J. Korean Math. Soc. **56** (2019), No. 5, pp. 1333–1354
https://doi.org/10.4134/JKMS.j180665
pISSN: 0304-9914 / eISSN: 2234-3008

BANACH FUNCTION ALGEBRAS OF n-TIMES CONTINUOUSLY DIFFERENTIABLE FUNCTIONS ON $\mathbf{R}^d$ VANISHING AT INFINITY AND THEIR BSE-EXTENSIONS

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Dedicated to Professor Kozo Yabuta on his 77th birth day

Abstract. In authors' paper in 2007, it was shown that the BSE-extension of $C_0^1(\mathbf{R})$, the algebra of continuously differentiable functions f on the real number space $\mathbf{R}$ such that f and df/dx vanish at infinity, is the Lipschitz algebra $Lip_1(\mathbf{R})$. This paper extends this result to the case of $C_0^n(\mathbf{R}^d)$ and $C_b^{n-1,1}(\mathbf{R}^d)$, where n and d represent arbitrary natural numbers. Here $C_0^n(\mathbf{R}^d)$ is the space of all n-times continuously differentiable functions f on $\mathbf{R}^d$ whose k-times derivatives are vanishing at infinity for $k = 0, \ldots, n$, and $C_b^{n-1,1}(\mathbf{R}^d)$ is the space of all $(n-1)$-times continuously differentiable functions on $\mathbf{R}^d$ whose k-times derivatives are bounded for $k = 0, \ldots, n-1$, and $(n-1)$-times derivatives are Lipschitz. As a byproduct of our investigation we obtain an important result that $C_b^{n-1,1}(\mathbf{R}^d)$ has a predual.

1. Introduction and preliminaries

In this paper $\mathbf{N}$ represents the set of natural numbers, and $\mathbf{C}$ the complex number field. We denote by $(A, \|\ \|_A)$ a commutative semisimple Banach algebra with Gelfand space Φ_A. $C_b(\Phi_A)$ and $C_0(\Phi_A)$ denote the space of all complex-valued continuous functions on Φ_A which are bounded and vanishing at infinity, respectively. The Gelfand transform of an element $a \in A$ is denoted by $\hat{a}$, and $\hat{A}$ represents the set of all Gelfand transforms of $a \in A$. A^* denotes the dual space of A, and $\mathrm{span}(\Phi_A)$ is the linear subspace of A^* generated by Φ_A. So, every element $p \in \mathrm{span}(\Phi_A)$ can be represented uniquely in the form $p = \sum_{\varphi \in \Phi_A} \hat{p}(\varphi)\varphi$, where $\hat{p}$ is a complex-valued function on Φ_A with a finite support; $\mathrm{supp}(\hat{p}) = \{\varphi \in \Phi_A : \hat{p}(\varphi) \neq 0\}$. A continuous function σ on Φ_A is

Received October 2, 2018; Accepted February 7, 2019.
2010 *Mathematics Subject Classification.* Primary 46J15; Secondary 46J40, 46J20.
Key words and phrases. natural commutative Banach function algebra, Lipschitz algebra, BSE-extension, BSE-algebra, BED-algebra, bounded approximate identity, multiplier algebra.

said to be a BSE-function if there exists a nonnegative constant β such that

$$\left|\sum_{\varphi\in\Phi_A}\hat{p}(\varphi)\sigma(\varphi)\right|\le\beta\|p\|_{A^*}\quad (p\in\operatorname{span}(\Phi_A)). \tag{1}$$

The infimum of β in (1) is denoted by $\|\sigma\|_{BSE(A)}$. The set of all BSE-functions on Φ_A is denoted by $C_{BSE}(\Phi_A)$. Obviously, $C_{BSE}(\Phi_A)$ forms a linear subspace of $C_b(\Phi_A)$. It turns out that $\|\ \|_{BSE(A)}$ is a complete algebra norm on $C_{BSE}(\Phi_A)$ ([9]). The Banach algebra $(C_{BSE}(\Phi_A), \|\ \|_{BSE(A)})$ has an important subalgebra $C^0_{BSE}(\Phi_A)$. Suppose $\sigma\in C_{BSE}(\Phi_A)$. We denote by $\mathcal{K}(\Phi_A)$ the directed set of all compact subsets of Φ_A with inclusion order. For $K\in\mathcal{K}(\Phi_A)$, we put

$$\|\sigma\|_{BSE(A),K}:=\sup_{p\in\operatorname{span}(\Phi_A\setminus K),\|p\|_{A^*}\le 1}\left|\sum_{\varphi\in\Phi_A}\hat{p}(\varphi)\sigma(\varphi)\right|.$$

$C^0_{BSE}(\Phi_A)$ is the set of all $\sigma\in C_{BSE}(\Phi_A)$ satisfying $\lim_{K\in\mathcal{K}(\Phi_A)}\|\sigma\|_{BSE(A),K}=0$. It follows that $C^0_{BSE}(\Phi_A)$ forms a closed ideal of $C_{BSE}(\Phi_A)$ ([4, Corollary 3.9]).

A bounded linear operator T of A is called a multiplier of A if $T(fg) = (Tf)g\ \ (f,g\in A)$ holds. The set of all multipliers of A is denoted by $M(A)$. $M(A)$ forms a commutative Banach algebra with respect to usual sum, scalar multiplication, the operator composition as multiplication, and the operator norm as norm. This algebra is called the multiplier algebra of A. It is well known that, for every $T\in M(A)$, there exists a unique bounded continuous function on Φ_A, denoted by $\hat{T}$, which satisfies $\widehat{Ta}=\hat{T}\hat{a}$ $(a\in A)$. We denote $\hat{M}(A)=\{\hat{T}:T\in M(A)\}$. $\hat{M}(A)$ forms a Banach function algebra on Φ_A, with $\|\hat{T}\|=\|T\|$ as norm.

Definition 1 (cf. [4,9])**.** Let A be a commutative semisimple Banach algebra.
(i) A is said to be a BSE-algebra if $C_{BSE}(\Phi_A)=\hat{M}(A)$ holds.
(ii) A is said to be a BED-algebra if $C^0_{BSE}(\Phi_A)=\hat{A}$ holds.

Lemma 1. *Suppose A is a Banach function algebra on a locally compact noncompact Hausdorff space X which satisfies the following* (i)*,* (ii)*, and* (iii)*.*
(i) $A\subseteq C_0(X)$;
(ii) *A is closed under taking the complex conjugation*;
(iii) *If $f\in A$ and $\lambda>0$ satisfy $\lambda-f(x)>0$ $(x\in X)$, we have $\frac{1}{\lambda-f}-\frac{1}{\lambda}\in A$.*
Then A is natural, that is, every $\varphi\in\Phi_A$ is represented, by some $x_\varphi\in X$, as

$$\varphi(f)=f(x_\varphi)\quad (f\in A).$$

Proof. Let $\tilde{X}=X\cup\{\infty\}$ be the one point compactification of X, and $A_e = A\oplus\mathbf{C}e$, the unitization of A. Every $f+\mu e\in A_e$ is considered as a function on $\tilde{X}$ by $(f+\mu e)(x)=f(x)+\mu$ if $x\in X$, and $=\mu$ if $x=\infty$, with $\|f+\mu e\|=\|f\|+|\mu|$ as norm. Then A_e is a Banach function algebra on $\tilde{X}$ which is also closed under taking the complex conjugation. We first show that A_e is natural. To do this, suppose contrary that A_e is not natural. Then there exists $\varphi_0\in\Phi_{A_e}$ which can not be given by any point $x \in \tilde{X}$, and Ker φ_0 is a maximal ideal of A_e which dose not contain any maximal ideal of A_e given by an element of $\tilde{X}$. Therefore, for each $x \in \tilde{X}$, there exists $f_x \in A_e$ such that $\varphi_0(f_x) = 0$ and $|f_x(x)| = 1$. Choose an open neighborhood $U_x \subseteq \tilde{X}$ of x such that $|f_x| > 0$ on U_x. Since $\tilde{X}$ is compact, there exist a finite number of elements $x_1, \dots, x_m \in \tilde{X}$ such that $\cup_{k=1}^m U_{x_k} = \tilde{X}$. Put $g := \sum_{k=1}^m \overline{f_{x_k}} f_{x_k} \in A_e$, where $\overline{f_{x_k}}$ is the complex conjugate of f_{x_k}. Then $0 < g(x) \ \ (x \in \tilde{X})$. Set $\lambda := g(\infty)$, and $f := \lambda e - g$. Then $f \in A$ with $0 < \lambda - f(x) (= g(x))$ for all $x \in X$. By (iii), it follows that $h := \frac{1}{\lambda - f} - \frac{1}{\lambda} \in A$. Then we have $1 = (\lambda - f(x))(h(x) + 1/\lambda)$ for all $x \in X$. From this we have

$$e = (\lambda e - f)(h + \frac{1}{\lambda} e). \tag{2}$$

Applying φ_0 to (2), we obtain

$$\begin{aligned} 1 &= \varphi_0(\lambda e - f)\varphi_0(h + \frac{1}{\lambda} e) \\ &= \varphi_0(g)\varphi_0(h + \frac{1}{\lambda} e) \\ &= \Bigg(\sum_{k=1}^m \varphi_0(\overline{f_{x_k}})\varphi_0(f_{x_k}) \Bigg) \varphi_0(h + \frac{1}{\lambda} e) = 0. \end{aligned} \tag{3}$$

Thus we arrive at a contradiction (3), hence A_e is natural.

Next, suppose $\varphi \in \Phi_A$. If we put $\tilde{\varphi}(f + \lambda e) = \varphi(f) + \lambda \ \ (f + \lambda e \in A_e)$, then $\tilde{\varphi} \in \Phi_{A_e}$. Since A_e is natural from the above argument, there exists $x_\varphi \in \tilde{X}$ such that

$$\tilde{\varphi}(f + \lambda e) = (f + \lambda e)(x_\varphi) \qquad (f + \lambda e \in A_e). \tag{4}$$

In this case $x_\varphi \neq \infty$. For, if $x_\varphi = \infty$, we have from (4) that $\varphi(f) = 0$ $(f \in A)$, which is impossible since φ is a nonzero complex homomorphism of A. Therefore $x_\varphi \in X$ follows, and from (4) we have $\varphi(f) = f(x_\varphi) \ \ (f \in A)$, which implies that A is natural. □

2. Algebras of differentiable functions, $C_b^n(\mathbf{R}^d)$ and $C_0^n(\mathbf{R}^d)$

Let n, d be given natural numbers. The symbol $S(\mathbf{R}^d)$ represents the unit sphere in $\mathbf{R}^d$, and $e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ $\in S(\mathbf{R}^d)$. We use the notation $|x| = (\sum_{i=1}^d |x_i|^2)^{1/2}$, $x = (x_1, \dots, x_d) \in \mathbf{R}^d$.

