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MULTIPLICITY RESULTS OF POSITIVE SOLUTIONS FOR SINGULAR GENERALIZED LAPLACIAN SYSTEMS

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Abstract. We study the homogeneous Dirichlet boundary value problem of generalized Laplacian systems with a singular weight which may not be in L^1. Using the well-known fixed point theorem on cones, we obtain the multiplicity results of positive solutions under two different asymptotic behaviors of the nonlinearities at 0 and ∞. Furthermore, a global result of positive solutions for one special case with respect to a parameter is also obtained.

1. Introduction

In this paper, we study the following nonlinear differential system

$$(P_\lambda)\qquad \begin{cases} -\mathbf{\Phi}(\mathbf{u}')' = \lambda \mathbf{h}(t)\cdot \mathbf{f}(\mathbf{u}), \quad t\in(0,1),\\ \mathbf{u}(0)=0=\mathbf{u}(1),\end{cases}$$

where $\mathbf{\Phi}(\mathbf{u}') = (\varphi(u_1'),\ldots,\varphi(u_N'))$ with $\varphi:\mathbb{R}\to\mathbb{R}$ an odd increasing homeomorphism, $\lambda>0$ a parameter, $\mathbf{h}(t)=(h_1(t),\ldots,h_N(t))$ with $h_i:(0,1)\to\mathbb{R}_+$ continuous, $h_i\not\equiv 0$ on any subinterval in $(0,1)$ and $\mathbf{f}(\mathbf{u})=(f^1(\mathbf{u}),\ldots,f^N(\mathbf{u}))$ with $f^i:\mathbb{R}_+^N\to\mathbb{R}_+$, here we denote $\mathbb{R}_+=[0,+\infty)$, $\mathbb{R}_+^N=\underbrace{\mathbb{R}_+\times\cdots\times\mathbb{R}_+}_{N}$ and $\mathbf{x}\cdot\mathbf{y}=(x_1y_1,x_2y_2,\ldots,x_Ny_N)$ the Hadamard product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$. Thus problem (P_λ) can be rewritten as

$$\begin{cases} -\varphi(u_1')' = \lambda h_1(t) f^1(\mathbf{u}),\\ \qquad\vdots\\ -\varphi(u_N')' = \lambda h_N(t) f^N(\mathbf{u}), \quad t\in(0,1),\\ u_i(0)=0=u_i(1), \quad i=1,\ldots,N.\end{cases}$$

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The generalized Laplacian problems like (P_λ) appear in various applications which describe reaction-diffusion systems, nonlinear elasticity, glaciology, population biology, combustion theory, and non-Newtonian fluids (see [8, 10, 11, 16]). They also have received growing attention in connection with positive radial solutions of elliptic problems in both annular and exterior domains (see [9, 21] and the references therein).

In recent years, existence and multiplicity of positive solutions of these problems have been extensively studied under various assumptions on the weight functions and nonlinearities (see [1–6], [9], [12], [14, 16–23]). For example, Wang [20] obtained the criteria of determining the number of positive solutions of problem (P_λ) with respect to the parameter λ when each $h_i : [0,1] \to \mathbb{R}_+$ is continuous and φ satisfies that there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(\sigma)\varphi(x) \leq \varphi(\sigma x) \leq \psi_2(\sigma)\varphi(x) \quad \text{for } \sigma,\ x > 0.$$

In this paper, we give assumptions on φ, $\mathbf{h}$ and $\mathbf{f}$ as follows.

(A) There exist an increasing homeomorphism ψ of $(0,\infty)$ onto $(0,\infty)$ and a function γ of $(0,\infty)$ into $(0,\infty)$ such that

$$\psi(\sigma) \leq \frac{\varphi(\sigma x)}{\varphi(x)} \leq \gamma(\sigma) \text{ for all } \sigma > 0,\ x \in \mathbb{R}/\{0\}.$$

(H) $h_i : (0,1) \to \mathbb{R}_+$ is a continuous function satisfying

$$\int_0^{\frac{1}{2}} \psi^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \psi^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)d\tau\right) ds < \infty,$$

for $i = 1, \ldots, N$.

(F_1) $f^i : \mathbb{R}_+^N \to \mathbb{R}_+$ is continuous for $i = 1, \ldots, N$.

(F_2) $f^i(\mathbf{u}) > 0$ for $\mathbf{u} \in \mathbb{R}_+^N$ with $\|\mathbf{u}\| > 0$, $i = 1, \ldots, N$.

(F_3) $f^i(u_1, \ldots, u_N) \leq f^i(v_1, \ldots, v_N)$, whenever $u_i = v_i$, $u_j \leq v_j$, $i \neq j$.

Note that φ covers the case of p-Laplace operator, namely $\varphi(x) = \varphi_p(x) := |x|^{p-2}x$, $x \in \mathbb{R}$, $p > 1$. Clearly, φ_p satisfies condition (A) with $\varphi_p \equiv \psi \equiv \gamma$. Specially, conditions (A), (H) on φ and h_i were introduced first by Xu and Lee [22] and more general than the ones given by Wang [20]. For convenience, we introduce a new class of weight functions. For a bijection $\iota : \mathbb{R} \to \mathbb{R}$, define $\mathcal{H}_\iota$ a subset of $C((0,1), \mathbb{R}_+)$ given by

$$\mathcal{H}_\iota = \left\{ g \in C((0,1), \mathbb{R}_+) \;\Big|\; \int_0^{\frac{1}{2}} \iota^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \iota^{-1}\left(\int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds < \infty \right\}.$$

By the notation, condition (H) means $h_i \in \mathcal{H}_\psi$.

Now we introduce some notations for the statement of the main theorem. Denote

$$\mathbf{f}_0 := \sum_{i=1}^N f_0^i, \quad \mathbf{f}_\infty := \sum_{i=1}^N f_\infty^i,$$

where

$$f_0^i := \lim_{\|\mathbf{u}\| \to 0} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \quad f_\infty^i := \lim_{\|\mathbf{u}\| \to \infty} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}$$

for $\mathbf{u} \in \mathbb{R}_+^N$ and $i = 1, \ldots, N$. For simplicity, we denote $\|\mathbf{u}\| = \sum_{i=1}^N |u_i|$ for $\mathbf{u} \in \mathbb{R}_+^N$ in this paper.

When $N = 1$, $\varphi = \varphi_p$, Agarwal-Lü-O'Regan [1] and Sánchez [18] proved the multiplicity of positive solutions of problem (P_λ) for λ belonging to some open interval if either $\mathbf{f}_0 = \mathbf{f}_\infty = 0$ or $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$. Later, Wang [20] extended the multiplicity results in [1, 18] to φ-Laplacian system with each $h_i \in C[0, 1]$. Recently, Xu and Lee [23] derived some explicit intervals for λ such that singular φ-Laplacian system (P_λ) has at least one positive solution if $0 < \mathbf{f}_0, \mathbf{f}_\infty < \infty$.

Our aim is to extend the multiplicity results of Wang [20] to singular φ-Laplacian system (P_λ) for the cases $\mathbf{f}_0 = \mathbf{f}_\infty = 0$ and $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$. Further, under the monotone-type assumption (F_3), we firstly obtain a global result of positive solutions for problem (P_λ) with respect to λ for the case $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$. More precisely, main results can be stated as follows.

Theorem 1.1. *Assume that* (A), (H), (F_1), *and* (F_2) *hold.*

(1) *If* $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, *then there exist* $\bar{\lambda} > \underline{\lambda} > 0$ *such that* (P_λ) *has at least two positive solutions for* $\lambda > \bar{\lambda}$, *and no positive solution for* $\lambda \in (0, \underline{\lambda})$, *where* $\bar{\lambda}$, $\underline{\lambda}$ *are given by* (3.2) *and* (3.9), *respectively.*

(2) *If* $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, *then there exist* $\bar{\lambda} > \underline{\lambda} > 0$ *such that* (P_λ) *has at least two positive solutions for* $\lambda \in (0, \underline{\lambda})$, *and no positive solution for* $\lambda > \bar{\lambda}$, *where* $\underline{\lambda}$, $\bar{\lambda}$ *are given by* (3.11) *and* (3.21), *respectively.*

Theorem 1.2. *Assume that* (A), (H), (F_1), (F_2) *and* (F_3) *hold. If* $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, *then there exist* $\lambda^* \geq \lambda_* > 0$ *such that* (P_λ) *has at least two positive solutions for* $\lambda \in (0, \lambda_*)$, *one positive solution for* $\lambda \in [\lambda_*, \lambda^*]$, *and no positive solution for* $\lambda > \lambda^*$, *where* λ^*, λ_* *are given by* (3.28) *and* (3.29), *respectively.*

Remark 1.3. If $f^{i_0}(\mathbf{0}) > 0$ for some $i_0 \in \{1, \ldots, N\}$, then we can get $\lambda_* = \lambda^*$ in Theorem 1.2. The proof can be easily completed by the similar arguments in [15].

Remark 1.4. Quasi-monotone condition (F_3) is redundant in one dimensional case so that Theorem 1.2 is valid for scalar φ-Laplacian problem without any monotonicity condition on f.

Remark 1.5. Under the same assumptions in Theorem 1.2, we expect a similar result for the case $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, but the analysis can not follow in a similar way.

