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GEOMETRIC REPRESENTATIONS OF FINITE GROUPS ON REAL TORIC SPACES

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Abstract. We develop a framework to construct geometric representations of finite groups G through the correspondence between real toric spaces $X^{\mathbb{R}}$ and simplicial complexes with characteristic matrices. We give a combinatorial description of the G-module structure of the homology of $X^{\mathbb{R}}$. As applications, we make explicit computations of the Weyl group representations on the homology of real toric varieties associated to the Weyl chambers of type A and B, which show an interesting connection to the topology of posets. We also realize a certain kind of Foulkes representation geometrically as the homology of real toric varieties.

1. Introduction

The fundamental theorem of toric geometry states that there is a bijective correspondence between the class of toric varieties of complex dimension n and the class of fans (consisting of strongly convex simplicial cones) in $\mathbb{R}^n$. This correspondence has opened up fertile research areas which reveal rich interactions between geometry and combinatorics. For example, many studies on the cohomology of toric varieties associated to root systems are found in the literature such as [1, 14, 22, 27, 28].

The real locus $X^{\mathbb{R}}$ of a toric variety X of complex dimension n, which is the fixed point set of the canonical involution, forms a real subvariety of real dimension n, and it is called a *real toric variety*. While many arguments for complex toric varieties apply verbatim, it turns out that real toric varieties sometimes

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exhibit essentially different nature. In [8] a generalization of real toric variety called real toric space is introduced. In the present paper, we develop a framework to construct geometric representations of finite groups through the correspondence between *real toric spaces* and *simplicial complexes with characteristic matrices*. More precisely, given an action of a finite group G on a simplicial complex, we describe a necessary and sufficient condition with which the action induces one on the corresponding real toric space $X^{\mathbb{R}}$. Furthermore, the G-module structure of the homology of $X^{\mathbb{R}}$ is described combinatorially (Theorem 2.1).

As an application, we investigate real toric varieties associated to Weyl chambers in Section 3. The result demonstrates sharp difference from the complex case studied in [14, 22, 27, 28]. In particular, we compute the Betti numbers for some exceptional cases, and we describe explicitly the W_R-module structures of $H_*(X_R^{\mathbb{R}})$ for $R = A_n$ and B_n. Remarkably, the real cases of type A and of type B reveal an interesting connection to the topology of posets studied in [17, 21, 23, 25, 26, 30] (see also [31] for survey). We note that the real toric varieties associated to the Weyl groups and their representations were already considered in [18, 19]. We work on these objects from different viewpoints. Especially, we provide simple combinatorial description for the representations of real toric varieties associated to Weyl groups of types A and B as sums of irreducible representations, whereas the known results in [18] for type A form alternating sums. The representation for type B is unknown before.

In Section 4, we investigate in the opposite direction; given a group representation, we construct a real toric space with an action of the group whose homology carries the representation. This is a typical situation in geometric representation theory, one of whose objectives is to seek for geometric objects realising particular representations. Precisely, here we realize a certain kind of Foulkes representation as the homology of real toric varieties. Our theory provides yet another application of toric topology in representation theory.

As we are primarily interested in finite Weyl groups, and all their irreducible representations over $\mathbb{C}$ are realized over $\mathbb{Q}$ (see [4] and the references therein), the coefficients of (co)homology in this paper is always taken to be the rational numbers $\mathbb{Q}$ unless otherwise stated.

2. Finite group action on real toric spaces

Real toric space is a topological generalization of (simplicial) real toric variety. In this section, we briefly review the definitions and notations regarding real toric spaces, following [8] and [11]. Then, we discuss the representation of a finite group G on the homology of a real toric space, which is induced by a G-action on combinatorial data.

Let K be a simplicial complex on m-vertices $[m] = \{1, \dots, m\}$. The *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ of K is defined as

$$\mathbb{R}\mathcal{Z}_K = (\underline{D^1}, \underline{S^0})^K = \bigcup_{\sigma \in K} \left\{ (x_1, \dots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma \right\},$$

where $D^1 = [-1, 1]$ is the closed interval and $S^0 = \{-1, 1\}$ is its boundary.

Let $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a linear map, where $\mathbb{Z}_2$ is the field with two elements. We regard Λ as an $n \times m$-matrix with elements in $\mathbb{Z}_2$, and call it a *mod* 2 *characteristic matrix*. The diagonal sign action of $\mathbb{Z}_2^m$ on $(D^1)^m$ induces an action of $\ker\Lambda \subset \mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$. The *real toric space* $M^{\mathbb{R}}(K, \Lambda)$ corresponding to the pair K and Λ is by definition the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$.

Indeed, a real (simplicial) toric variety is a real toric space. Let X be a (simplicial) toric variety, and $X^{\mathbb{R}}$ its real toric variety. By the fundamental theorem of toric geometry, there is a (simplicial) fan Σ corresponding to X. Let V be the set of rays of Σ and assume $V = \{v_1, \dots, v_m\}$. Then, the collection of subsets of V constituting the cones of Σ form a simplicial complex K on V. In addition, the primitive vectors in the direction of rays define a linear map $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, whose ith column is the mod 2 reduction of v_i. Then, $X^{\mathbb{R}}$ is homeomorphic to the real toric space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$.

