal{I}_{X'}(2)) = h^0(\mathcal{I}_X(2)) - 2b + 2 \geq 2$), except the statement that Y is not aCM. Thus until step (f) we assume $b = 1$ and write $L := L_1$. Fix a general $L \in A(T)$ and set $W := X \cup L$. Since $L \cap T \neq \emptyset$ and X is connected, W is connected.

(a) In this step we check that $p_a(W) = p_a(X)$. We have $|T \cap L| = 1$ because $r > 2$, $\dim A(T) = r$ and T has only ∞^2 secant lines. Since $\mathrm{Sing}(X)$ is a finite set, there are only ∞^1 lines containing a smooth point of T and a singular point of X. Thus $L \cap \mathrm{Sing}(X) = \emptyset$. Since $\dim A(T) > 1$, T has only ∞^1 tangent lines and $L \cap \mathrm{Sing}(X) = \emptyset$, L intersects quasi-transversally T. Since $|L \cap T| = 1$ and L intersects quasi-transversally T. Thus it is sufficient to prove that $L \cap E = \emptyset$ for each irreducible component E of X such that $E \neq T$ (if any). Indeed, for any $q \in \mathbb{P}^r$ and any reduced curve $F \subset \mathbb{P}^r$ with $q \notin F$ we have $R \cap F = \emptyset$ for the general line $R \subset \mathbb{P}^r$ containing q.

(b) Since $W \supset T$, we have $h^0(\mathcal{I}_W(t)) = 0$ for all $t \leq 1$. Thus to prove that W has maximal rank it is sufficient to prove that $h^1(\mathcal{I}_W(t)) = 0$ for all $t \geq 2$. In this step we prove that $h^0(\mathcal{O}_W(t)) = h^0(\mathcal{O}_X(t)) + t$. Consider the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_W(t) \to \mathcal{O}_X(t) \oplus \mathcal{O}_L(t) \to \mathcal{O}_{X\cap L}(t) \to 0, \tag{5.1}$$

which only requires that L is not an irreducible component of X and in which $X\cap L$ is the scheme-theoretic intersection. Thus $h^0(\mathcal{O}_{X\cap L}(t)) = \deg(X\cap T)$. By step (a) we have $\deg(X\cap L) = 1$. Since L is a line, $\deg(X\cap L) = 1$ and $t \ge 0$, we have $h^0(\mathcal{O}_L(t)) = t+1$ and the restriction map $H^0(\mathcal{O}_L(t)) \to H^0(\mathcal{O}_{X\cap L}(t)$ is surjective. Thus (5.1) gives $h^0(\mathcal{O}_W(t)) = h^0(\mathcal{O}_X(t)) + t$.

(c) Now we check that $h^0(\mathcal{I}_W(2)) = h^0(\mathcal{I}_X(2)) - 2$. Since $h^0(\mathcal{O}_W(2)) = h^0(\mathcal{O}_X(2)) + 2$ by step (a), we have $h^0(\mathcal{I}_W(2)) \ge h^0(\mathcal{I}_X(2)) - 2$ and so we only need to prove that $h^0(\mathcal{I}_W(2)) \le h^0(\mathcal{I}_X(2)) - 2$. Fix $q \in T_{\mathrm{reg}}$ and call R_q a general line containing q. It is sufficient to prove that $h^0(\mathcal{I}_{X\cup R_q}(2)) \le h^0(\mathcal{I}_X(2)) - 2$ for a general q. Since R_q contains a general point $p \in \mathbb{P}^r$ and $h^0(\mathcal{I}_X(2)) > 0$, we have $h^0(\mathcal{I}_{X\cup R_q}(2)) \le h^0(\mathcal{I}_{X\cup\{p\}}(2)) = h^0(\mathcal{I}_X(2)) - 1$ for any q. Thus it is sufficient to prove that a general element of $|\mathcal{I}_{X\cup\{p\}}(2)|$ does not contain R_q if q is general in T. Since $h^0(\mathcal{I}_X(2)) \ge 2$ and p is general in $\mathbb{P}^r$, this is the case if and only if a general $Q \in |\mathcal{I}_X(2)|$ is a cone with vertex containing q. This is not the case for a general $q \in T$, because T spans $\mathbb{P}^r$ and the singular locus of a quadric hypersurface of $\mathbb{P}^r$ is a proper linear subspace of $\mathbb{P}^r$.