As a benefit of a constructive technique used in this paper, we note that $\bar{\lambda}$, $\underline{\lambda}$ appeared in Theorem 1.1 can be computed explicitly (see examples in Section 4). For the proofs, we employ a newly developed solution operator introduced by Xu and Lee [22] and then we apply the fixed point theorem on cones for our main results.

Our paper is organized as follows. In Section 2, we establish a solution operator for problem (P_λ) and introduce some preliminary facts. In Section 3, we prove the main theorems and in Section 4, we give some examples.

2. Preliminaries

Main condition of weight function h_i in problem (P_λ) is of $\mathcal{H}_\psi$-class which includes singular functions specially on the boundary, i.e., h_i may not be integrable near the boundary, $t = 0$ and/or $t = 1$. In this case, solutions need not be in $C^1[0,1]$. So by a solution to problem (P_λ), we understand a function $\mathbf{u} \in C_0([0,1],\mathbb{R}^N) \cap C^1((0,1),\mathbb{R}^N)$ with $\Phi(\mathbf{u}')$ absolutely continuous which satisfies problem (P_λ).

Basic tool for proving our main results is the following well-known fixed point theorem ([7,13]).

Theorem 2.1. *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is completely continuous such that either*
$\|T\mathbf{u}\| \le \|\mathbf{u}\|$ for $\mathbf{u} \in K \cap \partial\Omega_1$ and $\|T\mathbf{u}\| \ge \|\mathbf{u}\|$ for $\mathbf{u} \in K \cap \partial\Omega_2$, or
$\|T\mathbf{u}\| \ge \|\mathbf{u}\|$ for $\mathbf{u} \in K \cap \partial\Omega_1$ and $\|T\mathbf{u}\| \le \|\mathbf{u}\|$ for $\mathbf{u} \in K \cap \partial\Omega_2$.
Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To set up the solution operator for (P_λ), let us define E the Banach space $\underbrace{C_0[0,1] \times \cdots \times C_0[0,1]}_{N}$ with norm $\|\mathbf{u}\|_\infty = \Sigma_{i=1}^N \|u_i\|_\infty$ and define a cone K by taking $K = \{\mathbf{u} \in E \mid u_i \text{ is concave on } [0,1],\ i = 1,\dots,N\}$.

Let us consider a simple scalar problem of the form

$$(W) \qquad -\varphi(w')' = g(t), \quad t \in (0,1),$$

$$(D) \qquad w(0) = w(1) = 0,$$

where φ satisfies (A) and $g \in \mathcal{H}_\varphi$. Note from condition (A) that $\mathcal{H}_\psi \subset \mathcal{H}_\varphi$ (see Remark 2.3). Let w be a solution of (W)+(D). Then integrating both sides of (W) on the interval $[s, \frac{1}{2}]$ for $s \in (0, \frac{1}{2}]$ and $[\frac{1}{2}, s]$ for $s \in [\frac{1}{2}, 1)$, respectively, we find that (W)+(D) is equivalent to

$$(2.1) \qquad \begin{cases} w'(s) = \varphi^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)d\tau\right), \ w(0) = 0, \quad s \in (0, \frac{1}{2}], \\ w'(s) = \varphi^{-1}\left(a - \int_{\frac{1}{2}}^s g(\tau)d\tau\right), \ w(1) = 0, \quad s \in [\frac{1}{2}, 1), \end{cases}$$

where $a = \varphi(w'(\frac{1}{2}))$. Showing the fact $\varphi^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)d\tau\right) \in L^1(0, \frac{1}{2}]$ is not obvious since g can not be in $L^1(0, \frac{1}{2}]$. One may refer to Xu and Lee [22] for the proof. Now we may integrate both sides of (2.1) on the interval $[0,t]$ for $t \in [0, \frac{1}{2}]$ and on the interval $[t, 1]$ for $t \in [\frac{1}{2}, 1]$, respectively. And we get

$$w(t) = \begin{cases} \int_0^t \varphi^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds, & t \in [0, \frac{1}{2}], \\ \int_t^1 \varphi^{-1}\left(-a + \int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds, & t \in [\frac{1}{2}, 1]. \end{cases}$$

To check $w(\frac{1}{2}^-) = w(\frac{1}{2}^+)$, define for $a \in \mathbb{R}$,

$$G(a) = \int_0^{\frac{1}{2}} \varphi^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds - \int_{\frac{1}{2}}^1 \varphi^{-1}\left(-a + \int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds. \tag{2.2}$$

Then the function $G : \mathbb{R} \to \mathbb{R}$ is well-defined and has a unique zero $a = a(g)$ in $\mathbb{R}$ (See Xu and Lee [22] for the proof). This implies $w(\frac{1}{2}^-) = w(\frac{1}{2}^+)$. Consequently, if φ satisfies (A) and $g \in \mathcal{H}_\varphi$, then the solution w of (W)+(D) can be represented by

$$w(t) = \begin{cases} \int_0^t \varphi^{-1}\left(a(g) + \int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds, & t \in [0, \frac{1}{2}], \\ \int_t^1 \varphi^{-1}\left(-a(g) + \int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds, & t \in [\frac{1}{2}, 1], \end{cases} \tag{2.3}$$

where $a(g) \in \mathbb{R}$ uniquely satisfies

$$\int_0^{\frac{1}{2}} \varphi^{-1}\left(a(g) + \int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds = \int_{\frac{1}{2}}^1 \varphi^{-1}\left(-a(g) + \int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds.$$

Replacing $g(t)$ with $\lambda h_i(t) f^i(\mathbf{u}(t))$ in (W)+(D), we may define

$$T_\lambda(\mathbf{u}) = \left(T_\lambda^1(\mathbf{u}), \ldots, T_\lambda^N(\mathbf{u})\right)$$

for $\lambda > 0$, $\mathbf{u} \in K$ and for $i = 1, \ldots, N$, given by

$$T_\lambda^i(\mathbf{u})(t) = \begin{cases} \int_0^t \varphi^{-1}\left(a^i(\lambda h_i f^i(\mathbf{u})) + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds, & t \in [0, \frac{1}{2}], \\ \int_t^1 \varphi^{-1}\left(-a^i(\lambda h_i f^i(\mathbf{u})) + \int_{\frac{1}{2}}^s \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds, & t \in [\frac{1}{2}, 1], \end{cases}$$

where $a^i(\lambda h_i f^i(\mathbf{u})) \in \mathbb{R}$ uniquely satisfies

$$\begin{aligned} &\int_0^{\frac{1}{2}} \varphi^{-1}\left(a^i(\lambda h_i f^i(\mathbf{u})) + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds \\ = &\int_{\frac{1}{2}}^1 \varphi^{-1}\left(-a^i(\lambda h_i f^i(\mathbf{u})) + \int_{\frac{1}{2}}^s \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds. \end{aligned}$$

One may show that $T_\lambda : K \to K$ is completely continuous (See Lemma 11 in Xu and Lee [22] for details). Thus we see that $\mathbf{u}$ is a positive solution of (P_λ) if and only if

$$\mathbf{u} = T_\lambda(\mathbf{u}) \text{ on } K.$$

We finally give some remarks and lemma for later use.

Remark 2.2. From condition (A), we get

$$\sigma x \le \varphi^{-1}[\gamma(\sigma)\varphi(x)],$$

and

$$\varphi^{-1}[\sigma\varphi(x)] \le \psi^{-1}(\sigma)x$$

for σ and $x > 0$.

Remark 2.3. Let $h \in L^1_{loc}((0,1),\mathbb{R}_+)$. Then for any fixed $s \in (0,\frac{1}{2})$, we know $\int_s^{\frac{1}{2}} h(\tau)d\tau < \infty$. Applying $\sigma = \int_s^{\frac{1}{2}} h(\tau)d\tau$ and $x = \varphi^{-1}(1)$ in Remark 2.2, we get

$$\varphi^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau\right) \le \varphi^{-1}(1)\psi^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau\right).$$

This implies $\mathcal{H}_\psi \subset \mathcal{H}_\varphi$.

Proposition 2.4. ([20]) *Let* $w \in C_0[0,1]\cap C^1(0,1)$ *satisfy* $\varphi(w')' \le 0$ *on* $(0,1)$. *Then* w *is concave on* $[0,1]$ *and* $\min_{t\in[\frac{1}{4},\frac{3}{4}]} w(t) \ge \frac{1}{4}\|w\|_\infty$, *where* $\|w\|_\infty$ *is the supremum norm of* w.

3. Proofs of main results

In this section, we need to give some lemmas which will play a crucial role in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. *Assume that* (A), (H), (F_1), *and* (F_2) *hold. If* $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, *then there exists* $\bar\lambda > 0$ *such that* (P_λ) *has at least two positive solutions for* $\lambda > \bar\lambda$.