For $a = (a_1, \dots, a_d) \in S(\mathbf{R}^d)$, T_a denotes the differential operator $T_a = \sum_{k=1}^d a_k \frac{\partial}{\partial x_k} = \sum_{k=1}^d a_k T_{e_k}$. We denote by $C_b^n(\mathbf{R}^d)$ (resp. $C_0^n(\mathbf{R}^d)$) the space of all complex-valued functions on $\mathbf{R}^d$ which are n-times continuously differentiable, and satisfy that all

$$T_{e_{i_1}, \dots, e_{i_k}} f (= T_{e_{i_k}}(\cdots (T_{e_{i_2}}(T_{e_{i_1}} f)) \cdots))$$

for $1 \leq i_1, \dots, i_k \leq d, k = 0, 1, \dots, n$, are bounded (resp. vanishing at infinity).

Suppose $f \in C_b^n(\mathbf{R}^d)$ (resp. $C_0^n(\mathbf{R}^d)$). For $a^k = (a_1^k, \ldots, a_d^k) \in S(\mathbf{R}^d)$, $k = 1, \ldots, n$, by applying $T_{a^1}, T_{a^2}, \ldots, T_{a^k}$ to f successively, we obtain

$$\begin{aligned} T_{a^1,\ldots,a^k} f &= \sum_{1\le i_1,\ldots,i_k\le d} a_{i_1}^1 \cdots a_{i_k}^k T_{e_{i_1},\ldots,e_{i_k}} f \\ &\in C_b^{n-k}(\mathbf{R}^d) \text{ (resp. } C_0^{n-k}(\mathbf{R}^d)), \end{aligned} \tag{5}$$

where $\sum_{1\le i_1,\ldots,i_k\le d}$ represents the sum over all choices of $i_1, \ldots, i_k$ in $\{1, \ldots, d\}$.

In the following, $\|f\|_\infty$ denotes the sup-norm of f on $\mathbf{R}^d$.

Definition 2. We define $\| \ \|_{C_b^n}$ on $C_b^n(\mathbf{R}^d)$ by

$$\begin{aligned} \|f\|_{C_b^n} &= \|f\|_\infty + \sum_{k=1}^n \frac{1}{k!} \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k} f\|_\infty \\ &= \sum_{k=0}^n \frac{1}{k!} \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k} f\|_\infty \quad (f \in C_b^n(\mathbf{R}^d)). \end{aligned}$$

Proposition 1.

$$\|f\|_{C_b^n} \le \|f\|_{n,\infty} \le d^n \|f\|_{C_b^n} \quad (f \in C_b^n(\mathbf{R}^d)),$$

where $\|f\|_{n,\infty} = \|f\|_\infty + \sum_{k=1}^n \frac{1}{k!} \sum_{1\le i_1,\ldots,i_k\le d} \|T_{e_{i_1},\ldots,e_{i_k}} f\|_\infty$.

Proof. The first inequality is a consequence of easy calculation using (5):

$$\begin{aligned} \|f\|_{C_b^n} &= \sum_{k=0}^n \frac{1}{k!} \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k} f\|_\infty \\ &= \sum_{k=0}^n \frac{1}{k!} \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \left\| \sum_{1\le i_1,\ldots,i_k\le d} a_{i_1}^1 \cdots a_{i_k}^k T_{e_{i_1},\ldots,e_{i_k}} f \right\|_\infty \\ &\le \sum_{k=0}^n \frac{1}{k!} \sum_{1\le i_1,\ldots,i_k\le d} \|T_{e_{i_1},\ldots,e_{i_k}} f\|_\infty = \|f\|_{n,\infty}, \end{aligned}$$

where $a^j = (a_1^j, \ldots, a_d^j) \in S(\mathbf{R}^d)$, $j = 1, \ldots, k,\ k = 1, \ldots, n$.

For the second inequality, fix $k (1 \le k \le n)$. Then, for each $1 \le i_1, \ldots, i_k \le d$, we have $\|T_{e_{i_1},\ldots,e_{i_k}} f\|_\infty \le \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k} f\|_\infty$, and from this we have

$$\begin{aligned} \|f\|_{n,\infty} &= \|f\|_\infty + \sum_{k=1}^n \sum_{1\le i_1,\ldots,i_k\le d} \frac{1}{k!} \|T_{e_{i_1},\ldots,e_{i_k}} f\|_\infty \\ &\le \|f\|_\infty + \sum_{k=1}^n d^k \frac{1}{k!} \sup_{a^1,\ldots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k} f\|_\infty \\ &\le d^n \|f\|_{C_b^n}. \end{aligned}$$

□

By Proposition 1, two norms on $C_b^n(\mathbf{R}^d)$, $\| \|_{n,\infty}$ and $\| \|_{C_b^n}$ are equivalent, and since it is obvious that $\| \|_{n,\infty}$ is complete, it follows that $\| \|_{C_b^n}$ is also complete.

Lemma 2. *Suppose $f, g \in C_b^k(\mathbf{R}^d)$, $a^1, \dots, a^k \in S(\mathbf{R}^d)$, and $0 \le N$. Then*

$$\sup_{N \le |x|} \left| T_{a^1,\dots,a^k}(fg)(x) \right| \le \sum_{j=0}^{k} \binom{k}{j} \sup_{b^1,\dots,b^j \in S(\mathbf{R}^d)} \sup_{N \le |x|} \left| T_{b^1,\dots,b^j} f(x) \right| \cdot \sup_{c^1,\dots,c^{k-j} \in S(\mathbf{R}^d)} \sup_{N \le |x|} \left| T_{c^1,\dots,c^{k-j}} g(x) \right|, \tag{6}$$

where $\binom{k}{j}$ represents the binomial coefficient.

Proof. We observe that

$$T_{a^1,\dots,a^k}(fg)(x) = f(x) T_{a^1,\dots,a^k} g(x) + \sum_{j=1}^{k-1} \sum_{(\#)} T_{a^{s_1},\dots,a^{s_j}} f(x) T_{a^{t_1},\dots,a^{t_{k-j}}} g(x) + (T_{a^1,\dots,a^k} f(x)) g(x) \quad \text{for all } x \in \mathbf{R}^d, \tag{7}$$

where

$$(\#) = \Big\{ 1 \le s_1 \le \cdots \le s_j \le k, \quad 1 \le t_1 \le \cdots \le t_{k-j} \le k, \quad \{s_1, \dots, s_j, t_1, \dots, t_{k-j}\} = \{1, 2 \dots, k\} \Big\}.$$

With easy calculation, we obtain (6) from (7). □

Proposition 2. *$(C_b^n(\mathbf{R}^d), \| \|_{C_b^n})$ is a Banach algebra, and $C_0^n(\mathbf{R}^d)$ is its closed ideal.*

Proof. Let $f, g \in C_b^n(\mathbf{R}^d)$ be given arbitrarily. By Lemma 2 with $N = 0$, we have

$$\begin{aligned}
&\|fg\|_{C_b^n} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \sup_{a^1,\dots,a^k \in S(\mathbf{R}^d)} \|T_{a^1,\dots,a^k}(fg)\|_\infty \\
&\le \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \sup_{b^1,\dots,b^j \in S(\mathbf{R}^d)} \|T_{b^1,\dots,b^j} f\|_\infty \sup_{c^1,\dots,c^{k-j} \in S(\mathbf{R}^d)} \|T_{c^1,\dots,c^{k-j}} g\|_\infty \\
&\le \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{1}{j!} \sup_{\substack{b^1,\dots,b^j \\ \in S((\mathbf{R}^d)}} \|T_{b^1,\dots,b^j} f\|_\infty \frac{1}{(k-j)!} \sup_{\substack{c^1,\dots,c^{k-j} \\ \in S(\mathbf{R}^d)}} \|T_{c^1,\dots,c^{k-j}} g\|_\infty \\
&\le \left(\sum_{k=0}^{n} \frac{1}{k!} \sup_{b^1,\dots,b^k \in S(\mathbf{R}^d)} \|T_{b^1,\dots,b^k} f\|_\infty \right) \left(\sum_{k=0}^{n} \frac{1}{k!} \sup_{c^1,\dots,c^k \in S(\mathbf{R}^d)} \|T_{c^1,\dots,c^k} g\|_\infty \right)
\end{aligned}$$

$= \|f\|_{C_b^n} \|g\|_{C_b^n}.$

Thus, the norm $\| \ \|_{C_b^n}$ is submultiplicative, and hence $(C_b^n(\mathbf{R}^d), \| \ \|_{C_b^n})$ is a Banach algebra.

Let $\{f_N\}_N$ be a Cauchy sequence in $C_0^n(\mathbf{R}^d)$. Then there exists $f \in C_b^n(\mathbf{R}^d)$ such that $\lim_{N\to\infty} \|f - f_N\|_{C_b^n} = 0$. Let $k (0 \le k \le n), 1 \le i_1, \ldots, i_k \le d$ and $\varepsilon > 0$ be given arbitrarily. Then there exists $N_0 \in \mathbf{N}$ such that $\|f - f_{N_0}\|_{C_b^n} \le \varepsilon/(2n!)$. Choose $M > 0$ such that $\sup_{M \le |x|} |T_{e_{i_1},\ldots,e_{i_k}} f_{N_0}(x)| \le \varepsilon/2$. Then we have

$$\begin{aligned} \sup_{M\le|x|} |T_{e_{i_1},\ldots,e_{i_k}} f(x)| &\le \sup_{M\le|x|} |T_{e_{i_1},\ldots,e_{i_k}} f_{N_0}(x)| + n!\|f - f_{N_0}\|_{C_b^n} \\ &\le \varepsilon/2 + n!(\varepsilon/(2n!)) = \varepsilon. \end{aligned}$$

Hence $T_{e_{i_1},\ldots,e_{i_k}} f$ vanishes at infinity for all $0 \le k \le n$ and $1 \le i_1, \ldots, i_k \le d$, that is, $f \in C_0^n(\mathbf{R}^d)$. This implies that $C_0^n(\mathbf{R}^d)$ is closed.

Suppose $f \in C_0^n(\mathbf{R}^d)$ and $g \in C_b^n(\mathbf{R}^d)$. For any $1 \le i_1, \ldots, i_n \le d$, $1 \le k \le n$, $T_{e_{i_1},\ldots,e_{i_k}}(fg)$ is a sum of the functions of forms

$$fT_{e_{i_1},\ldots,e_{i_k}} g, \quad T_{e_{j_1},\ldots,e_{j_r}} f T_{e_{j_{r+1}},\ldots,e_{j_k}} g, \quad (T_{e_{i_1},\ldots,e_{i_k}} f) g \quad (1 \le r \le k-1),$$

which belong to $C_0(\mathbf{R}^d)$, where $\{j_1, \ldots, j_r\}$ and $\{j_{r+1}, \ldots j_k\}$ are some subsequences of $\{i_1, \ldots, i_k\}$.