(d) Now we check that for each integer $t \ge 2$ we have $h^0(\mathcal{I}_W(t)) = h^0(\mathcal{I}_X(t)) - t$. Set $\{q\} := X\cap L$. Since $h^0(\mathcal{O}_W(t)) = h^0(\mathcal{O}_X(t)) + t$ by step (a), we have $h^0(\mathcal{I}_W(t)) \ge h^0(\mathcal{I}_X(t)) - t$ and so we only need to prove that $h^0(\mathcal{I}_W(t)) \le h^0(\mathcal{I}_X(t)) - t$. In step (c) we proved the case $t = 2$. Thus we may assume $t > 2$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_X(t))$ is surjective and $L\cap X = \{q\}$ as schemes, it is sufficient to prove the surjectivity of the restriction map $\lambda_t : H^0(\mathbb{P}^r, \mathcal{I}_X(t)) \to H^0(L, \mathcal{I}_{\{q\},L}(t))$. We use induction on t. Fix a general $S \subset L$ such that $|S| = t$ and write $S = \{o_1, \dots, o_t\}$ and set $S_i := \{o_1, \dots, o_i\}$. Since $h^0(L, \mathcal{I}_{\{q\},L}(t)) = t$, it is sufficient to prove that $h^0(\mathcal{I}_{X\cup S_i}(t)) \le h^0(\mathcal{I}_X(t)) - i$ for $i = 1, \dots, t$. Step (c) gives the existence of $Q \in |\mathcal{I}_X(2)|$ such that $X \cap L = \{q, o_1\}$. The union of Q and $t-2$ general hyperplanes gives $h^0(\mathcal{I}_{X\cup\{o_1\}}(t)) = h^0(\mathcal{I}_X(t)) - 1$. Let H_i be a general hyperplane of $\mathbb{P}^r$ containing $\{o_i\}$. Let M_i be a general hyperplane (so $S \cap M_i = \emptyset$ for all i). The degree t hypersurface $Q \cup (\bigcup_{h=1}^{i} H_i) \cup (\bigcup_{h=i+1}^{t} M_h)$ gives $h^0(\mathcal{I}_{X\cup S_i}(t)) < h^0(\mathcal{I}_{X\cup S_{i-1}}(t))$.

(e) Steps (a) and (d) prove that W has maximal rank.

(f) Now for any $b \ge 1$ we prove that $X \cup L_1 \cup \cdots \cup L_b$ is not aCM. As in step (a) we get $h^0(\mathcal{O}_{X\cup L_1\cup\cdots\cup L_b}(1)) = h^0(\mathcal{O}_X(1)) + b \ge r+1+b$ and so $X \cup L_1 \cup \cdots \cup L_b$ is not linearly normal. □

Remark 5.2. Take $r \ge 5$ and a pair (d, g) such that $d \ge r+1$ and $g = \pi(d, r+1)$. By part (2) of Theorem 1.2 there is a smooth, integral and non-degenerate curve $X \subset \mathbb{P}^r$ with $\deg(X) = d$, $p_a(X) = g$ and maximal rank, but it is not aCM. We saw before Proposition 5.1 that $h^0(\mathcal{I}_X(2)) \ge 2$ and hence the case $b = 1$ of Proposition 5.1 shows that $(d+1, g)$ is realized by some reducible curve with maximal rank, but not aCM. We need to find (d, g, r) for which $(d+1, g, r)$ is not realized by any X. Obviously $\pi_1(d+1, r+1) > \pi_1(d, r+1)$. For $d \gg 0$

we have $\pi_1(d+1, r+1) \sim d^2/(2r+2)$ and hence $\pi_1(d+1, r+1) < \pi(d, r+1)$. Apply part (1) of Proposition 1.2.