Proof. For any $r > 0$, define

$$\hat m_r = \min\{f^i(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N_+,\ \frac{r}{4} \le \|\mathbf{x}\| \le r,\ i = 1,\dots,N\}.$$

We see that $\hat m_r > 0$, by (F_2). For $K_r \triangleq \{\mathbf{u} \in K \mid \|\mathbf{u}\|_\infty < r\}$, let $\mathbf{u} \in \partial K_r$, then by Proposition 2.4, for $t \in [\frac{1}{4},\frac{3}{4}]$,

$$r = \|\mathbf{u}\|_\infty \ge \|\mathbf{u}(t)\| = \sum_{i=1}^N u_i(t) \ge \min_{t\in[\frac{1}{4},\frac{3}{4}]} \sum_{i=1}^N u_i(t) \ge \frac{1}{4}\|\mathbf{u}\|_\infty = \frac{r}{4},$$

and

$$f^i(\mathbf{u}(t)) \ge \hat m_r \text{ for } i = 1,\dots,N. \tag{3.1}$$

For simplicity, denote $a^i_{\lambda,\mathbf{u}} \triangleq a^i(\lambda h_i f^i(\mathbf{u}))$. Then for $\mathbf{u} \in \partial K_r$, we get

$$\begin{aligned} 2T^i_\lambda(\mathbf{u})(\frac{1}{2}) = &\int_0^{\frac{1}{2}} \varphi^{-1}\left(a^i_{\lambda,\mathbf{u}} + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds \\ &+ \int_{\frac{1}{2}}^1 \varphi^{-1}\left(-a^i_{\lambda,\mathbf{u}} + \int_{\frac{1}{2}}^s \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds. \end{aligned}$$

If $a^i_{\lambda,\mathbf{u}} \ge 0$, then

$$\begin{aligned} &\int_0^{\frac{1}{2}} \varphi^{-1}\left(a^i_{\lambda,\mathbf{u}} + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds \\ &\ge \int_0^{\frac{1}{2}} \varphi^{-1}\left(\int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds, \end{aligned}$$

and by the definition of $a^i_{\lambda,\mathbf{u}}$,

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(-a^i_{\lambda,\mathbf{u}}+\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\\ =\ &\int_{0}^{\frac{1}{2}}\varphi^{-1}\left(a^i_{\lambda,\mathbf{u}}+\int_{s}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\geq 0.\end{aligned}$$

Thus

$$2T^i_\lambda(\mathbf{u})(\frac{1}{2})\geq\int_{0}^{\frac{1}{2}}\varphi^{-1}\left(\int_{s}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds.$$

If $a^i_{\lambda,\mathbf{u}}<0$, then $-a^i_{\lambda,\mathbf{u}}>0$ and

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(-a^i_{\lambda,\mathbf{u}}+\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds\\ \geq\ &\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds,\end{aligned}$$

and by the same argument, we get

$$2T^i_\lambda(\mathbf{u})(\frac{1}{2})\geq\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds.$$

Thus, we obtain

$$\begin{aligned}&2T^i_\lambda(\mathbf{u})(\frac{1}{2})\\ \geq\ &\min\left\{\int_{0}^{\frac{1}{2}}\varphi^{-1}\left(\int_{s}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds,\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\right\}.\end{aligned}$$

By using (3.1), we get

$$\begin{aligned}&2\|T^i_\lambda(\mathbf{u})\|_\infty\\ \geq\ &2T^i_\lambda(\mathbf{u})(\frac{1}{2})\\ \geq\ &\min\left\{\int_{0}^{\frac{1}{2}}\varphi^{-1}\left(\int_{s}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds,\int_{\frac{1}{2}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\right\}\\ \geq\ &\min\left\{\int_{0}^{\frac{1}{4}}\varphi^{-1}\left(\int_{s}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds,\int_{\frac{3}{4}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{s}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\right\}\\ \geq\ &\min\left\{\int_{0}^{\frac{1}{4}}\varphi^{-1}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds,\int_{\frac{3}{4}}^{1}\varphi^{-1}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\lambda h_i(\tau)f^i(\mathbf{u}(\tau))d\tau\right)ds\right\}\\ \geq\ &\min\left\{\int_{0}^{\frac{1}{4}}\varphi^{-1}\left(\lambda\hat{m}_r\int_{\frac{1}{4}}^{\frac{1}{2}}h_i(\tau)d\tau\right)ds,\int_{\frac{3}{4}}^{1}\varphi^{-1}\left(\lambda\hat{m}_r\int_{\frac{1}{2}}^{\frac{3}{4}}h_i(\tau)d\tau\right)ds\right\}\end{aligned}$$

$$= \frac{1}{4}\varphi^{-1}\left(\lambda\hat{m}_r \min\left\{\int_{\frac{1}{4}}^{\frac{1}{2}} h_i(\tau)d\tau, \int_{\frac{1}{2}}^{\frac{3}{4}} h_i(\tau)d\tau\right\}\right)$$
$$\geq \frac{1}{4}\varphi^{-1}\left(\lambda\hat{m}_r\Gamma\right),$$

where $\Gamma \triangleq \min\{\min\{\int_{\frac{1}{4}}^{\frac{1}{2}} h_i(\tau)d\tau, \int_{\frac{1}{2}}^{\frac{3}{4}} h_i(\tau)d\tau\} \mid i = 1, \ldots, N\}$. Define

$$p(r) = \frac{\varphi(8r)}{\hat{m}_r\Gamma},$$

then $p : (0, \infty) \to (0, \infty)$ is continuous. Since $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, we get

$$\lim_{r\to 0} p(r) = \lim_{r\to\infty} p(r) = \infty.$$

Thus, there exists $r_* \in (0, \infty)$ such that

$$p(r_*) = \inf\{p(r) \mid r > 0\} \triangleq \bar{\lambda}. \tag{3.2}$$

Then for any $\lambda > \bar{\lambda}$, there exist r_1, $r_2 > 0$ such that $0 < r_1 < r_* < r_2 < \infty$ with $p(r_1) = p(r_2) = \lambda$. Therefore, if $\mathbf{u} \in \partial K_{r_1}$, then for any $\lambda > \bar{\lambda}$,

$$2\|T^i_\lambda(\mathbf{u})\|_\infty \geq 2T^i_\lambda(\mathbf{u})(\frac{1}{2}) \geq \frac{1}{4}\varphi^{-1}(\frac{\varphi(8r_1)}{\hat{m}_{r_1}\Gamma}\hat{m}_{r_1}\Gamma) = 2r_1 = 2\|\mathbf{u}\|_\infty,$$

and thus

$$\|T_\lambda(\mathbf{u})\|_\infty \geq \|T^i_\lambda(\mathbf{u})\|_\infty \geq \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_1}, \lambda > \bar{\lambda}. \tag{3.3}$$

Similarly,

$$\|T_\lambda(\mathbf{u})\|_\infty \geq \|T^i_\lambda(\mathbf{u})\|_\infty \geq \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_2}, \lambda > \bar{\lambda}. \tag{3.4}$$

Let $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, then $f^i_0 = f^i_\infty = 0$, $i = 1, \ldots, N$. For $\lambda > \bar{\lambda}$, we can choose $\epsilon(= \epsilon(\lambda)) > 0$ sufficiently small so that

$$\psi^{-1}(\lambda\epsilon)\Upsilon \leq \frac{1}{N},$$

where

$$\Upsilon \triangleq \max\left\{\max\{\int_0^{\frac{1}{2}} \psi^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds, \int_{\frac{1}{2}}^1 \psi^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)d\tau\right) ds\} \mid i = 1, \ldots, N\right\}.$$

Since $f^i_0 = 0$, there exists $r^i_3(= r^i_3(\epsilon)) > 0$ such that for $\mathbf{x} \in \mathbb{R}^N_+$ with $\|\mathbf{x}\| \leq r^i_3$,

$$f^i(\mathbf{x}) \leq \epsilon\varphi(\|\mathbf{x}\|) \quad \text{for } i = 1, \ldots, N.$$

Take $0 < r_3 < \min\{r_1, \min\{r^i_3 \mid i = 1, \ldots, N\}\}$. Then for $\mathbf{u} \in \partial K_{r_3}$, we get

$$f^i(\mathbf{u}(t)) \leq \epsilon\varphi(\|\mathbf{u}(t)\|) \leq \epsilon\varphi(r_3) \quad \text{for } i = 1, \ldots, N. \tag{3.5}$$

Since $f^i_\infty = 0$, we define a function $\hat{f}^i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\hat{f}^i(t) = \max\{f^i(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N_+, \|\mathbf{x}\| \leq t\}.$$

By Lemma 2.8 in Wang [20], we have

$$\hat{f}^i_\infty = \lim_{t\to\infty} \frac{\hat{f}^i(t)}{\varphi(t)} = f^i_\infty = 0.$$

Since $\hat{f^i_\infty} = 0$, then for ϵ given above, there exists $r_4^i(= r_4^i(\epsilon)) > 0$ such that for $t \in \mathbb{R}_+$ with $t \ge r_4^i$,

$$\hat{f}^i(t) \le \epsilon\varphi(t) \quad \text{for } i = 1, \dots, N.$$

Take $r_4 > \max\big\{r_2, \max\{r_4^i \mid i = 1, \dots, N\}\big\}$. Then for $\mathbf{u} \in \partial K_{r_4}$, we get

$$f^i(\mathbf{u}(t)) \le \hat{f}^i(r_4) \le \epsilon\varphi(r_4) \quad \text{for } i = 1, \dots, N. \tag{3.6}$$