Hence $fg, T_{e_{i_1},\ldots,e_{i_k}}(fg) \in C_0(\mathbf{R}^d)$ for $k = 1, \ldots, n$. Therefore $fg \in C_0^n(\mathbf{R}^d)$. □

Proposition 3. $(C_0^n(\mathbf{R}^d), \| \ \|_{C_b^n})$ *is a natural Banach function algebra on* $\mathbf{R}^d$, *and by the identification of* $\varphi \in \Phi_{C_0^n(\mathbf{R}^d)}$ *with the corresponding* $x_\varphi \in \mathbf{R}^d$, $\mathbf{R}^d$ *is its Gelfand space and the identity map is the Gelfand transform.*

From this, it follows easily that $C_0^n(\mathbf{R}^d)$ *is regular.*

Proof. Suppose $\lambda > 0$, and f a real function in $C_0^n(\mathbf{R}^d)$ such that $\lambda - f(x) > 0$ for all $x \in \mathbf{R}^d$. Since $f \in C_0(\mathbf{R}^d)$, there is $\delta > 0$ such that $\lambda - f(x) \ge \delta$ for all $x \in \mathbf{R}^d$. Put $F = \frac{1}{\lambda - f} - \frac{1}{\lambda} = \frac{f}{\lambda(\lambda - f)}$.

That $F \in C_0(\mathbf{R}^d)$ is clear. Let $1 \le i_1, \ldots, i_n \le d$ be arbitrarily chosen. We claim here that $T_{e_{i_1},\ldots,e_{i_k}} F = \frac{G_k}{(\lambda - f)^{k+1}}$, where $G_k \in C_0^{n-k}(\mathbf{R}^d)$ for $k = 1, \ldots, n$. Since $T_{e_{i_1}} F = (T_{e_{i_1}} f)\frac{1}{(\lambda - f)^2}$, the claim is true for $k = 1$. If the claim is true for $k (< n)$, then it is easy to see by elementary calculation that the claim is true for $k+1$. By induction, the claim is true for $k = 1, \ldots, n$, which prove that $F \in C_0^n(\mathbf{R}^d)$. Since $C_0^n(\mathbf{R}^d)$ is closed under taking the complex conjugation, we can apply Lemma 1, to conclude that $C_0^n(\mathbf{R}^d)$ is a natural Banach function algebra on $\mathbf{R}^d$. □

Theorem 1. *The algebra* $C_0^n(\mathbf{R}^d)$ *has a bounded approximate identity composed of elements with compact supports.*

Proof. Let $u \in C_0^n(\mathbf{R})$ be such that $\mathrm{supp}(u) \subset [-1,1]$, and $\int_{-1}^1 u(x)dx = 1$. For each $N \in \mathbf{N}$, define a function u_N on $\mathbf{R}$ by

$$u_N(x) = \left(\int_{-\infty}^x u(t+N+1)dt\right) \cdot \left(\int_x^\infty u(t-N-1)dt\right) \quad (-\infty < x < \infty).$$

Then $u_N \in C_0^n(\mathbf{R})$ with $\mathrm{supp}(u_N) \subset [-N-2, N+2]$ and $u_N(x) = 1$ $(-N \leq x \leq N)$. Therefore if we define

$$u_N^{(d)}(x) = u_N(|x|) \quad (N \in \mathbf{N}, x \in \mathbf{R}^d),$$

we have $u_N^{(d)} \in C_0^n(\mathbf{R}^d)$ with $\mathrm{supp}(u_N^{(d)}) \subseteq \{x \in \mathbf{R}^d : |x| \leq N+2\}$ which satisfy

$$u_N^{(d)}(x) = 1 \ (|x| \leq N), \text{ and } \|u_N^{(d)}\|_{C_b^n} = \cdots = \|u_1^{(d)}\|_{C_b^n}.$$

Let $f \in C_0^n(\mathbf{R}^d)$ and $\varepsilon > 0$ be given. There exists $N_0 \in \mathbf{N}$ such that if $N_0 \leq N$ then, for each $j = 0, \ldots, n$, we have

$$\begin{aligned}
&\frac{1}{j!} \sup_{\substack{b^1,\ldots,b^j \\ \in S(\mathbf{R}^d)}} \sup_{N \leq |x|} |T_{b^1,\ldots,b^j} f(x)| \\
&\leq \frac{1}{j!} \sup_{\substack{b^1,\ldots,b^j \\ S(\mathbf{R}^d)}} \sum_{1 \leq i_1,\ldots,i_j \leq d} \langle b_{i_1}^1 \cdots b_{i_j}^j \rangle \sup_{N \leq |x|} |T_{e_{i_1},\ldots,e_{i_j}} f(x)| \\
&\leq \frac{2\varepsilon}{(n+1)(n+2)\|1 - u_{N_0}^{(d)}\|_{C_b^n}}, \qquad (8)
\end{aligned}$$

where $b^\ell = (b_1^\ell, \ldots, b_d^\ell)$, $\ell = 1, \ldots, j$. By (8), Lemma 2(6), and the facts that $(f(1-u_N^{(d)}))(x) = 0$ if $|x| \leq N$, $\|1 - u_N^{(d)}\|_{C_b^n} = \cdots = \|1 - u_1^{(d)}\|_{C_b^n}$, we have

$$\begin{aligned}
&\|f - f u_N^{(d)}\|_{C_b^n} \\
&= \sum_{k=0}^n \frac{1}{k!} \sup_{a^1,\ldots,a^k \in S(\mathbf{R}^d)} \sup_{N \leq |x|} |T_{a^1,\ldots,a^k}(f(1-u_N^{(d)}))(x)| \\
&\leq \sum_{k=0}^n \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \sup_{\substack{b^1,\ldots,b^j \\ \in S(\mathbf{R}^d)}} \sup_{N \leq |x|} |T_{b^1,\ldots,b^j} f(x)| \sup_{\substack{c^1,\ldots,c^{k-j} \\ S(\mathbf{R}^d)}} \|T_{c^1,\ldots,c^{k-j}}(1-u_N^{(d)})\|_\infty \\
&\leq \sum_{k=0}^n \sum_{j=0}^k \frac{1}{j!} \sup_{\substack{b^1,\ldots,b^j \\ \in S(\mathbf{R}^d)}} \sup_{N \leq |x|} |T_{b^1,\ldots,b^j} f(x)| \frac{\sup_{\substack{c^1,\ldots,c^{k-j} \\ S(\mathbf{R}^d)}} \|T_{c^1,\ldots,c^{k-j}}(1-u_N^{(d)})\|_\infty}{(k-j)!} \\
&\leq \sum_{k=0}^n (k+1) \frac{2\varepsilon}{(n+1)(n+2)\|1-u_{N_0}^{(d)}\|_{C_b^n}} \|1 - u_N^{(d)}\|_{C_b^n} = \varepsilon \quad (N_0 \leq N).
\end{aligned}$$

Thus $\{u_N^{(d)}\}_{N \in \mathbf{N}}$ is a bounded approximate identity of $C_0^n(\mathbf{R}^d)$ such that $\mathrm{supp}(u_N^{(d)}) \subset \{x \in \mathbf{R}^d : |x| \leq N+2\}$ for $N = 1, 2, \ldots$. □

3. Algebras $C_b^{n-1,1}(\mathbf{R}^d)$ and $C_0^{n-1,1}(\mathbf{R}^d)$

We denote by $Lip_1(\mathbf{R})$ (resp. $Lip_1^0(\mathbf{R})$) the Lipschitz algebra on $\mathbf{R}$; that is, the space of complex-valued continuous functions on $\mathbf{R}$ which are bounded (resp. vanishing at infinity) and satisfy

$$\rho(f) = \sup_{\substack{x,y\in\mathbf{R},\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} < \infty$$

$$\left(\text{resp. } \lim_{M\to\infty} \rho_M(f)\left(= \sup_{\substack{x,y\in\mathbf{R},x\neq y\\ M\leq |x|,|y|}} \frac{|f(x)-f(y)|}{|x-y|}\right) = 0\right).$$

With $\|f\|_{Lip_1} = \|f\|_\infty + \rho(f)$ as norm, the space $Lip_1(\mathbf{R})$ becomes a Banach function algebra on $\mathbf{R}$.

It is shown in [4, p. 123] that the BSE-extension of $C_0^1(\mathbf{R})$ is $Lip_1(\mathbf{R})$, and that $Lip_1^0(\mathbf{R})$ is its closed ideal, which itself is a natural Banach function algebra on $\mathbf{R}$. Moreover it is shown that $C^0_{BSE}(\Phi_{C_0^1(\mathbf{R})}) = Lip_1^0(\mathbf{R})$ and $M(C_0^1(\mathbf{R})) = C_b^1(\mathbf{R})$.

Definition 3. For $f \in C_b(\mathbf{R}^d)$, we write $\rho(f) = \sup_{x,y\in\mathbf{R}^d,x\neq y} \frac{|f(y)-f(x)|}{|y-x|}$, and

$$\rho[f](x) = \sup_{y\in\mathbf{R}^d,x\neq y} \frac{|f(y)-f(x)|}{|y-x|} \qquad (x\in\mathbf{R}^d).$$

We put

$$Lip_1(\mathbf{R}^d) := \{f\in C_b(\mathbf{R}^d) : \rho(f)<\infty\}, \text{ and}$$
$$Lip_1^0(\mathbf{R}^d) := \{f\in Lip_1(\mathbf{R}^d) : f \text{ and } \rho[f] \text{ vanish at infinity}\}.$$

One can verify easily that the definitions of $Lip_1^0(\mathbf{R})$ in [4] and in Definition 3 above are consistent.

Note that, for $f, g \in Lip_1(\mathbf{R}^d)$ we have

$$\begin{aligned}\rho[fg](x) &= \sup_{y\in\mathbf{R}^d,y\neq x} \frac{|f(y)g(y)-f(x)g(x)|}{|y-x|}\\ &\leq \sup_{y\in\mathbf{R}^d,y\neq x}\left[|g(y)|\frac{|f(y)-f(x)|}{|y-x|} + |f(x)|\frac{|g(y)-g(x)|}{|y-x|}\right]\\ &\leq \rho[f](x)\|g\|_\infty + |f(x)|\rho(g) \quad (x\in\mathbf{R}^d). \end{aligned} \tag{9}$$

Lemma 3. (i) *For $f\in Lip_1(\mathbf{R}^d)$, we have $\rho(f) = \sup_{x\in\mathbf{R}^d} \rho[f](x)$.*

(ii) *For $f \in C_b^1(\mathbf{R}^d)$, we have $\rho(f) = \sup_{a\in S(\mathbf{R}^d)} \|T_a f\|_\infty$.*

(iii) *For $\alpha,\beta\in\mathbf{C}$, and $f,g\in Lip_1(\mathbf{R}^d)$, we have*

$$\rho[\alpha f+\beta g](x) \leq |\alpha|\rho[f](x) + |\beta|\rho[g](x) \quad (x\in\mathbf{R}^d).$$

(iv) *$Lip_1(\mathbf{R}^d)$ is an algebra and $Lip_1^0(\mathbf{R}^d)$ is its ideal.*

Proof. (i) Suppose $f \in Lip_1(\mathbf{R}^d)$. For any $x, y \in \mathbf{R}^d$ with $x \neq y$, we have

$$\frac{|f(y) - f(x)|}{|y - x|} \leq \rho[f](x). \tag{10}$$

From (10), we have $\rho(f) \leq \sup_{x \in \mathbf{R}^d} \rho[f](x)$. Conversely, it is easy to see that $\rho[f](x) \leq \rho(f)$, $x \in \mathbf{R}^d$ and hence $\sup_{x \in \mathbf{R}^d} \rho[f](x) \leq \rho(f)$.