References

[1] A. Alzati and F. Russo, *On the k-normality of projected algebraic varieties*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 1, 27–48. https://doi.org/10.1007/s005740200001

[2] E. Ballico and P. Ellia, *On projections of ruled and Veronese surfaces*, J. Algebra **121** (1989), no. 2, 477–487. https://doi.org/10.1016/0021-8693(89)90078-1

[3] E. Ballico, P. Ellia, and C. Fontanari, *Maximal rank of space curves in the range A*, Eur. J. Math. **4** (2018), no. 3, 778–801. https://doi.org/10.1007/s40879-018-0235-z

[4] M. Brodmann and P. Schenzel, *Arithmetic properties of projective varieties of almost minimal degree*, J. Algebraic Geom. **16** (2007), no. 2, 347–400. https://doi.org/10.1090/S1056-3911-06-00442-5

[5] I. A. Cheltsov, *Del Pezzo surfaces with nonrational singularities*, Math. Notes **62** (1997), no. 3-4, 377–389 (1998); translated from Mat. Zametki **62** (1997), no. 3, 451–467. https://doi.org/10.1007/BF02360880

[6] L. Chiantini and F. Orecchia, *Plane sections of arithmetically normal curves in* $\mathbb{P}^3$, in Algebraic curves and projective geometry (Trento, 1988), 32–42, Lecture Notes in Math., 1389, Springer, Berlin, 1989. https://doi.org/10.1007/BFb0085922

[7] M. Demazure, *Surfaces de Del Pezzo, I, II, III, IV, V*, Séminaire sur les Singularités des Surfaces, Palaiseau, France 1976–1977, Lect. Notes in Mathematics **777**, Springer, Berlin, 1980.

[8] A. Dolcetti, *Maximal rank space curves of high genus are projectively normal*, Ann. Univ. Ferrara Sez. VII (N.S.) **35** (1989), 17–23 (1990).

[9] G. Ellingsrud, *Sur le schéma de Hilbert des variétés de codimension* 2 *dans* $\mathbf{P}^e$ *à cône de Cohen-Macaulay*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 4, 423–431.

[10] T. Fujisawa, *On non-rational numerical del Pezzo surfaces*, Osaka J. Math. **32** (1995), no. 3, 613–636. http://projecteuclid.org/euclid.ojm/1200786269

[11] J. Harris, *Curves in projective space*, Séminaire de Mathématiques Supérieures, 85, Presses de l'Université de Montréal, Montreal, QC, 1982.

[12] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[13] ______, *Genre des courbes algébrique dans l'espace projectif (d'après L. Gruson et C. Peskine)*, Bourbaki Seminar, Vol. 1981/1982, pp. 301–313, Astérisque, 92–99, Soc. Math. France, Paris, 1983.

[14] R. Hartshorne and A. Hirschowitz, *Nouvelles courbes de bon genre dans l'espace projectif*, Math. Ann. **280** (1988), no. 3, 353–367. https://doi.org/10.1007/BF01456330

[15] F. Hidaka and K. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. **4** (1981), no. 2, 319–330. https://doi.org/10.3836/tjm/1270215157

[16] F. Sakai, *Anticanonical models of rational surfaces*, Math. Ann. **269** (1984), no. 3, 389–410. https://doi.org/10.1007/BF01450701

[17] T. Sauer, *Smoothing projectively Cohen-Macaulay space curves*, Math. Ann. **272** (1985), no. 1, 83–90. https://doi.org/10.1007/BF01455929

Edoardo Ballico
Department of Mathematics
University of Trento
38123 Povo (TN), Italy
Email address: ballico@science.unitn.it