Since $T_\lambda(\mathbf{u}) \in K$ for $\mathbf{u} \in \partial K_{r_j} (j = 3, 4)$, there exists a unique $\sigma_i \in (0, 1)$ such that $T^i_\lambda(\mathbf{u})(\sigma_i) = \max_{t\in[0,1]} T^i_\lambda(\mathbf{u})(t)$ and $T^i_\lambda(\mathbf{u})'(\sigma_i) = 0$. We first consider the case $\sigma_i \in (0, \frac{1}{2}]$.

$$0 = T^i_\lambda(\mathbf{u})'(\sigma_i) = \varphi^{-1}\left(a^i_{\lambda,\mathbf{u}} + \int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right).$$

Since φ is an odd homeomorphism, $a^i_{\lambda,\mathbf{u}} = -\int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau$. Applying (3.5), (3.6) and Remark 2.2 with $\sigma = \lambda\epsilon$, $x = \varphi^{-1}\left(\varphi(r_j)\int_s^{\frac{1}{2}} \lambda h_i(\tau)d\tau\right)$ and then $\sigma = \int_s^{\frac{1}{2}} h_i(\tau)d\tau$, $x = r_j$ consecutively, we obtain

$$\begin{aligned}
\|T^i_\lambda(\mathbf{u})\|_\infty &= T^i_\lambda(\mathbf{u})(\sigma_i)\\
&= \int_0^{\sigma_i} \varphi^{-1}\left(a^i_{\lambda,\mathbf{u}} + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds\\
&= \int_0^{\sigma_i} \varphi^{-1}\left(-\int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds\\
&= \int_0^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds\\
&\le \int_0^{\frac{1}{2}} \varphi^{-1}\left(\int_s^{\frac{1}{2}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))d\tau\right) ds\\
&\le \int_0^{\frac{1}{2}} \varphi^{-1}\left(\lambda\epsilon\varphi(r_j)\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds\\
&\le \psi^{-1}(\lambda\epsilon)\int_0^{\frac{1}{2}} \varphi^{-1}\left(\varphi(r_j)\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds\\
&\le \psi^{-1}(\lambda\epsilon)\,[\int_0^{\frac{1}{2}} \psi^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds]\, r_j.
\end{aligned}$$

Similarly for the case $\sigma_i \in [\frac{1}{2}, 1)$, we get

$$\|T^i_\lambda(\mathbf{u})\|_\infty \le \psi^{-1}(\lambda\epsilon)\,[\int_{\frac{1}{2}}^{1} \psi^{-1}\left(\int_{\frac{1}{2}}^{s} h_i(\tau)d\tau\right) ds]\, r_j.$$

Combining the above two inequalities and using the choice of ϵ, we get

$$\|T^i_\lambda(\mathbf{u})\|_\infty \le \psi^{-1}(\lambda\epsilon)\,\Upsilon r_j \le \frac{r_j}{N}$$

for $i = 1, \dots, N$, $j = 3, 4$, and thus

$$\|T_\lambda(\mathbf{u})\|_\infty = \sum_{i=1}^{N} \|T^i_\lambda(\mathbf{u})\|_\infty \le \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_j} (j = 3, 4). \tag{3.7}$$

Combining (3.3), (3.4) and (3.7), we conclude that problem (P_λ) has at least two positive solutions $\mathbf{u}_1$, $\mathbf{u}_2$ with $r_3 \le \|\mathbf{u}_1\|_\infty \le r_1 < r_2 \le \|\mathbf{u}_2\|_\infty \le r_4$ for $\lambda > \bar{\lambda}$. □

Lemma 3.2. *Assume that (A), (H), and (F_1) hold. If $\mathbf{f}_0 = \mathbf{f}_\infty = 0$, then there exists $\underline{\lambda} \in (0, \bar{\lambda})$ such that (P_λ) has no positive solution for $\lambda \in (0, \underline{\lambda})$.*

Proof. Since $\mathbf{f}_0 = \mathbf{f}_\infty = 0 < \infty$, then $f^i_0 < \infty$ and $f^i_\infty < \infty$, $i = 1, \dots, N$. Thus, for any $i = 1, \dots, N$, there exist positive numbers β^i_1, β^i_2, R^i_1, R^i_2 such that $R^i_1 < R^i_2$, $\beta^i_1 > f^i_0$, $\beta^i_2 > f^i_\infty$,

$$f^i(\mathbf{x}) \le \beta^i_1 \varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \le R^i_1,$$

and

$$f^i(\mathbf{x}) \le \beta^i_2 \varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \ge R^i_2.$$

Let

$$\beta^i = \max\{\beta^i_1, \beta^i_2, \max\{\frac{f^i(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} \mid \mathbf{x} \in \mathbb{R}^N_+,\ R^i_1 \le \|\mathbf{x}\| \le R^i_2\}\},$$

and

$$\beta = \max\{\max\{\beta^i \mid i = 1, \dots, N\},\ \inf\{\beta \mid \beta > 0, \frac{\psi(\frac{1}{N\Upsilon})}{\beta} < \bar{\lambda}\}\}.$$

Thus, we have

$$f^i(\mathbf{x}) \le \beta\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ i = 1, \dots, N. \tag{3.8}$$

Assume that $\mathbf{v}(t)$ is a positive solution of (P_λ). We prove that if (P_λ) has a positive solution, then $\lambda \ge \underline{\lambda}$, where

$$\underline{\lambda} := \frac{\psi(\frac{1}{N\Upsilon})}{\beta}. \tag{3.9}$$

Indeed, on the contrary, suppose that (P_λ) has a positive solution $\mathbf{v}$ for $0 < \lambda < \underline{\lambda}$. Since $\mathbf{v}(t) = T_\lambda(\mathbf{v})(t)$ for $t \in [0, 1]$, applying the same argument in the proof of Lemma 3.1 with aid of (3.8) and Remark 2.2 with $\sigma = \lambda\beta$,

$x = \varphi^{-1}\left(\varphi(\|\mathbf{v}\|_\infty)\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right)$ and $\sigma = \int_s^{\frac{1}{2}} h_i(\tau)d\tau$, $x = \|\mathbf{v}\|_\infty$ consecutively, we get for $0 < \lambda < \underline{\lambda}$,

$$\|\mathbf{v}\|_\infty = \|T_\lambda(\mathbf{v})\|_\infty = \sum_{i=1}^{N} \|T_\lambda^i(\mathbf{v})\|_\infty \leq N \cdot \psi^{-1}(\lambda\beta)\Upsilon\|\mathbf{v}\|_\infty < \|\mathbf{v}\|_\infty,$$

which is a contradiction. □

Lemma 3.3. *Assume that (A), (H), (F_1), and (F_2) hold. If $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, then there exists $\underline{\lambda} > 0$ such that (P_λ) has at least two positive solutions for $\lambda \in (0, \underline{\lambda})$.*

Proof. For any $r > 0$, define

$$\hat{M}_r = \max\{f^i(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}_+^N,\ \|\mathbf{x}\| \leq r,\ i = 1, \dots, N\}.$$

By (F_2), then $\hat{M}_r > 0$. Let $\mathbf{u} \in \partial K_r$, then for $t \in [0, 1]$,

$$\|\mathbf{u}(t)\| \leq \|\mathbf{u}\|_\infty = r,$$

and

$$f^i(\mathbf{u}(t)) \leq \hat{M}_r \text{ for } i = 1, \dots, N. \tag{3.10}$$

Since $T_\lambda(\mathbf{u}) \in K$ for $\mathbf{u} \in \partial K_r$, there exists a unique $\sigma_i \in (0, 1)$ such that $T_\lambda^i(\mathbf{u})(\sigma_i) = \max_{t\in[0,1]} T_\lambda^i(\mathbf{u})(t)$ and $T_\lambda^i(\mathbf{u})'(\sigma_i) = 0$. We also consider two cases $\sigma_i \in (0, \frac{1}{2}]$ and $\sigma_i \in [\frac{1}{2}, 1)$ with the similar argument in the proof of Lemma 3.1 with aid of (3.10), we get

$$\|T_\lambda^i(\mathbf{u})\|_\infty \leq \varphi^{-1}(\lambda\hat{M}_r)\Upsilon \quad \text{for } i = 1, \dots, N.$$

Define

$$q(r) = \frac{\varphi\left(\frac{r}{N\Upsilon}\right)}{\hat{M}_r},$$

then $q : (0, \infty) \to (0, \infty)$ is continuous clearly. Since $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, we get

$$\lim_{r\to 0} q(r) = \lim_{r\to\infty} q(r) = 0.$$

Thus, there exists $r^* \in (0, \infty)$ such that

$$q(r^*) = \sup\{q(r) \mid r > 0\} \triangleq \underline{\lambda}. \tag{3.11}$$

Then for any $\lambda \in (0, \underline{\lambda})$, there exist r_1, $r_2 > 0$ such that $0 < r_1 < r^* < r_2 < \infty$ with $q(r_1) = q(r_2) = \lambda$. Therefore, if $\mathbf{u} \in \partial K_{r_1}$, then for $\lambda \in (0, \underline{\lambda})$,

$$\|T_\lambda^i(\mathbf{u})\|_\infty \leq \varphi^{-1}\left(\frac{\varphi(\frac{r_1}{N\Upsilon})}{\hat{M}_{r_1}}\hat{M}_{r_1}\right)\Upsilon = \frac{r_1}{N} \quad \text{for } i = 1, \dots, N,$$

and thus

$$\|T_\lambda(\mathbf{u})\|_\infty = \sum_{i=1}^{N} \|T_\lambda^i(\mathbf{u})\|_\infty \leq \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_1},\ \lambda \in (0, \underline{\lambda}). \tag{3.12}$$