(ii) Suppose $f \in C_b^1(\mathbf{R}^d)$. For any $x \in \mathbf{R}^d$ and $a \in S(\mathbf{R}^d)$,

$$\begin{aligned} |T_a f(x)| &= \lim_{\substack{0 \neq h \in \mathbf{R}, \\ h \to 0}} \frac{|f(x + ha) - f(x)|}{|h|} \\ &= \lim_{\substack{0 \neq h \in \mathbf{R}, \\ h \to 0}} \frac{|f(x + ha) - f(x)|}{|(x + ha) - x|} \leq \rho[f](x), \end{aligned}$$

and hence

$$\sup_{a \in S(\mathbf{R}^d)} \|T_a f\|_\infty \leq \rho(f). \tag{11}$$

On the other hand, for any $x, y \in \mathbf{R}^d$ with $x \neq y$, if we put $a = \frac{y-x}{|y-x|}$ and $h = |y - x|$, then $a \in S(\mathbf{R}^d)$ and

$$\frac{|f(y) - f(x)|}{|y - x|} = \frac{|f(x + ha) - f(x)|}{|h|} = |T_a f(x + \theta ha)|$$

for some $0 < \theta < 1$, and hence

$$\rho(f) \leq \sup_{a \in S(\mathbf{R}^d)} \|T_a f\|_\infty. \tag{12}$$

From (11) and (12), we have (ii).

(iii) We can get this inequality by straightforward calculation.

(iv) This follows easily from (9). □

Definition 4. Let n, d be given natural numbers. We define;

$$C_b^{n-1,1}(\mathbf{R}^d) := \left\{ f \in C_b^{n-1}(\mathbf{R}^d) : T_{e_{i_1}, \ldots, e_{i_{n-1}}} f \in Lip_1(\mathbf{R}^d),\ 1 \leq i_1, \ldots, i_{n-1} \leq d \right\},$$

$$C_0^{n-1,1}(\mathbf{R}^d) := \left\{ f \in C_0^{n-1}(\mathbf{R}^d) :\ T_{e_{i_1}, \ldots, e_{i_{n-1}}} f \in Lip_1^0(\mathbf{R}^d)\ ,\ 1 \leq i_1, \ldots, i_{n-1} \leq d \right\},$$

$$\|f\|_{C_b^{n-1,1}} := \|f\|_{C_b^{n-1}} + \frac{1}{n!} \sup_{a^1, \ldots, a^{n-1} \in S(\mathbf{R}^d)} \rho(T_{a^1, \ldots, a^{n-1}} f) \qquad (f \in C_b^{n-1,1}(\mathbf{R}^d)). \tag{13}$$

In particular, we have $C_b^{0,1}(\mathbf{R}^d) = Lip_1(\mathbf{R}^d)$ and $C_0^{0,1}(\mathbf{R}^d) = Lip_1^0(\mathbf{R}^d)$.

Proposition 4. *$\|f\|_{C_b^{n-1,1}} \leq \|f\|_{n-1,\infty,\rho} \leq d^{n-1} \|f\|_{C_b^{n-1,1}}$ $(f \in C_b^{n-1,1}(\mathbf{R}^d))$, where $\|f\|_{n-1,\infty,\rho} = \|f\|_{n-1,\infty} + \frac{1}{n!} \sum_{1 \leq i_1, \ldots, i_{n-1} \leq d} \rho(T_{e_{i_1}, \ldots, e_{i_{n-1}}} f)$.*

Proof. Suppose $f \in C_b^{n-1,1}(\mathbf{R}^d)$. Let $a^j = (a_1^j, \ldots, a_d^j) \in S(\mathbf{R}^d)$, $j = 1, \ldots, n-1$ be given arbitrarily. By Lemma 3 and (5), we have

$$\begin{aligned} \rho(T_{a^1,\ldots,a^{n-1}} f) &= \sup_{x \in \mathbf{R}^d} \rho[T_{a^1,\ldots,a^{n-1}} f](x) \\ &\leq \sum_{1 \leq i_1,\ldots,i_{n-1} \leq d} |a_{i_1}^1 \cdots a_{i_{n-1}}^{n-1}| \sup_{x \in \mathbf{R}^d} \rho[T_{e_{i_1},\ldots,e_{i_{n-1}}} f](x) \\ &\leq \sum_{1 \leq i_1,\ldots,i_{n-1} \leq d} \rho(T_{e_{i_1},\ldots,e_{i_{n-1}}} f). \end{aligned} \tag{14}$$

Therefore, by Proposition 1, (13), and (14), we obtain

$$\begin{aligned} \|f\|_{C_b^{n-1,1}} &= \|f\|_{C_b^{n-1}} + \frac{1}{n!} \sup_{a^1,\ldots,a^{n-1} \in S(\mathbf{R}^d)} \rho(T_{a^1,\ldots,a^{n-1}} f) \\ &\leq \|f\|_{n-1,\infty} + \frac{1}{n!} \sum_{1 \leq i_1,\ldots,i_{n-1} \leq d} \rho(T_{e_{i_1},\ldots,e_{i_{n-1}}} f) = \|f\|_{n-1,\infty,\rho}. \end{aligned}$$

Next, we consider the second inequality. From Proposition 1, we have

$$\|f\|_{n-1,\infty} \leq d^{n-1} \|f\|_{C_b^{n-1}}. \tag{15}$$

Further, since $\rho(T_{e_{i_1},\ldots,e_{i_{n-1}}} f) \leq \sup_{a^1,\ldots,a^{n-1} \in S(\mathbf{R}^d)} \rho(T_{a^1,\ldots,a^{n-1}} f)$ for each $1 \leq i_1, \ldots, i_{n-1} \leq d$, we have

$$\sum_{1 \leq i_1,\ldots,i_{n-1} \leq d} \rho(T_{e_{i_1},\ldots,e_{i_{n-1}}} f) \leq d^{n-1} \sup_{a^1,\ldots,a^{n-1} \in S(\mathbf{R}^d)} \rho(T_{a^1,\ldots,a^{n-1}} f). \tag{16}$$

From (15), (16), and the definitions of $\|f\|_{n-1,\infty,\rho}$ and $\|f\|_{C_b^{n-1,1}}$, we get the desired result. □

Proposition 4 shows that the two norms $\| \ \|_{n-1,\infty,\rho}$ and $\| \ \|_{C_b^{n-1,1}}$ are equivalent. Obviously $\| \ \|_{n-1,\infty,\rho}$ is complete, and hence $\| \ \|_{C_b^{n-1,1}}$ is also complete.

Lemma 4. (i) *For $f \in C_b^n(\mathbf{R}^d)$ and $a^1, \ldots, a^{n-1} \in S(\mathbf{R}^d)$, we have*

$$\rho(T_{a^1,\ldots,a^k} f) = \sup_{a \in S(\mathbf{R}^d)} \|T_{a^1,\ldots,a^k,a} f\|_\infty, \ k = 1, \ldots, n-1,$$

hence $T_{a^1,\ldots,a^k} f \in Lip_1(\mathbf{R}^d)$.

(ii) *If $f \in C_0^{n-1,1}(\mathbf{R}^d)$ and $1 \leq i_1, \ldots, i_{n-1} \leq d$, then $T_{e_{i_1},\ldots,e_{i_k}} f \in Lip_1^0(\mathbf{R}^d)$ for $k = 0, 1, \ldots, n-1$.*

Proof. (i) Obviously, (i) follows from Lemma 3(ii).

(ii) If $n = 1$, the assertion is trivial. So we consider the case $2 \leq n$. Suppose $f \in C_0^{n-1,1}(\mathbf{R}^d)$. Then $T_{e_{i_1},\ldots,e_{i_{n-1}}} f \in C_0^{0,1}(\mathbf{R}^d) = Lip_1^0(\mathbf{R}^d)$ from the definition of $f \in C_0^{n-1,1}(\mathbf{R}^d)$.

Let $\varepsilon > 0$ be given. For each $k(0 \le k \le n-2)$, $T_{e_{i_1},\dots,e_{i_k}} f$ belongs to $C_0^1(\mathbf{R}^d)$, so by Lemma 3(ii), we have

$$\rho(T_{e_{i_1},\dots,e_{i_k}} f) = \sup_{a\in S(\mathbf{R}^d)} \|T_{e_{i_1},\dots,e_{i_k},a} f\|_\infty \le (k+1)!\|f\|_{C_b^{k+1}} < \infty.$$

Therefore $\rho[T_{e_{i_1},\dots,e_{i_k}} f]$ is bounded by Lemma 3(i). To show that $\rho[T_{e_{i_1},\dots,e_{i_k}} f]$ vanishes at infinity, choose $M > 0$, by (5), such that

$$\text{(17)} \qquad \sup_{a\in S(\mathbf{R}^d)} \sup_{M\le|x|} |T_{e_{i_1},\dots,e_{i_k},a} f(x)| \le \varepsilon, \quad 2\|T_{e_{i_1},\dots,e_{i_k}} f\|_\infty \le M\varepsilon.$$

Then, if $2M \le |x|$ and $y \in \mathbf{R}^d$ with $y \ne x$, we put $a = \frac{y-x}{|y-x|}, h = |y-x|$, then for some $0 < \theta < 1$,

$$\text{(18)} \qquad \frac{|T_{e_{i_1},\dots,e_{i_k}} f(y) - T_{e_{i_1},\dots,e_{i_k}} f(x)|}{|y-x|} \le \begin{cases} |T_{e_{i_1},\dots,e_{i_k},a} f(x+\theta h a)| & \text{if } M > h, \\ 2\|T_{e_{i_1},\dots,e_{i_k}} f\|_\infty / M & \text{if } M \le h. \end{cases}$$

From (17) and (18), we have $\sup_{2M\le|x|} \rho[T_{e_{i_1},\dots,e_{i_k}} f](x) \le \varepsilon$, which implies that $T_{e_{i_1},\dots,e_{i_k}} f \in Lip_1^0(\mathbf{R}^d)$. □

Corollary 1. *The algebra $C_b^n(\mathbf{R}^d)$ is contained in $C_b^{n-1,1}(\mathbf{R}^d)$, and the identity map of $(C_b^n(\mathbf{R}^d), \|\ \|_{C_b^n})$ into $(C_b^{n-1,1}(\mathbf{R}^d), \|\ \|_{C_b^{n-1,1}})$ is an isometry.*

Proof. Obviously, this follows from Lemma 4(i). □

In the following Lemma 5 and Proposition 5, we use the following notations;

$$\alpha_0(f) = \|f\|_\infty, \ \alpha_k(f) = \sup_{a^1,\dots,a^k\in S(\mathbf{R}^d)} \|T_{a^1,\dots,a^k} f\|_\infty, \quad \text{and}$$
$$\beta_0(f) = \rho(f), \quad \beta_k(f) = \sup_{a^1,\dots,a^k\in S(\mathbf{R}^d)} \rho(T_{a^1,\dots,a^k} f) \quad (1 \le k \le n-1).$$

Lemma 5. *Suppose $f, g \in C_b^{n-1,1}(\mathbf{R}^d)$. Then we have*

(i) $\alpha_k(fg) \le \sum_{j=0}^k \binom{k}{j} \alpha_j(f)\alpha_{k-j}(g)$ $(0 \le k \le n-1)$.