Similarly,

$$\|T_\lambda(\mathbf{u})\|_\infty = \sum_{i=1}^{N} \|T_\lambda^i(\mathbf{u})\|_\infty \le \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_2},\ \lambda \in (0, \underline{\lambda}). \tag{3.13}$$

Let $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, then $f_0^{i_0} = f_\infty^{j_0} = \infty$, where

$$f_0^{i_0} := \max\{f_0^i \mid i = 1, \ldots, N\}, \quad f_\infty^{j_0} := \max\{f_\infty^i \mid i = 1, \ldots, N\}$$

for some $i_0, j_0 \in \{1, \ldots, N\}$. For $\lambda \in (0, \underline{\lambda})$, we can take $M = \frac{\gamma(32)}{\lambda\Gamma} > 0$. Since $f_0^{i_0} = \infty$, there exists $r_M > 0$ such that for $\mathbf{x} \in \mathbb{R}_+^N$ with $\|\mathbf{x}\| \le r_M$, we have

$$f^{i_0}(\mathbf{x}) \ge M\varphi(\|\mathbf{x}\|).$$

If $\mathbf{u} \in K$ with $\|\mathbf{u}\|_\infty \le r_M$, then by Proposition 2.4, for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\|\mathbf{u}(t)\| \le \|\mathbf{u}\|_\infty \le r_M,$$

and

$$f^{i_0}(\mathbf{u}(t)) \ge M\varphi(\|\mathbf{u}(t)\|) \ge M\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty). \tag{3.14}$$

Take $0 < r_3 < \min\{r_1, r_M\}$. Then for $\mathbf{u} \in \partial K_{r_3}$, we get

$$f^{i_0}(\mathbf{u}(t)) \ge M\varphi(\|\mathbf{u}(t)\|) \ge M\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty). \tag{3.15}$$

Since $\mathbf{f}_\infty^{j_0} = \infty$, for M given above, there exists $R_M > 0$ such that for $\mathbf{x} \in \mathbb{R}_+^N$ with $\|\mathbf{x}\| \ge R_M$, we have

$$f^{j_0}(\mathbf{x}) \ge M\varphi(\|\mathbf{x}\|).$$

If $\mathbf{u} \in K$ with $\|\mathbf{u}\|_\infty \ge 4R_M$, then by Proposition 2.4, for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\|\mathbf{u}(t)\| = \sum_{i=1}^{N} u_i(t) \ge \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \sum_{i=1}^{N} u_i(t) \ge \frac{1}{4}\|\mathbf{u}\|_\infty \ge R_M,$$

and

$$f^{j_0}(\mathbf{u}(t)) \ge M\varphi(\|\mathbf{u}(t)\|) \ge M\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty). \tag{3.16}$$

Take $r_4 > \max\{r_2, 4R_M\}$. Then for $\mathbf{u} \in \partial K_{r_4}$, we get

$$f^{j_0}(\mathbf{u}(t)) \ge M\varphi(\|\mathbf{u}(t)\|) \ge M\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty). \tag{3.17}$$

We also consider two cases $a_{\lambda,\mathbf{u}}^i \ge 0$ and $a_{\lambda,\mathbf{u}}^i < 0$ ($i = i_0, j_0$). Applying the same argument in the proof of Lemma 3.1 with aids of (3.15), (3.17) and by the definition of M, we get

$$\begin{aligned} 2\|T_\lambda^i(\mathbf{u})\|_\infty \ge 2T_\lambda^i(\mathbf{u})(\frac{1}{2}) &= \frac{1}{4}\varphi^{-1}\left(\lambda M\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty)\Gamma\right) \\ &\ge \frac{1}{4}\varphi^{-1}\left(\gamma(32)\varphi(\frac{1}{4}\|\mathbf{u}\|_\infty)\right). \end{aligned}$$

Applying Remark 2.2 with $\sigma = 32$ and $x = \frac{1}{4}\|\mathbf{u}\|_\infty$, we get

$$2\|T^i_\lambda(\mathbf{u})\|_\infty \geq \frac{1}{4} \times 32 \times \frac{1}{4}\|\mathbf{u}\|_\infty = 2\|\mathbf{u}\|_\infty.$$

Thus, for $i = i_0, j_0$, we have

$$\|T_\lambda(\mathbf{u})\|_\infty \geq \|T^i_\lambda(\mathbf{u})\|_\infty \geq \|\mathbf{u}\|_\infty \text{ for } \mathbf{u} \in \partial K_{r_j} (j = 3, 4). \tag{3.18}$$

Combining (3.12), (3.13) and (3.18), we conclude that problem (P_λ) has at least two positive solutions $\mathbf{u}_1$, $\mathbf{u}_2$ with $r_3 \leq \|\mathbf{u}_1\|_\infty \leq r_1 < r_2 \leq \|\mathbf{u}_2\|_\infty \leq r_4$ for $\lambda \in (0, \underline{\lambda})$. □

Lemma 3.4. *Assume that (A), (H), and (F_1) hold. If $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, then there exists $\bar{\lambda} \in (\underline{\lambda}, \infty)$ (here $\underline{\lambda}$ is given in Lemma 3.3) such that (P_λ) has no positive solution for $\lambda > \bar{\lambda}$.*

Proof. Since $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$, we can easily get $f^{i_0}_0 > 0$ and $f^{j_0}_\infty > 0$. Thus, there exist positive numbers η_1, η_2, r'_1 and r'_2 such that $r'_1 < r'_2$, $0 < \eta_1 < f^{i_0}_0$, $0 < \eta_2 < f^{j_0}_\infty$,

$$f^{i_0}(\mathbf{x}) \geq \eta_1\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \leq r'_1,$$

and

$$f^{j_0}(\mathbf{x}) \geq \eta_2\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \geq r'_2.$$

Let

$$\eta_3 = \min\{\eta_1, \eta_2, \min\{\frac{f^{j_0}(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} \mid \mathbf{x} \in \mathbb{R}^N_+,\ \frac{r'_1}{4} \leq \|\mathbf{x}\| \leq r'_2\},$$
$$\sup\{\eta \mid \eta > 0, \frac{\gamma(32)}{\eta\Gamma} > \underline{\lambda}\}\} > 0.$$

Then, we have

$$f^{i_0}(\mathbf{x}) \geq \eta_3\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \leq r'_1, \tag{3.19}$$

and

$$f^{j_0}(\mathbf{x}) \geq \eta_3\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}^N_+,\ \|\mathbf{x}\| \geq \frac{r'_1}{4}. \tag{3.20}$$

Assume that $\mathbf{v}$ is a positive solution of (P_λ), we prove that if (P_λ) has a positive solution, then $\lambda \leq \bar{\lambda}$, where

$$\bar{\lambda} := \frac{\gamma(32)}{\eta_3\Gamma}. \tag{3.21}$$

Indeed, on the contrary, suppose that (P_λ) has a positive solution $\mathbf{v}$ for $\lambda > \bar{\lambda}$. If $\|\mathbf{v}\|_\infty \leq r'_1$, then by (3.19) and Proposition 2.4, we get for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$f^{i_0}(\mathbf{v}(t)) \geq \eta_3\varphi(\|\mathbf{v}(t)\|) \geq \eta_3\varphi(\frac{1}{4}\|\mathbf{v}\|_\infty). \tag{3.22}$$

On the other hand, if $\|\mathbf{v}\|_\infty > r_1'$, then by Proposition 2.4 and (3.20),

$$\|\mathbf{v}(t)\| = \sum_{i=1}^{N} v_i(t) \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} \sum_{i=1}^{N} v_i(t) \geq \frac{1}{4}\|\mathbf{v}\|_\infty > \frac{r_1'}{4},$$

and

$$f^{j_0}(\mathbf{v}(t)) \geq \eta_3\varphi(\|\mathbf{v}(t)\|) \geq \eta_3\varphi(\frac{1}{4}\|\mathbf{v}\|_\infty) \tag{3.23}$$

for $t \in [\frac{1}{4}, \frac{3}{4}]$. Since $\mathbf{v}(t) = T_\lambda(\mathbf{v})(t)$ for $t \in [0, 1]$, applying the same argument in the proof of Lemma 3.1 with aids of (3.22), (3.23) and Remark 2.2 with $\sigma = 32$, $x = \frac{1}{4}\|\mathbf{v}\|_\infty$, then for $\lambda > \bar{\lambda}$,

$$\begin{aligned}\|\mathbf{v}\|_\infty = \|T_\lambda(\mathbf{v})\|_\infty &\geq \frac{1}{8}\varphi^{-1}\left(\lambda\eta_3\varphi(\frac{1}{4}\|\mathbf{v}\|_\infty)\Gamma\right)\\ &> \frac{1}{8}\varphi^{-1}\left(\gamma(32)\varphi(\frac{1}{4}\|\mathbf{v}\|_\infty)\right)\\ &\geq \frac{1}{8}\times 32\times\frac{1}{4}\|\mathbf{v}\|_\infty = \|\mathbf{v}\|_\infty,\end{aligned}$$

which is a contradiction. □

Proof of Theorem 1.1. Theorem 1.1(1) follows from Lemma 3.1 and Lemma 3.2. Theorem 1.1(2) follows from Lemma 3.3 and Lemma 3.4. □

Lemma 3.5. *Assume that* (A), (H), (F_1), (F_3), *and* $\mathbf{f}_0 = \infty$ *hold. If* (P_λ) *has a positive solution at* $\lambda = \hat{\lambda}$, *then* (P_λ) *has at least one positive solution for* $\lambda \in (0, \hat{\lambda})$.

Proof. Let $\hat{\mathbf{u}}$ be a positive solution of (P_λ) at $\lambda = \hat{\lambda}$ and let $\lambda \in (0, \hat{\lambda})$ be fixed. Consider the following modified problem

$$(P_\lambda^*) \qquad \begin{cases}-\mathbf{\Phi}(\mathbf{u}')' = \lambda\mathbf{h}(t)\cdot\mathbf{f}_*(\mathbf{u}), & t\in(0,1),\\ \mathbf{u}(0) = 0 = \mathbf{u}(1),\end{cases}$$

where $\mathbf{f}_* = (f_*^1, \ldots, f_*^N)$ and each $f_*^i : \mathbb{R}_+^N \to \mathbb{R}_+$ is defined by $f_*^i(u_1, \ldots, u_N) = f^i(\gamma_1(u_1), \ldots, \gamma_N(u_N))$ with

$$\gamma_i(u_i) = \begin{cases}\hat{u}_i, & \text{if } u_i > \hat{u}_i,\\ u_i, & \text{if } 0 \leq u_i \leq \hat{u}_i.\end{cases}$$

First, we show that (P_λ^*) has at least one positive solution. Define T_λ^* the same as T_λ replacing $\mathbf{f}$ by $\mathbf{f}_*$. Then $T_\lambda^* : K \to K$ is also completely continuous. By the fact that $\mathbf{f}_*$ is bounded, there exists $R > 0$ such that $\|T_\lambda^*(\mathbf{u})\|_\infty \leq R$, for any $\mathbf{u} \in K$, i.e.,

$$\|T_\lambda^*(\mathbf{u})\|_\infty \leq \|\mathbf{u}\|_\infty \quad \text{for } \mathbf{u} \in \partial K_R. \tag{3.24}$$

Let $\mathbf{f}_0 = \infty$, then $f_0^{i_0} = \infty$. Applying the similar argument in Lemma 3.3 with $0 < r < \min\{\|\hat{\mathbf{u}}\|_\infty, R\}$, we get

$$\|T_\lambda^*(\mathbf{u})\|_\infty \geq \|(T_\lambda^{i_0})^*(\mathbf{u})\|_\infty \geq \|\mathbf{u}\|_\infty \tag{3.25}$$

for $\mathbf{u} \in \partial K_r$. Combing (3.24) and (3.25), we conclude that (P_λ^*) has at least one solution $\mathbf{u}$ with $r \leq \|\mathbf{u}\|_\infty \leq R$, i.e., $\mathbf{u}$ is a positive solution.

Next, we show that if $\mathbf{u}$ is a solution of (P_λ^*), then $\mathbf{0} \leq \mathbf{u}(t) \leq \hat{\mathbf{u}}(t)$ for $t \in [0, 1]$. If it is true, then (P_λ^*) and (P_λ) are equivalent and the proof is complete. Clearly, $\mathbf{u}(t) \geq \mathbf{0}$ for $t \in [0, 1]$. We also need show that $\mathbf{u}(t) \leq \hat{\mathbf{u}}(t)$ for $t \in [0, 1]$. If it is not true, then $u_i(t) \not\leq \hat{u}_i(t)$ for some $i \in \{1, \ldots, N\}$. By the boundary values of u_i and $\hat{u}_i$, there exist $T_1, T_2 \in (0, 1)$ such that

$$u_i(t) - \hat{u}_i(t) > 0 \text{ on } (T_1, T_2) \text{ and } u_i(T_1) - \hat{u}_i(T_1) = u_i(T_2) - \hat{u}_i(T_2) = 0.$$

Thus, by (F_3), we have for $t \in (T_1, T_2)$,

$$\begin{aligned} -\varphi(u_i'(t))' &= \lambda h_i(t) f_*^i(u_1, \ldots, u_i, \ldots, u_N) \\ &= \lambda h_i(t) f^i(\gamma_1(u_1), \ldots, \hat{u}_i, \ldots, \gamma_i(u_N)) \\ &\leq \hat{\lambda} h_i(t) f^i(\hat{u_1}, \ldots, \hat{u}_i, \ldots, \hat{u_N}) \\ &= -\varphi(\hat{u}_i{}'(t))', \end{aligned}$$

i.e.,

$$\varphi(u_i'(t))' \geq \varphi(\hat{u}_i{}'(t))'. \tag{3.26}$$

Since $u_i - \hat{u}_i \in C_0[T_1, T_2]$, there exist $t_0 \in (T_1, T_2)$ and $0 < \delta < T_2 - t_0$ such that

$$u_i(t_0) - \hat{u}_i(t_0) = \max_{t \in [T_1, T_2]} \{u_i(t) - \hat{u}_i(t)\},$$

and

$$u_i'(t_0) - \hat{u}_i{}'(t_0) = 0, \quad u_i'(t) - \hat{u}_i{}'(t) < 0, \quad t \in (t_0, t_0 + \delta).$$

Integrating both sides of (3.26) from t_0 to $t \in (t_0, t_0 + \delta)$, then we get

$$\varphi(u_i'(t)) - \varphi(u_i'(t_0)) \geq \varphi(\hat{u}_i{}'(t)) - \varphi(\hat{u}_i{}'(t_0)).$$

Since φ is increasing, we have $u_i'(t) \geq \hat{u}_i{}'(t)$, $t \in (t_0, t_0 + \delta)$, which is a contradiction. □

Lemma 3.6. *Assume that* (A), (H), (F_1), *and* $\mathbf{f}_\infty = \infty$ *hold. Let* I *be a compact interval of* $(0, \infty)$. *Then there exists a constant* $b_I > 0$ *such that all possible positive solutions* $\mathbf{u}$ *of* (P_λ) *at* $\lambda \in I$ *satisfy* $\|\mathbf{u}\|_\infty < b_I$.

Proof. Suppose on the contrary that there exists a sequence $\{\mathbf{u}_n\}$ of positive solutions of (P_{λ_n}) with $\{\lambda_n\} \subset I = [\alpha, \beta] \subset (0, \infty)$ and $\|\mathbf{u}_n\|_\infty \to \infty$ as $n \to \infty$. Take $M = \frac{2\gamma(32)}{\alpha\Gamma}$. Let $\mathbf{f}_\infty = \infty$, then $f_\infty^{j_0} = \infty$. Since $f_\infty^{j_0} = \infty$, for M given above, there exists $R_M > 0$ such that for $\mathbf{x} \in \mathbb{R}_+^N$ with $\|\mathbf{x}\| \geq R_M$, we have

$$f^{j_0}(\mathbf{x}) \geq M\varphi(\|\mathbf{x}\|).$$

From the assumption, we can get $\|\mathbf{u}_n\|_\infty \geq 4R_M$ for sufficiently large n. Thus, by Proposition 2.4, we have

$$\|\mathbf{u}_n(t)\| = \sum_{i=1}^{N} u_i^n(t) \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} \sum_{i=1}^{N} u_i^n(t) \geq \frac{1}{4}\|\mathbf{u}_n\|_\infty \geq R_M,$$

and

$$f^{j_0}(\mathbf{u}_n(t)) \geq M\varphi(\|\mathbf{u}_n(t)\|) \geq M\varphi(\frac{1}{4}\|\mathbf{u}_n\|_\infty) \tag{3.27}$$

for $t \in [\frac{1}{4}, \frac{3}{4}]$ and sufficiently large n. Since $\mathbf{u}_n(t) = T_{\lambda_n}(\mathbf{u}_n)(t)$ for $t \in [0, 1]$, applying the same argument in Lemma 3.1 with aid of (3.27) and by the definition of M and Remark 2.2 with $\sigma = 32$, $x = \frac{1}{4}\|\mathbf{u}_n\|_\infty$, we get

$$\begin{aligned}
\|\mathbf{u}_n\|_\infty = \|T_{\lambda_n}(\mathbf{u}_n)\|_\infty &\geq \frac{1}{8}\varphi^{-1}(\lambda_n M\varphi(\frac{1}{4}\|\mathbf{u}_n\|_\infty)\Gamma)\\
&\geq \frac{1}{8}\varphi^{-1}(\alpha M\varphi(\frac{1}{4}\|\mathbf{u}_n\|_\infty)\Gamma)\\
&\geq \frac{1}{8}\varphi^{-1}(2\gamma(32)\varphi(\frac{1}{4}\|\mathbf{u}_n\|_\infty))\\
&> \frac{1}{8}\varphi^{-1}(\gamma(32)\varphi(\frac{1}{4}\|\mathbf{u}_n\|_\infty))\\
&\geq \frac{1}{8} \times 32 \times \frac{1}{4}\|\mathbf{u}_n\|_\infty = \|\mathbf{u}_n\|_\infty
\end{aligned}$$

for $\lambda_n \in I$ with sufficiently large n. This is a contradiction. □

Proof of Theorem 1.2. Define

$$\lambda^* := \sup\{\lambda \mid (P_\lambda) \text{ has at least one positive solution}\}. \tag{3.28}$$

$$\lambda_* := \sup\{\tilde{\lambda} \mid (P_\lambda) \text{ has at least two positive solutions for } \lambda \in (0, \tilde{\lambda})\}. \tag{3.29}$$

By Lemma 3.3 and Lemma 3.4, λ_* and λ^* are both well-defined and $0 < \lambda_* \leq \lambda^* \leq \bar{\lambda}$. By the definitions of λ_* and λ^*, and Lemma 3.5, we get that (P_λ) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, one positive solution for $\lambda \in [\lambda_*, \lambda^*)$, and no positive solution for $\lambda > \lambda^*$.