(ii) $\beta_{n-1}(fg) \le \sum_{k=0}^{n-1} \binom{n-1}{k} \big(\alpha_k(f)\beta_{n-1-k}(g) + \beta_k(f)\alpha_{n-1-k}(g)\big)$.

Proof. (i) By Lemma 2(6) with $N = 0$, the inequality follows.

(ii) For any choice of $a^1, \dots, a^{n-1} \in S(\mathbf{R}^d)$, we have

$$\text{(19)} \qquad T_{a^1,\dots,a^{n-1}}(fg) = \sum_{k=0}^{n-1} \sum_{(\#)} T_{a^{s_1},\dots,a^{s_k}} f T_{a^{t_1},\dots,a^{t_{n-1-k}}} g,$$

where $(\#) = \Big\{ 1 \le s_1 \le \cdots \le s_k \le n-1, \ 1 \le t_1 \le \cdots \le t_{n-1-k} \le n-1,$
$\{s_1, \dots, s_k, t_1, \dots, t_{n-1-k}\} = \{1, \dots, n-1\} \Big\}.$

Operating ρ to the both sides of the equation (19), we have (cf. (9))

$$\rho(T_{a^1,\dots,a^{n-1}}(fg)) \le \sum_{k=0}^{n-1}\sum_{(\#)}\Big(\|T_{a^{s_1},\dots,a^{s_k}}f\|_\infty \rho(T_{a^{t_1},\dots,a^{t_{n-1-k}}}g) + \rho(T_{a^{s_1},\dots,a^{s_k}}f)\|T_{a^{t_1},\dots,a^{t_{n-1-k}}}g\|_\infty\Big). \tag{20}$$

Taking the supremum in (20) over all choices of $a^1,\dots,a^{n-1} \in S(\mathbf{R}^d)$, we obtain (ii). □

Proposition 5. $(C_b^{n-1,1}(\mathbf{R}^d), \|\ \|_{C_b^{n-1,1}})$ *is a Banach algebra, and* $C_0^{n-1,1}(\mathbf{R}^d)$ *is its closed ideal.*

Proof. In the following calculation, we use the relations $\beta_k(h) = \alpha_{k+1}(h)$ for $h \in C_b^{k+1}(\mathbf{R}^d)$, $k = 0,\dots,n-2$ (cf. Lemma 4(i)). Let $f, g \in C_b^{n-1,1}(\mathbf{R}^d)$. Then by Lemma 5 and the formula $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$, we have

$$\begin{aligned}
\|fg\|_{C_b^{n-1,1}} &= \sum_{k=0}^{n-1}\frac{1}{k!}\alpha_k(fg) + \frac{1}{n!}\beta_{n-1}(fg)\\
&\le \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{1}{k!}\tbinom{k}{j}\alpha_j(f)\alpha_{k-j}(g)\\
&\quad + \frac{1}{n!}\Big(\sum_{k=0}^{n-1}\tbinom{n-1}{k}\alpha_k(f)\beta_{n-1-k}(g) + \tbinom{n-1}{k}\beta_k(f)\alpha_{n-1-k}(g)\Big)\\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{1}{k!}\tbinom{k}{j}\alpha_j(f)\alpha_{k-j}(g) + \frac{1}{n!}\|f\|_\infty\beta_{n-1}(g)\\
&\quad + \frac{1}{n!}\sum_{k=1}^{n-1}\tbinom{n-1}{k}\alpha_k(f)\alpha_{n-k}(g)\\
&\quad + \frac{1}{n!}\sum_{k=0}^{n-2}\tbinom{n-1}{k}\alpha_{k+1}(f)\alpha_{n-1-k}(g) + \frac{1}{n!}\beta_{n-1}(f)\|g\|_\infty\\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{1}{j!}\alpha_j(f)\frac{1}{(k-j)!}\alpha_{k-j}(g) + \frac{1}{n!}\|f\|_\infty\beta_{n-1}(g)\\
&\quad + \frac{1}{n!}\Big(\sum_{k=1}^{n-1}\Big(\tbinom{n-1}{k} + \tbinom{n-1}{k-1}\Big)\alpha_k(f)\alpha_{n-k}(g)\Big)\\
&\quad + \frac{1}{n!}\beta_{n-1}(f)\|g\|_\infty\\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{1}{j!}\alpha_j(f)\frac{1}{(k-j)!}\alpha_{k-j}(g) + \frac{1}{n!}\|f\|_\infty\beta_{n-1}(g)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-1}\frac{1}{k!}\alpha_k(f)\frac{1}{(n-k)!}\alpha_{n-k}(g)+\frac{1}{n!}\beta_{n-1}(f)\|g\|_\infty\\
&\le\left(\sum_{k=0}^{n-1}\frac{1}{k!}\alpha_k(f)+\frac{1}{n!}\beta_{n-1}(f)\right)\left(\sum_{k=0}^{n-1}\frac{1}{k!}\alpha_k(g)+\frac{1}{n!}\beta_{n-1}(g)\right)\\
&=\|f\|_{C_b^{n-1,1}}\|g\|_{C_b^{n-1,1}}.
\end{aligned}
$$

Thus $\|\ \|_{C_b^{n-1,1}}$ is submultiplicative, hence $(C_b^{n-1,1}(\mathbf{R}^d),\|\ \|_{C_b^{n-1,1}})$ is a Banach algebra.

Let $\{f_N\}_N$ be a Cauchy sequence in $C_0^{n-1,1}(\mathbf{R}^d)$. Then there exists $f\in C_b^{n-1,1}(\mathbf{R}^d)$ such that $\lim_{N\to\infty}\|f-f_N\|_{C_b^{n-1,1}}=0$. Since

$$\|f-f_N\|_{C_b^{n-1}}\le\|f-f_N\|_{C_b^{n-1,1}}\to 0\ \ (N\to\infty),$$

we have $f\in C_0^{n-1}(\mathbf{R}^d)$ from Proposition 2. Let $1\le i_1,\dots,i_{n-1}\le d$ and $\varepsilon>0$ be given arbitrarily. Let $N_0\in\mathbf{N}$ be such that $\|f-f_{N_0}\|_{C_b^{n-1,1}}\le\varepsilon/(2n!)$, and choose $M>0$ such that $\sup_{M\le|x|}\rho[T_{e_{i_1},\dots,e_{i_{n-1}}}f_{N_0}](x)\le\varepsilon/2$. Then we have

$$
\begin{aligned}
\sup_{M\le|x|}\rho[T_{e_{i_1},\dots,e_{i_{n-1}}}f](x)&\le\sup_{M\le|x|}\rho[T_{e_{i_1},\dots,e_{i_{n-1}}}f_{N_0}](x)+n!\|f_{N_0}-f\|_{C_b^{n-1,1}}\\
&\le\varepsilon/2+n!(\varepsilon/(2n!))=\varepsilon.
\end{aligned}
$$

Thus $\rho[T_{e_{i_1},\dots,e_{i_{n-1}}}f]$ vanishes at infinity. This implies that $f\in C_0^{n-1,1}(\mathbf{R}^d)$ and hence $C_0^{n-1,1}(\mathbf{R}^d)$ is closed.

Suppose $f\in C_0^{n-1,1}(\mathbf{R}^d)$ and $g\in C_b^{n-1,1}(\mathbf{R}^d)$. By Proposition 2, fg belongs to $C_0^{n-1}(\mathbf{R}^d)$. Let $1\le i_1,\dots,i_{n-1}\le d$ be arbitrarily chosen. Then $T_{e_{i_1},\dots,e_{i_{n-1}}}(fg)$ is a sum of the functions in the following (a) or (b);

(a) $T_{e_{j_1},\dots,e_{j_k}}fT_{e_{j_{k+1}},\dots,e_{j_{n-1}}}g$ with $1\le k\le n-2, 1\le j_1\le\cdots\le j_k\le n-1$, $1\le j_{k+1}\le\cdots\le j_{n-1}\le n-1$ and $\{j_1,\dots,j_{n-1}\}=\{1,\dots,n-1\}$,

(b) $fT_{e_{i_1},\dots,e_{i_{n-1}}}g, (T_{e_{i_1},\dots,e_{i_{n-1}}}f)g$.

In (a), $T_{e_{j_1},\dots,e_{j_k}}f\in Lip_1^0(\mathbf{R}^d)$ and $T_{e_{j+1},\dots,e_{j_{n-1}}}g\in Lip_1(\mathbf{R}^d)$ from Lemma 4, hence functions in (a) belong to $Lip_1^0(\mathbf{R}^d)$ from Lemma 3(iv). That functions in (b) belong to $Lip_1^0(\mathbf{R}^d)$ follows from Definition 4 and Lemma 3(iv).

Therefore $T_{e_{i_1},\dots,e_{i_{n-1}}}(fg)\in Lip_1^0(\mathbf{R}^d)$ for all choices of $1\le i_1,\dots,i_{n-1}\le d$, and hence $fg\in C_0^{n-1,1}(\mathbf{R}^d)$ follows. □

Proposition 6. *$C_0^{n-1,1}(\mathbf{R}^d)$ is a natural Banach function algebra on $\mathbf{R}^d$, and by the identification of $\varphi\in\Phi_{C_0^{n-1,1}(\mathbf{R}^d)}$ with the corresponding $x_\varphi\in\mathbf{R}^d$, $\mathbf{R}^d$ is its Gelfand space and the identity map is the Gelfand transform.*

From this, it follows easily that $C_0^{n-1,1}(\mathbf{R}^d)$ is regular.

Proof. Suppose $f\in C_0^{n-1,1}(\mathbf{R}^d)$ and $\lambda>0$ such that $\lambda-f(x)>0$ for all $x\in\mathbf{R}^d$. Since $f\in C_0^{n-1}(\mathbf{R}^d)$, there exists $\delta>0$ such that $\lambda-f(x)\ge\delta$ for all

$x \in \mathbf{R}^d$. Put $F = \frac{1}{\lambda - f} - \frac{1}{\lambda}$. Then it follows from the proof of Proposition 3 that $F = \frac{f}{\lambda(\lambda - f)} \in C_0^{n-1}(\mathbf{R}^d)$.

We claim that, for any choice of $i_1, \ldots, i_{n-1} \in \{1, \ldots, d\}$, we have $T_{e_{i_1}, \ldots, e_{i_k}} F = \frac{G_k}{(\lambda - f)^{k+1}}$ with $G_k \in C_0^{n-1-k,1}(\mathbf{R}^d)$ for $k = 1, \ldots, n-1$. Since $T_{e_{i_1}} F = T_{e_{i_1}} f \frac{1}{(\lambda - f)^2}$, the claim is true for $k = 1$. If the claim is true for $k (< n-1)$, then it is easy to see by elementary calculation that the claim is true for $k+1$. By induction we can conclude that the claim is true for $k = 1, 2, \ldots, n-1$; in particular

$$T_{e_{i_1}, \ldots, e_{i_{n-1}}} F = \frac{1}{(\lambda - f)^n} G_{n-1}. \tag{21}$$

From (21) we have that

$$T_{e_{i_1}, \ldots, e_{i_{n-1}}} F \in C_0^{0,1}(\mathbf{R}^d) = Lip_1^0(\mathbf{R}^d).$$

Since $C_0^{n-1,1}(\mathbf{R}^d)$ is closed under taking the complex conjugation, we can apply Lemma 1 to conclude that $C_0^{n-1,1}(\mathbf{R}^d)$ is natural. □

Lemma 6. *Suppose $f \in Lip_1^0(\mathbf{R}^d)$, and $\{e_N\}_{N=1}^{\infty}$ is a bounded sequence in $Lip_1(\mathbf{R}^d)$ such that $e_N(x) = 0$ if $|x| \leq N$.*

Then, for given $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that $\rho(fe_N) \leq \varepsilon$ $(N_0 \leq N)$, that is, $\rho(fe_N)$ vanishes as $N \to \infty$.

Proof. Put $\beta = \sup_{N \in \mathbf{N}} (\|e_N\|_\infty + \rho(e_N)) < \infty$. Choose $N_0 \in \mathbf{N}$ such that

$$\max\left\{ \sup_{N_0/2 \leq |x|} |f(x)|, \sup_{N_0/2 \leq |x|} \rho[f](x) \right\} \leq \varepsilon/(2\beta), \tag{22}$$

$$3\beta \|f\|_\infty \leq (N_0/2)\varepsilon. \tag{23}$$

Suppose $N_0 \leq N$. Then, for any $x, y \in \mathbf{R}^d$, $x \neq y$, we have

$$\begin{aligned}
&\frac{|f(y)e_N(y) - f(x)e_N(x)|}{|y - x|} \\
&\leq |e_N(y)| \frac{|f(y) - f(x)|}{|y - x|} + |f(x)| \frac{|e_N(y) - e_N(x)|}{|y - x|} \\
&\leq \begin{cases}
0 \cdot \frac{|f(y) - f(x)|}{|y - x|} + |f(x)| \frac{|0 - 0|}{|y - x|} = 0 & \text{if } |x| < N/2, |y| \leq N, \\
\beta \cdot \frac{2\|f\|_\infty}{N/2} + \|f\|_\infty \frac{\beta}{N/2} \leq \varepsilon & \text{if } |x| < N/2, N < |y| \text{ (from (23))}, \\
0 \cdot \frac{|f(y) - f(x)|}{|y - x|} + |f(x)|\beta \leq \varepsilon/2 & \text{if } N/2 \leq |x|, |y| \leq N \text{ (from (22))}, \\
\beta\rho[f](x) + |f(x)|\beta \leq \varepsilon & \text{if } N/2 \leq |x|, N < |y| \text{ (from (22))}.
\end{cases}
\end{aligned}$$

Therefore $\rho(fe_N) \leq \varepsilon$ if $N_0 \leq N$, that is, $\lim_{N \to \infty} \rho(fe_N) = 0$. □

Theorem 2. *The algebra $(C_0^{n-1,1}(\mathbf{R}^d), \| \|_{C_b^{n-1,1}})$ has a bounded approximate identity composed of elements with compact supports.*

Proof. We will show that the bounded approximate identity $\{u_N^{(d)}\}$ for $C_0^n(\mathbf{R}^d)$, which is defined in Theorem 1, is also valid as a bounded approximate identity for $C_0^{n-1,1}(\mathbf{R}^d)$.

That $\{u_N^{(d)}\}$ is a bounded sequence in $C_0^{n-1,1}(\mathbf{R}^d)$ is clear by Corollary 1. Let $f \in C_0^{n-1,1}(\mathbf{R}^d)$ be given arbitrarily. Then we have

$$
\begin{aligned}
&\|f - fu_N^{(d)}\|_{C_b^{n-1,1}} \\
&= \|f - fu_N^{(d)}\|_{C_b^{n-1}} + \frac{1}{n!} \sup_{\substack{a^1,\ldots,a^{n-1} \\ \in S(\mathbf{R}^d)}} \rho(T_{a^1,\ldots,a^{n-1}}(f(1-u_N^{(d)}))) \\
&\leq \|f - fu_N^{(d)}\|_{C_b^{n-1}} \\
&\quad + \frac{1}{n!} \sup_{\substack{a^1,\ldots,a^{n-1} \\ \in S(\mathbf{R}^d)}} \sum_{1\leq i_1,\ldots,i_{n-1}\leq d} | a^1_{i_1}\cdots a^{n-1}_{i_{n-1}}|\rho(T_{e_{i_1},\ldots,e_{i_{n-1}}}(f(1-u_N^{(d)}))) \\
&\quad \text{where } a^i = (a^i_1,\ldots,a^i_d),\ i = 1,\ldots,n-1, \\
&\leq \|f - fu_N^{(d)}\|_{C_b^{n-1}} + \frac{1}{n!} \sum_{1\leq i_1,\ldots,i_{n-1}\leq d} \rho(T_{e_{i_1},\ldots,e_{i_{n-1}}}(f(1-u_N^{(d)}))) \\
&\leq \|f - fu_N^{(d)}\|_{C_b^{n-1}} \\
&\quad + \frac{1}{n!} \sum_{1\leq i_1,\ldots,i_{n-1}\leq d} \sum_{k=0}^{n-1} \sum_{(\#)} \rho(T_{e_{i_{s_1}},\ldots,e_{i_{s_k}}} f \cdot T_{e_{i_{t_1}},\ldots,e_{i_{t_{n-1-k}}}}(1-u_N^{(d)})),
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
(\#) = \big\{&1 \leq s_1 \leq \cdots \leq s_k \leq n-1,\ 1 \leq t_1 \leq \cdots \leq t_{n-1-k} \leq n-1, \\
&\{s_1,\ldots,s_k,t_1,\ldots,t_{n-1-k}\} = \{1,2,\ldots,n-1\}\big\}.
\end{aligned}
$$

In the last line of (24), for $1 \leq i_1,\ldots,i_{n-1} \leq d,\ 0 \leq k \leq n-1$, we have

$$T_{e_{i_{s_1}},\ldots,e_{i_{s_k}}} f \in Lip_1^0(\mathbf{R}^d)$$

by Lemma 4(ii), and $\{T_{e_{i_{t_1}},\ldots,e_{i_{t_{n-1-k}}}}(1-u_N^{(d)})\}_{N=1}^{\infty}$ is a sequence of bounded functions in $Lip_1(\mathbf{R}^d)$ which satisfy $T_{e_{i_{t_1}},\ldots,e_{i_{t_{n-1-k}}}}(1-u_N^{(d)})(x) = 0$ if $|x| \leq N$, so we have by Lemma 6 that $\rho(T_{e_{i_1},\ldots,e_{i_k}} f \cdot T_{e_{i_{k+1}},\ldots,e_{i_{n-1}}}(1-u_N^{(d)}))$ vanishes as $N \to \infty$. Of course, $\|f - fu_N^{(d)}\|_{C_b^{n-1}}$ vanishes as $N \to \infty$ by Theorem 1. Therefore in the last line of (24) each term vanishes as $N \to \infty$. Then (24) implies that $\|f - fe_N^{(d)}\|_{C_b^{n-1,1}}$ vanishes as $N \to \infty$. □

4. BSE-extension of $C_0^n(\mathbf{R}^d)$, BSE-, BED-properties of $C_0^{n-1,1}(\mathbf{R}^d)$, and the multiplier algebras of $C_0^n(\mathbf{R}^d)$ and $C_0^{n-1,1}(\mathbf{R}^d)$

In this section, we prefer to use the expressions $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$ and $C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d))$ instead of $C_{BSE}(\Phi_{C_0^n(\mathbf{R}^d)})$ and $C_{BSE}(\Phi_{C_0^{n-1,1}(\mathbf{R}^d)})$, respectively, because in either of the two cases the Gelfand space is identified with $\mathbf{R}^d$ by Propositions 3 and 6.

Theorem 3. *The BSE-extension of* $C_0^n(\mathbf{R}^d)$ *is* $C_b^{n-1,1}(\mathbf{R}^d)$; *that is,*

$$C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d) = C_b^{n-1,1}(\mathbf{R}^d).$$

Proof. We show first the inclusion $\subseteq$. Suppose $\sigma \in C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$. We observe that, for any $k(0 \le k \le n)$, any choice of $1 \le i_1, \dots, i_n \le d$ and $x \in \mathbf{R}^d$, the map $C_0^n(\mathbf{R}^d) \to \mathbf{C} : f \mapsto \frac{1}{k!} T_{e_{i_1},\dots,e_{i_k}} f(x)$ is a bounded linear functional which is contained in the unit ball of $C_0^n(\mathbf{R}^d)^*$. By [9, Theorem 4(i)], there exists a bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $C_0^n(\mathbf{R}^d)$ of a bound, say β, such that $\lim_{\lambda\in\Lambda} f_\lambda(x) = \sigma(x)$ for all $x \in \mathbf{R}^d$. By the natural embedding of $C_0^n(\mathbf{R}^d)$ into its second dual, $\{f_\lambda\}_{\lambda\in\Lambda}$ is a net in the β-ball of $C_0^n(\mathbf{R}^d)^{**}$. Since the β-ball of $C_0^n(\mathbf{R}^d)^{**}$ is weak*-compact, there exists a weak*-convergent subnet $\{f_{\lambda'}\}_{\lambda'\in\Lambda'}$ of $\{f_\lambda\}_{\lambda\in\Lambda}$. Hence, for any $k(0 \le k \le n)$ and any choice $1 \le i_1, \dots, i_k \le d$, there exists a bounded function $\tau_{i_1,\dots,i_k}$ on $\mathbf{R}^d$ such that

$$\lim_{\lambda'\in\Lambda'} T_{e_{i_1},\dots,e_{i_k}} f_{\lambda'}(x) = \tau_{i_1,\dots,i_k}(x) \quad (x \in \mathbf{R}^d).$$