Finally, it is enough to show that (P_λ) has at least one positive solution at $\lambda = \lambda^*$. By the definition of λ^* and Lemma 3.4, we can choose a sequence $\{\lambda_n\}$ with $\frac{\lambda^*}{2} \leq \lambda_n < \lambda^* \leq \bar{\lambda}$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$, and then by Lemma 3.6 with $I = [\frac{\lambda^*}{2}, \bar{\lambda}]$, there exists $b_I > 0$ such that the corresponding positive solutions $\mathbf{u}_n$ satisfying $\|\mathbf{u}_n\|_\infty < b_I$, i.e., $\{\mathbf{u}_n\}$ is bounded.

By the fact that T_{λ_n} is completely continuous, we get $\{T_{\lambda_n}(\mathbf{u}_n)\}$ is equicontinuous. This implies that $\{\mathbf{u}_n\}$ is equicontinuous, since $\mathbf{u}_n = T_{\lambda_n}(\mathbf{u}_n)$. By the Ascoli-Arzela theorem, $\{\mathbf{u}_n\}$ is relatively compact. Hence, there exists a convergent subsequence $\{\mathbf{u}_n\}$, denoted again by $\{\mathbf{u}_n\}$ and $\mathbf{u}^* \in K$ such that $\mathbf{u}_n \to \mathbf{u}^*$ as $n \to \infty$. Since $\mathbf{u}_n = T_{\lambda_n}(\mathbf{u}_n)$, by the Lebesgue Dominated Convergence Theorem, we can get $\mathbf{u}^* = T_{\lambda^*}(\mathbf{u}^*)$, i.e., $\mathbf{u}^*$ is a solution of (P_{λ^*}).

Moreover, by $\mathbf{f}_0 = \infty$ and applying the similar argument in Lemma 3.6, we see that $\mathbf{u}^* \not\equiv \mathbf{0}$. Therefore mainly due to condition (F_2) and the Maximal Principle, it is not hard to see that $\mathbf{u}^*$ is a positive solution of (P_{λ^*}). □

4. Applications

In this section, we give some examples applicable to our main results.

Example 4.1. Consider the following scalar φ-Laplacian problem

$$(E_1) \qquad \begin{cases} \varphi(u')' + \lambda t^{-\frac{3}{2}} f(u) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $\varphi(x) = |x|x + x$, $x \in \mathbb{R}$, and

$$f(u) = \begin{cases} u^3, & \text{if } 0 \le u < 1, \\ u, & \text{if } u \ge 1. \end{cases}$$

We easily see that φ is an odd increasing homeomorphism. Define functions ψ and γ given as

$$\psi(\sigma) = \begin{cases} \sigma^2, & \text{if } 0 < \sigma \le 1, \\ \sigma, & \text{if } \sigma > 1, \end{cases}$$

and

$$\gamma(\sigma) = \begin{cases} 1, & \text{if } 0 < \sigma \le 1, \\ \sigma^2, & \text{if } \sigma > 1. \end{cases}$$

Then $\psi, \gamma : (0,\infty) \to (0,\infty)$ and ψ is an increasing homeomorphism with

$$\psi^{-1}(\sigma) = \begin{cases} \sigma^{\frac{1}{2}}, & \text{if } 0 < \sigma \le 1, \\ \sigma, & \text{if } \sigma > 1. \end{cases}$$

We may see that (E_1) satisfies assumptions (A), (H), (F_1) and (F_2) (see Xu and Lee [22] for details). In addition,

$$f_0 = \lim_{\|u\|\to 0} \frac{f(u)}{\varphi(\|u\|)} = \lim_{\|u\|\to 0} \frac{u^3}{u^2+u} = 0,$$

$$f_\infty = \lim_{\|u\|\to \infty} \frac{f(u)}{\varphi(\|u\|)} = \lim_{\|u\|\to \infty} \frac{u}{u^2+u} = 0.$$

For any $r > 0$,

$$\hat{m}_r = \max\{f(x) \mid x \in \mathbb{R}_+, \frac{r}{4} \le x \le r\} = f(r),$$

where

$$f(r) = \begin{cases} r^3, & \text{if } 0 < r < 1, \\ r, & \text{if } r \ge 1. \end{cases}$$

If $0 < r < 1$, then

$$p(r) = \frac{\varphi(8r)}{\hat{m}_r\Gamma} = \frac{(8r)^2 + 8r}{0.49r^3} = \frac{64r + 8}{0.49r^2},$$

and

$$p'(r) = \frac{-31.36r - 7.84}{0.2401r^3} < 0.$$

If $r \geq 1$, then

$$p(r) = \frac{\varphi(8r)}{\hat{m}_r\Gamma} = \frac{(8r)^2 + 8r}{0.49r} = \frac{64r + 8}{0.49},$$

and

$$p'(r) = \frac{64}{0.49} > 0.$$

Thus, we get

$$\bar{\lambda} = \inf\{p(r) \mid r > 0\} = p(1) = \frac{64 \times 1 + 8}{0.49} \doteq 146.94.$$

Since $f_0 = f_\infty = 0$, there exist $\beta_1 = 1 > f_0$, $\beta_2 = \frac{1}{10000} > f_\infty$, $R_1 = 1$, $R_2 = 10000$ such that

$$f(x) \leq \varphi(x) \text{ for } 0 \leq x \leq 1,$$

and

$$f(x) \leq \frac{1}{10000}\varphi(x) \text{ for } x \geq 10000.$$

Since for $x \geq 1$,

$$\frac{f(x)}{\varphi(x)} = \frac{x}{x^2 + x} = \frac{1}{x + 1},$$

we get

$$\max\{\frac{f(x)}{\varphi(x)} \mid x \in \mathbb{R}_+, 1 \leq x \leq 10000\} = \frac{1}{2}.$$

From

$$\frac{\psi(\frac{1}{N\Upsilon})}{\beta} < \bar{\lambda},$$

we get

$$\frac{(\frac{1}{1\times 1.46})^2}{\beta} < 146.94,$$

i.e., $\beta > 0.0031$ and thus

$$\inf\{\beta \mid \beta > 0, \frac{\psi(\frac{1}{N\Upsilon})}{\beta} < \bar{\lambda}\} > 0.0031.$$

Therefore, we obtain

$$\beta = \max\{\beta_1, \beta_2, \max\{\frac{f(x)}{\varphi(x)} \mid x \in \mathbb{R}_+,\ 1 \leq x \leq 10000\},$$
$$\inf\{\beta \mid \beta > 0, \frac{\psi(\frac{1}{N\Upsilon})}{\beta} < \bar{\lambda}\}\} = 1,$$

and

$$\underline{\lambda} = \frac{\psi(\frac{1}{N\Upsilon})}{\beta} = \frac{(\frac{1}{1\times 1.46})^2}{1} = 0.46.$$

Consequently, by Theorem 1.1(1), we get the following Conclusion.

Conclusion. Problem (E_1) has at least two positive solutions for $\lambda > 146.94$, and no positive solution for $\lambda \in (0, 0.46)$.

Example 4.2. Consider the following φ-Laplacian system

$$(E_2) \qquad \begin{cases} \varphi(u')' + \lambda t^{-\frac{5}{4}} f^1(u,v) = 0, \\ \varphi(v')' + \lambda t^{-\frac{6}{5}} f^2(u,v) = 0, \quad t \in (0,1), \\ u(0) = v(0) = u(1) = v(1) = 0, \end{cases}$$

where $\varphi(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$, $f^1(u,v) = e^{-u}(v+1)^{\frac{1}{2}}$, $f^2(u,v) = (u+v+2)^{\frac{1}{2}}$. Then φ is an odd increasing homeomorphism. By the homogeneity of φ, taking $\psi(\sigma) = \gamma(\sigma) \equiv \varphi(\sigma)$. We can easily check that (E_2) satisfies assumptions (A), (H), (F_1) and (F_2) (see Xu and Lee [22] for details) and exactly obtain

$$\Gamma = \min\{\min\{\int_{\frac{1}{4}}^{\frac{1}{2}} h_i(\tau)d\tau, \int_{\frac{1}{2}}^{\frac{3}{4}} h_i(\tau)d\tau\} \mid i = 1,2\} = 0.4473.$$

In fact,

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{1}{2}} h_1(\tau)d\tau &= \int_{\frac{1}{4}}^{\frac{1}{2}} \tau^{-\frac{5}{4}}d\tau \\
&= -4\tau^{-\frac{1}{4}}\Big|_{\frac{1}{4}}^{\frac{1}{2}} = -4[(\frac{1}{2})^{-\frac{1}{4}} - (\frac{1}{4})^{-\frac{1}{4}}] = -4(2^{\frac{1}{4}} - 4^{\frac{1}{4}}) \doteq 0.9000, \\
\int_{\frac{1}{2}}^{\frac{3}{4}} h_1(\tau)d\tau &= \int_{\frac{1}{2}}^{\frac{3}{4}} \tau^{-\frac{5}{4}}d\tau \\
&= -4\tau^{-\frac{1}{4}}\Big|_{\frac{1}{2}}^{\frac{3}{4}} = -4[(\frac{3}{4})^{-\frac{1}{4}} - (\frac{1}{2})^{-\frac{1}{4}}] = -4((\frac{3}{4})^{-\frac{1}{4}} - 2^{\frac{1}{4}}) \doteq 0.4585, \\
\int_{\frac{1}{4}}^{\frac{1}{2}} h_2(\tau)d\tau &= \int_{\frac{1}{4}}^{\frac{1}{2}} \tau^{-\frac{6}{5}}d\tau \\
&= -5\tau^{-\frac{1}{5}}\Big|_{\frac{1}{4}}^{\frac{1}{2}} = -5[(\frac{1}{2})^{-\frac{1}{5}} - (\frac{1}{4})^{-\frac{1}{5}}] = -5(2^{\frac{1}{5}} - 4^{\frac{1}{5}}) \doteq 0.8540, \\
\int_{\frac{1}{2}}^{\frac{3}{4}} h_2(\tau)d\tau &= \int_{\frac{1}{2}}^{\frac{3}{4}} \tau^{-\frac{6}{5}}d\tau \\
&= -5\tau^{-\frac{1}{5}}\Big|_{\frac{1}{2}}^{\frac{3}{4}} = -5[(\frac{3}{4})^{-\frac{1}{5}} - (\frac{1}{2})^{-\frac{1}{5}}] = -5((\frac{3}{4})^{-\frac{1}{5}} - 2^{\frac{1}{5}}) \doteq 0.4473, \\
\Upsilon &= \max\{\max\{H_0^i, H_1^i\} \mid i = 1,2\} = 53.8174.
\end{aligned}$$

In fact,

$$H_0^1=\int_0^{\frac12}\psi^{-1}(\int_s^{\frac12}h_1(\tau)d\tau)ds=\int_0^{\frac12}(\int_s^{\frac12}\tau^{-\frac54}d\tau)^3ds\doteq 53.8174,$$
$$H_1^1=\int_{\frac12}^1\psi^{-1}(\int_{\frac12}^s h_1(\tau)d\tau)ds=\int_{\frac12}^1(\int_{\frac12}^s\tau^{-\frac54}d\tau)^3ds\doteq 0.0690,$$
$$H_0^2=\int_0^{\frac12}\psi^{-1}(\int_s^{\frac12}h_2(\tau)d\tau)ds=\int_0^{\frac12}(\int_s^{\frac12}\tau^{-\frac65}d\tau)^3ds\doteq 23.6831,$$
$$H_1^2=\int_{\frac12}^1\psi^{-1}(\int_{\frac12}^s h_2(\tau)d\tau)ds=\int_{\frac12}^1(\int_{\frac12}^s\tau^{-\frac65}d\tau)^3ds\doteq 0.0648.$$

In addition,

$$\begin{aligned}f_0^1&=\lim_{\|(u,v)\|\to0}\frac{f^1(u,v)}{\varphi(\|(u,v)\|)}\\&=\lim_{\|(u,v)\|\to0}\frac{e^{-u}(v+1)^{\frac12}}{(u+v)^{\frac13}}=\lim_{\|(u,v)\|\to0}\frac{(v+1)^{\frac12}}{e^u(u+v)^{\frac13}}=\infty,\end{aligned}$$

$$\begin{aligned}0\le f_\infty^1&=\lim_{\|(u,v)\|\to\infty}\frac{f^1(u,v)}{\varphi(\|(u,v)\|)}=\lim_{\|(u,v)\|\to\infty}\frac{e^{-u}(v+1)^{\frac12}}{(u+v)^{\frac13}}\\&\le\lim_{\|(u,v)\|\to\infty}\frac{(v+1)^{\frac12}}{(u+v)^{\frac13}}\le\lim_{\|(u,v)\|\to\infty}\frac{(u+v+1)^{\frac12}}{(u+v)^{\frac13}}\\&=\lim_{\|(u,v)\|\to\infty}\frac{(u+v+1)^{\frac13}(u+v+1)^{\frac16}}{(u+v)^{\frac13}}\\&=\lim_{\|(u,v)\|\to\infty}(1+\frac{1}{u+v})^{\frac13}(u+v+1)^{\frac16}=\infty,\end{aligned}$$

$$f_0^2=\lim_{\|(u,v)\|\to0}\frac{f^2(u,v)}{\varphi(\|(u,v)\|)}=\lim_{\|(u,v)\|\to0}\frac{(u+v+2)^{\frac12}}{(u+v)^{\frac13}}=\infty,$$

$$\begin{aligned}f_\infty^2&=\lim_{\|(u,v)\|\to\infty}\frac{f^2(u,v)}{\varphi(\|(u,v)\|)}=\lim_{\|(u,v)\|\to\infty}\frac{(u+v+2)^{\frac12}}{(u+v)^{\frac13}}\\&\ge\lim_{\|(u,v)\|\to\infty}(u+v)^{\frac16}=\infty.\end{aligned}$$

Thus,

$$\mathbf{f}_0=f_0^1+f_0^2=\infty,\qquad \mathbf{f}_\infty=f_\infty^1+f_\infty^2=\infty.$$

For any $r>0$,

$$\hat{M}_r=\max\{f^i(\mathbf{x})\mid\mathbf{x}\in\mathbb{R}_+^2,\|\mathbf{x}\|\le r,i=1,2\}=(r+2)^{\frac12}.$$

Then we can easily get

$$q(r) = \frac{\varphi(\frac{r}{N\Upsilon})}{\hat{M}_r} = \frac{\varphi(\frac{1}{N\Upsilon})\varphi(r)}{\hat{M}_r} = \frac{(\frac{1}{2\times 53.8174})^{\frac{1}{3}} r^{\frac{1}{3}}}{(r+2)^{\frac{1}{2}}} \doteq \frac{0.2102 r^{\frac{1}{3}}}{(r+2)^{\frac{1}{2}}},$$

and

$$q'(r) \begin{cases} > 0, & \text{if } 0 < r < 4, \\ = 0, & \text{if } r = 4, \\ < 0, & \text{if } r > 4. \end{cases}$$

Thus, we get

$$\underline{\lambda} = \sup\{q(r) \mid r > 0\} = q(4) \doteq 0.1362.$$

Since $f_0^2 = f_\infty^2 = \infty$, there exist $\eta_1 = 1 < f_0^2$, $\eta_2 = 10 < f_\infty^2$, $r_1^{'} = 1$, $r_2^{'} = 10^6$ such that

$$f^2(\mathbf{x}) \geq \varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}_+^2, \|\mathbf{x}\| \leq 1,$$

and

$$f^2(\mathbf{x}) \geq 10\varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}_+^2, \|\mathbf{x}\| \geq 10^6.$$

Since

$$\frac{f^2(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} = \frac{(\|\mathbf{x}\|+2)^{\frac{1}{2}}}{\|\mathbf{x}\|^{\frac{1}{3}}},$$

we get

$$\min\{\frac{f^2(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} \mid \mathbf{x} \in \mathbb{R}_+^2, \frac{1}{4} \leq \|\mathbf{x}\| \leq 10^6\} = \frac{(4+2)^{\frac{1}{2}}}{4^{\frac{1}{3}}} \doteq 1.5438.$$

From

$$\frac{\gamma(32)}{\eta\Gamma} > \underline{\lambda},$$

we get

$$\frac{3.1748}{\eta \cdot 0.4473} > 0.1362,$$

i.e., $\eta < 52.1123$ and thus

$$\sup\{\eta \mid \eta > 0, \frac{\gamma(32)}{\eta\Gamma} > \underline{\lambda}\} < 52.1123.$$

Therefore, we obtain

$$\eta_3 = \min\{\eta_1, \eta_2, \min\{\frac{f^2(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} \mid \mathbf{x} \in \mathbb{R}_+^2,\ \frac{1}{4} \leq \|\mathbf{x}\| \leq 10^6\}, \sup\{\eta \mid \eta > 0, \frac{\gamma(32)}{\eta\Gamma} > \underline{\lambda}\}\} = 1,$$

and

$$\bar{\lambda} = \frac{\gamma(32)}{\eta_3\Gamma} = \frac{3.1748}{1 \times 0.4473} \doteq 7.0977.$$

Consequently, by Theorem 1.1(2), we get the following conclusion.

Conclusion. Problem (E_2) has at least two positive solutions for $\lambda \in (0, 0.1362)$, and no positive solution for $\lambda > 7.0977$.

Clearly, problem (E_2) also satisfies assumption (F_3). By Theorem 1.2, there must exist $\lambda^* \geq \lambda_* > 0$ such that problem (E_2) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, one positive solution for $\lambda \in [\lambda_*, \lambda^*]$, and no positive solution for $\lambda > \lambda^*$.

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