We claim that $\sigma \in C_b^{n-1}(\mathbf{R}^d)$, and that

$$T_{e_{i_1},\dots,e_{i_k}}\sigma(x) = \tau_{i_1,\dots,i_k}(x) \qquad (x \in \mathbf{R}^d, 1 \le i_1, \dots, i_k \le d,\ k = 1, \dots, n-1),$$
$$\rho(T_{e_{i_1},\dots,e_{i_{n-1}}}\sigma) < \infty, \qquad 1 \le i_1, \dots, i_{n-1} \le d.$$

To see this, let $x \in \mathbf{R}^d$ and $h \in \mathbf{R} \setminus \{0\}$ be given arbitrarily. Then, by a mean value theorem, we have

$$\begin{aligned}\frac{\sigma(x + he_{i_1}) - \sigma(x)}{h} &= \lim_{\lambda'} \frac{f_{\lambda'}(x + he_{i_1}) - f_{\lambda'}(x)}{h} = \lim_{\lambda'} T_{e_{i_1}} f_{\lambda'}(x + \theta_{\lambda'} he_{i_1}) \\ &= \lim_{\lambda'} \left[T_{e_{i_1}} f_{\lambda'}(x) + \frac{T_{e_{i_1}} f_{\lambda'}(x + \theta_{\lambda'} he_{i_1}) - T_{e_{i_1}} f_{\lambda'}(x)}{\theta_{\lambda'} h} \theta_{\lambda'} h \right] \\ &= \tau_{i_1}(x) + \lim_{\lambda'} T_{e_{i_1},e_{i_1}} f_{\lambda'}(x + \tilde{\theta}_{\lambda'}\theta_{\lambda'} he_{i_1})\theta_{\lambda'} h \\ &\quad (0 < \theta_{\lambda'}, \tilde{\theta}_{\lambda'} < 1).\end{aligned}$$

Hence

$$\left| \frac{\sigma(x + e_{i_1}h) - \sigma(x)}{h} - \tau_{i_1}(x) \right| \le \sup_{\lambda'} \left\| T_{e_{i_1},e_{i_1}} f_{\lambda'} \right\|_\infty |\theta_{\lambda'} h| \to 0 (h \to 0).$$

This implies that σ is partially differentiable with respect to x_{i_1} and that $T_{e_{i_1}}\sigma = \tau_{i_1}$. We can repeat this procedure, with respect to $x_{i_2}, \dots, x_{i_{n-1}}$

successively, to obtain

$$T_{e_{i_1},\dots,e_{i_k}}\sigma = \tau_{i_1,\dots,i_k} \ (1 \le k \le n-1).$$

Further, we must show that $\rho(T_{e_{i_1},\dots,e_{i_{n-1}}}\sigma) < \infty$. (This will also make sure that $T_{e_{i_1},\dots,e_{i_{n-1}}}\sigma \in C_b(\mathbf{R}^d)$.) For any $x, y \in \mathbf{R}^d$, $x \ne y$, we have

$$\begin{aligned}
&\left|\frac{T_{e_{i_1},\dots,e_{i_{n-1}}}\sigma(x) - T_{e_{i_1},\dots,e_{i_{n-1}}}\sigma(y)}{x-y}\right| \\
&= \left|\frac{\tau_{i_1,\dots,i_{n-1}}(x) - \tau_{i_1,\dots,i_{n-1}}(y)}{x-y}\right| \\
&= \lim_{\lambda'\in\Lambda'}\left|\frac{T_{e_{i_1},\dots,e_{i_{n-1}}}f_{\lambda'}(x) - T_{e_{i_1},\dots,e_{i_{n-1}}}f_{\lambda'}(y)}{x-y}\right| \\
&\le \sup_{\lambda'\in\Lambda'} \rho\big(T_{e_{i_1},\dots,e_{i_{n-1}}}f_{\lambda'}\big) \le \sup_{\lambda'\in\Lambda'} n!\|f_{\lambda'}\|_{C_b^n} \le n!\beta < \infty.
\end{aligned}$$

Thus we obtain $\sigma \in C_b^{n-1,1}(\mathbf{R}^d)$, that is, $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d) \subseteq C_b^{n-1,1}(\mathbf{R}^d)$.

Next, we show the reverse inclusion $\supseteq$. Let $\sigma \in C_b^{n-1,1}(\mathbf{R}^d)$ be given arbitrarily. Choose a nonnegative function $v \in C_0^n(\mathbf{R}^d)$ such that $\mathrm{supp}(v) \subseteq B_1(= \{x \in \mathbf{R}^d : |x| \le 1\})$ and $\int_{B_1} v(x)dx = 1$. Set $v_\ell(x) = \ell^d v(\ell x)$ $(x \in \mathbf{R}^d)$, and put $\sigma_\ell = \sigma * v_\ell$, $\ell = 1, 2, 3, \dots$. Obviously, $\{\sigma_\ell\}$ is a sequence of n-times continuously differentiable functions on $\mathbf{R}^d$ which converges pointwisely to σ. We will show that $\{\sigma_\ell\}_\ell$ is a bounded sequence in $C_b^n(\mathbf{R}^d)$. Let $a^1, \dots, a^{n-1} \in S(\mathbf{R}^d)$ be given arbitrarily. Then

$$|\sigma_\ell(x)| \le \int_{\mathbf{R}^d} |\sigma(x-y)|v_\ell(y)dy \le \|\sigma\|_\infty \int_{\mathbf{R}^d} v_\ell(y)dy = \|\sigma\|_\infty \quad (x \in \mathbf{R}^d), \tag{25}$$

and

$$\big|T_{a^1,\dots,a^k}\sigma_\ell(x)\big| = \left|\int_{\mathbf{R}^d} T_{a^1,\dots,a^k}\sigma(x-y)v_\ell(y)dy\right| \le \|T_{a^1,\dots,a^k}\sigma\|_\infty \tag{26}$$

for $k = 1, \dots, n-1$. Also we have

$$\begin{aligned}
&\rho(T_{a^1,\dots,a^{n-1}}\sigma_\ell)(x) \\
&= \sup_{x,y\in\mathbf{R}^d, x\ne y} \frac{1}{|x-y|}\left|\int_{\mathbf{R}^d} T_{a^1,\dots,a^{n-1}}\sigma(x-z)v_\ell(z)dz \right.\\
&\qquad \left. - \int_{\mathbf{R}^d} T_{a^1,\dots,a^{n-1}}\sigma(y-z)v_\ell(z)dz\right| \\
&\le \int_{\mathbf{R}^d} \sup_{x,y\in\mathbf{R}^d, x\ne y}\left|\frac{T_{a^1,\dots,a^{n-1}}\sigma(x-z) - T_{a^1,\dots,a^{n-1}}\sigma(y-z)}{x-y}\right| v_\ell(z)dz \\
&\le \rho\big(T_{a^1,\dots,a^{n-1}}\sigma\big) \le n!\|\sigma\|_{C_b^{n-1,1}} < \infty.
\end{aligned} \tag{27}$$

By (25), (26), (27), and the properties of functions $\{u_\ell^{(d)}\}_{\ell=1}^\infty$ constructed in Theorem 1, it follows that $\{u_\ell^{(d)}\sigma_\ell\}_{\ell=1}^\infty$ is a bounded sequence of functions in $C_0^n(\mathbf{R}^d)$ which converges pointwisely to σ. So $\sigma \in C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$ by [9, Theorem 4(i)]. □

Theorem 4. (i) $C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d) = C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$.

(ii) $C_0^{n-1,1}(\mathbf{R}^d)$ *is a BSE-algebra, that is,*

$$C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d) = M(C_0^{n-1,1}(\mathbf{R}^d)).$$

(iii) $C_0^{n-1,1}(\mathbf{R}^d)$ *is a BED-algebra, that is,*

$$C^0_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d) = C_0^{n-1,1}(\mathbf{R}^d).$$

Proof. (i) We prove first the inclusion $\subseteq$. Suppose $\sigma \in C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d)$. By [9, Theorem 4(i)], there exist $\beta < \infty$ and a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $C_0^{n-1,1}(\mathbf{R}^d)$ which satisfy $\|f_\lambda\|_{C_b^{n-1,1}} \leq \beta$ $(\lambda \in \Lambda)$, and $\lim_\lambda f_\lambda(x) = \sigma(x)$ $(x \in \mathbf{R}^d)$. By Theorem 3, there exists a constant γ such that

$$\|f\|_{BSE(C_0^n(\mathbf{R}^d))} \leq \gamma\|f\|_{C_b^{n-1,1}} \quad (f \in C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)). \tag{28}$$

Here, we denote by Ω the directed set of all finite subsets of $\mathbf{R}^d$ with inclusion order, and $\Lambda \times \Omega$ is the directed set with the order: $(\lambda_1, \omega_1) \leq (\lambda_2, \omega_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $\omega_1 \leq \omega_2$.

We claim here that, for each $(\lambda, \omega) \in \Lambda \times \Omega$, we can choose $f_{\lambda,\omega} \in C_0^n(\mathbf{R}^d)$ which satisfies (a) $f_\lambda(x) = f_{\lambda,\omega}(x)$ $(x \in \omega)$, and (b) $\|f_{\lambda,\omega}\|_{C_b^n} \leq \gamma\beta + 1$. To show this, we observe that each element f_λ is a BSE-function of $C_0^n(\mathbf{R}^d)$, and by (28), that $\|f_\lambda\|_{BSE(C_0^n(\mathbf{R}^d))} \leq \gamma\|f_\lambda\|_{C_b^{n-1,1}} \leq \gamma\beta$. Hence by Helly's theorem we can choose $f_{\lambda,\omega} \in C_0^n(\mathbf{R}^d)$ which satisfies (a) and (b).

We assert that $\{f_{\lambda,\omega}\}_{(\lambda,\omega)\in\Lambda\times\Omega}$ is a bounded net in $C_0^n(\mathbf{R}^d)$ which converges pointwisely to σ. Indeed, take $x \in \mathbf{R}^d$ arbitrarily and put $\omega_0 = \{x\} \in \Omega$. Let $\varepsilon > 0$ be given arbitrarily. Then we can choose $\lambda_0 \in \Lambda$ such that $|f_\lambda(x) - \sigma(x)| \leq \varepsilon$ $(\lambda_0 \leq \lambda)$. Then we have

$$|f_{\lambda,\omega}(x) - \sigma(x)| = |f_\lambda(x) - \sigma(x)| \leq \varepsilon \quad ((\lambda, \omega) \geq (\lambda_0, \omega_0)).$$

This implies that $\sigma \in C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$. Hence $C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d)$ is contained in $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$. The reverse inclusion "$\supseteq$" is easily proved by Corollary 1 and [9, Theorem 4], so we obtain (i).

(ii) By Proposition 5, we have

$$C_b^{n-1,1}(\mathbf{R}^d) \subseteq M(C_0^{n-1,1}(\mathbf{R}^d)). \tag{29}$$

On the other hand, since $C_0^{n-1,1}(\mathbf{R}^d)$ has a bounded approximate identity by Theorem 2, it follows from [9, Corollary 5] that

$$M(C_0^{n-1,1}(\mathbf{R}^d)) \subseteq C_{BSE(C_0^{n-1,1}(\mathbf{R}^d))}(\mathbf{R}^d). \tag{30}$$

From (29), (30), (i), and Theorem 3, we get the desired result.

(iii) Since $C_0^{n-1,1}(\mathbf{R}^d)$ is regular and has a bounded approximate identity composed of elements with compact supports from Proposition 6 and Theorem 2, (iii) follows from [4, Theorem 4.7]. □

Remark 1. By Theorems 1 and 2, $C_0^n(\mathbf{R}^d)$ and $C_0^{n-1,1}(\mathbf{R}^d)$ are in the class of commutative Banach algebras B with the properties (α_B) and (β_B) in [5, p. 539] (see also [5, p. 543, Examples 3.3]). Hence we can define and investigate Segal algebras in $C_0^n(\mathbf{R}^d)$ and $C_0^{n-1,1}(\mathbf{R}^d)$.

Theorem 5. (i) $M(C_0^n(\mathbf{R}^d)) = C_b^n(\mathbf{R}^d)$.
(ii) *The algebra* $C_0^n(\mathbf{R}^d)$ *is neither of BSE nor of BED.*

Proof. (i) The inclusion $\supseteq$ follows from Proposition 2. To prove the reverse inclusion, let $f \in M(C_0^n(\mathbf{R}^d))$ be given. For each $x \in \mathbf{R}^d$ and a compact neighborhood U_x of x, there exists $u_x \in C_0^n(\mathbf{R}^d)$ such that $u_x = 1$ on U_x. Then $fu_x \in C_0^n(\mathbf{R}^d)$ and $f = fu_x$ on U_x. Therefore f is n-times continuously differentiable. Since $C_0^n(\mathbf{R}^d)$ has a bounded approximate identity by Theorem 1, $M(C_0^n(\mathbf{R}^d))$ is contained in $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d)$ by [9, Corollary 5]. Also $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d) = C_b^{n-1,1}(\mathbf{R}^d)$ by Theorem 3, and hence $f \in C_b^{n-1,1}(\mathbf{R}^d)$. Then, for any $1 \le i_1, \ldots, i_n \le d$, we have

$$\begin{aligned}\|T_{e_{i_1},\ldots,e_{i_n}} f\|_\infty &= \sup_{x\in\mathbf{R}^d} \lim_{h\in\mathbf{R}, h\to 0} \frac{|T_{e_{i_1},\ldots,e_{i_{n-1}}} f(he_{i_n}+x) - T_{e_{i_1},\ldots,e_{i_{n-1}}} f(x)|}{|h|} \\ &\le \sup_{x\in\mathbf{R}^d} \rho[T_{e_{i_1},\ldots,e_{i_{n-1}}} f](x) = \rho(T_{e_{i_1},\ldots,e_{i_{n-1}}} f) < \infty,\end{aligned}$$

and hence $f \in C_b^n(\mathbf{R}^d)$.

(ii) $C_0^n(\mathbf{R}^d)$ is not of BSE since $C_{BSE(C_0^n(\mathbf{R}^d))}(\mathbf{R}^d) = C_b^{n-1,1}(\mathbf{R}^d) \neq C_b^n(\mathbf{R}^d) = M(C_0^n(\mathbf{R}^d))$ by Theorem 3 and (i). Then, since $C_0^n(\mathbf{R}^d)$ is regular and has a bounded approximate identity composed of elements with compact supports, we can apply [4, Theorem 4.7] to conclude that $C_0^n(\mathbf{R}^d)$ is not of BED. □

5. $C_0^{n-1,1}(\mathbf{R}^d)$ as Birtel's commutative extension of $C_0^n(\mathbf{R}^d)$

Birtel [1] introduced the notion of commutative extension of commutative semisimple Banach algebras:

Definition 5 ([1])**.** Suppose that A is a commutative semisimple Banach algebra. Denote by A' the norm closed subspace of A^* generated by Φ_A, and A'^* the Banach space dual of A'. Arens type products $A \times A' \to A' : (f,p) \mapsto f \cdot p$; $A' \times A'^* \to A' : (p, F) \mapsto p \cdot F$; $A'^* \times A'^* \to A'^* : (F, G) \mapsto F \cdot G$; are defined by

(i) $\langle f \cdot p, g\rangle = \langle fg, p\rangle = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\varphi(f)\varphi(g)$

$$(f, g \in A, p = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\varphi \in \mathrm{span}(\Phi_A));$$

(ii) $\langle f, p \cdot F\rangle = \langle f \cdot p, F\rangle = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)F(\varphi)\varphi(f)$

$$(f \in A, p = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\varphi \in \operatorname{span}(\Phi_A)), F \in A'^*);$$

(iii) $\langle p, F \cdot G\rangle = \langle p \cdot F, G\rangle = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)F(\varphi)G(\varphi)$

$$(p = \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\varphi \in \operatorname{span}(\Phi_A), F, G \in A'^*).$$

Since $\operatorname{span}(\Phi_A)$ is dense in A', above (i), (ii), and (iii) are enough to define products.

Birtel showed that A'^* is a commutative Banach algebra with respect to the Arens type product and that the natural embedding of A into A'^* is a continuous isomorphism, and called A'^* the commutative extension of A.

Definition 6. Let $D_{BSE}(\Phi_A)$ be the space of bounded complex-valued functions σ on Φ_A which satisfy BSE-condition with respect to A with norm

$$\|\sigma\|_{BSE} := \sup_{p\in\operatorname{span}(\Phi_A), \|p\|_{A^*}\le 1} \left| \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\sigma(\varphi) \right| < \infty. \tag{31}$$

Using Helly's theorem, we can prove easily that a bounded function σ on Φ_A belongs to $D_{BSE}(\Phi_A)$ if and only if there exists a bounded net in A converging pointwisely to σ, (cf. the proof of Theorem 4(i) of [9]).

If we consider $F \in A'^*$ as a function on A' defined by $F(\zeta) = \langle F, \zeta\rangle$ $(\zeta \in A')$, $\pi(F) := F|\Phi_A$ is a bounded function on Φ_A with a bound $\|F\|_{A'^*}$.

Lemma 7. *For each $F \in A'^*$, we have $\pi(F) \in D_{BSE}(\Phi_A)$, and $\pi : A'^* \to D_{BSE}(\Phi_A) : F \mapsto F|\Phi_A$ is a surjective isometric isomorphism. Hence we can identify A'^* with $D_{BSE}(\Phi_A)$ through this representation, that is, $A'^* = D_{BSE}(\Phi_A)$.*

Proof. Let $F \in A'^*$ be given arbitrarily. Then we have

$$\begin{aligned}\|\pi(F)\|_{BSE} &= \sup_{p\in\operatorname{span}(\Phi_A), \|p\|_{A^*}\le 1} \left| \sum_{\varphi\in\Phi_A} \hat{p}(\varphi)\sigma(\varphi) \right| \\ &= \sup_{p\in\operatorname{span}(\Phi_A), \|p\|_{A^*}\le 1} |\langle p, F\rangle| = \|F\|_{A'^*}.\end{aligned}$$

Therefore π is an isometric map from A'^* to $D_{BSE}(\Phi_A)$. Also if $\sigma \in D_{BSE}(\Phi_A)$, we see from (31) that σ corresponds to an element of A'^*, which implies that π is surjective.

By (iii) of Definition 5, Arens type product in $D_{BSE(A)}(\Phi_A)$ is equal to pointwise multiplication on Φ_A. This proves that π is a homomorphism. □

We can see by Lemma 7 that $D_{BSE}(\Phi_A)$ is a representation of A'^* as a Banach function algebra on Φ_A. Note that $C_{BSE}(\Phi_A)$ is the set of complex-valued continuous functions σ with $\|\sigma\|_{BSE} < \infty$. In general, $D_{BSE}(\Phi_A)$ is not equal to $C_{BSE}(\Phi_A)$. For example, in the case where $A = L^1(\mathbf{R})$ with $\Phi_A = \mathbf{R}, C_{BSE}(\Phi_A)$ is the set of all the Fourier-Stieltjes transforms of elements in $M(\mathbf{R})$. On the other hand, $D_{BSE}(\Phi_A)$ is the set of Fourier-Stieltjes transforms of elements in $M(\overline{\mathbf{R}})$, where $\overline{\mathbf{R}}$ is the Bohr compactification of $\mathbf{R}$, and they are not equal. But in our case where $A = C_0^n(\mathbf{R}^d)$, we have the following result.

Theorem 6. $D_{BSE}(\Phi_{C_0^n(\mathbf{R}^d)}) = C_{BSE}(\Phi_{C_0^n(\mathbf{R}^d)})$.

Proof. Suppose $\sigma \in D_{BSE}(\Phi_{C_0^n(\mathbf{R}^d)})$. Since σ is a bounded function on $\mathbf{R}^d$ which satisfies the BSE-condition with respect to $C_0^n(\mathbf{R}^d)$, by Helly's theorem, there is a bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $C_0^n(\mathbf{R}^d)$ (with a bound β) converging pointwisely to σ on $\mathbf{R}^d$. Let $x, y \in \mathbf{R}^d$, $x \neq y$. We put $a = \frac{x-y}{|x-y|}, h = |x - y|$. Then we have

$$\begin{aligned}\frac{|\sigma(x) - \sigma(y)|}{|x-y|} &= \lim_{\lambda\in\Lambda} \frac{|f_\lambda(x) - f_\lambda(y)|}{|x-y|} \\ &= \lim_{\lambda\in\Lambda} |T_a f_\lambda(y + \theta_{x,y} ha)|, \quad (\text{with } 0 < \theta_{x,y} < 1) \\ &\le \sup_{\lambda\in\Lambda} \|T_a f_\lambda\|_\infty \le \beta.\end{aligned}$$

This implies that $\sigma \in C_b(\mathbf{R}^d)$ and hence $\sigma \in C_{BSE}(\Phi_{C_0^n(\mathbf{R}^d)})$. The reverse inclusion is obvious. □

Corollary 2. *$C_b^{n-1,1}(\mathbf{R}^d) = C_0^n(\mathbf{R}^d)'^*$, that is, $C_b^{n-1,1}(\mathbf{R}^d)$ has a predual.*

Proof. The proof follows by an obvious combination of Lemma 7, Theorems 3 and 6. □

Remark 2. A Banach algebra of n-times continuously differentiable functions on $[0, 1]$ are treated in [8, p. 300] (see also, [2,3,6,7]). But as far as the authors know, there are no articles in which $C_0^n(\mathbf{R}^d)$ or $C_0^{n-1,1}(\mathbf{R}^d)$ is investigated as a Banach algebra.

Acknowledgements. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University. Authors are grateful to the referee for the precise evaluation to the paper, and for giving helpful suggestions to improve the paper.

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