Denote by $\mathrm{Row}(\Lambda)$ the subspace of $\mathbb{Z}_2^m$ spanned by the row vectors of Λ; that is, $\mathrm{Row}(\Lambda) = \mathrm{Im}(\Lambda^T) = (\ker\Lambda)^{\perp}$, where $(\ker\Lambda)^{\perp}$ is the orthogonal complement to $\ker\Lambda$. Notice that $\mathrm{Row}(\Lambda) = \mathrm{Row}(A\Lambda)$ and $\ker\Lambda = \ker(A\Lambda)$ for any invertible $n \times n$-matrix A; that is, it is independent of the choice of the basis of $\mathbb{Z}_2^n$. Throughout the paper, we identify the power set of $[m]$ with $\mathbb{Z}_2^m$ in the standard way; each element of $[m]$ corresponds to a coordinate of $\mathbb{Z}_2^m$.

The group of simplicial automorphisms $\mathrm{Aut}(K)$ of K acts on the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$, which is extensively studied by [2]. Our first theorem considers finite group representations on the cohomology of real toric spaces, which connects combinatorics, topology, and representation theory.

Theorem 2.1. *Suppose G is a subgroup of $\mathrm{Aut}(K)$ which satisfies one of the following two equivalent conditions*

1. *G preserves $\ker\Lambda$; that is, for any $g \in G$, $\ker\Lambda = \ker\Lambda P_g^{-1}$, where P_g is the permutation matrix $\mathbb{Z}_2^m \to \mathbb{Z}_2^m$ corresponding to g;*
2. *for each element $g \in G$, there exists an $n\times n$-matrix A_g such that $\Lambda P_g^{-1} = A_g\Lambda$.*

Then, the action of G on $\mathbb{R}\mathcal{Z}_K$ induces one on $M^{\mathbb{R}}(K, \Lambda)$ and we have the following isomorphism of G-modules

$$(*) \qquad H_*(M^{\mathbb{R}}(K, \Lambda)) \simeq \bigoplus_{S\in\mathrm{Row}\Lambda} \widetilde{H}_{*-1}(K_S).$$

Proof. We first prove the equivalence of two conditions (1) and (2). The row space $\mathrm{Row}(\Lambda)$ and $\ker\Lambda$ are the orthogonal complements to each other in $\mathbb{Z}_2^m$. Since the G-action on $\mathbb{Z}_2^m$ is an isometry, $\ker\Lambda$ is G-stable if and only if $\mathrm{Row}(\Lambda)$ is G-stable. This, in turn, is equivalent to that each row is mapped to a linear combination of row vectors.

Assume one of (1) and (2) is satisfied. Since $M^{\mathbb{R}}(K, \Lambda)$ is the quotient $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$, the action induces one on $M^{\mathbb{R}}(K, \Lambda)$, which in turn induces a

G-module structure on $H_*(M^{\mathbb{R}}(K,\Lambda))$. Define a map H as the composition

$$H : \Sigma\mathbb{R}\mathcal{Z}_K \longrightarrow \bigvee_{S\subset[m]} \Sigma\mathbb{R}\mathcal{Z}_{K_S} \longrightarrow \bigvee_{S\subset[m]} \Sigma^2|K_S|,$$

where K_S is the induced subcomplex of K with respect to $S \subset [m]$. The map H is shown to be a homotopy equivalence in [3]. Moreover, it is shown to be G-equivariant on the homology in [2, Corollary 2.3.11], where the action of $g \in G$ on the target space is induced by $g\colon K_S \to K_{gS}$. Recall from [8] the rational homotopy equivalence

$$\psi\colon \Sigma \bigvee_{S\in \mathrm{Row}(\Lambda)} \Sigma|K_S| \xrightarrow{\Sigma\iota} \bigvee_{S\subset[m]} \Sigma^2|K_S| \xrightarrow{H^{-1}} \Sigma\mathbb{R}\mathcal{Z}_K \xrightarrow{\Sigma q} \Sigma M^{\mathbb{R}}(K,\Lambda),$$

where ι is the inclusion and q is the quotient. As all the maps appearing in the definition of ψ are equivariant on the (co)homology, ψ induces an isomorphism of G-representations $(*)$. □

This theorem provides a handy way to construct various geometric representations of finite groups, which can be described explicitly in a combinatorial way as we will see in later sections.

Example 2.2. Let K be a 4-gon on $\{1,2,3,4\}$, and G a cyclic group of order 4 generated by g. Consider the G-action on K defined by

$$g\cdot i = \begin{cases} i+1, & i = 1,2,3; \\ 1, & i = 4. \end{cases}$$

Put $\Lambda_1 := \left(\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$ and $\Lambda_2 := \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$. Then, one can see that G does not preserve $\ker\Lambda_1$ because

$$g\cdot\Lambda_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

has a different row space from that of Λ_1. On the other hand, G preserves $\ker\Lambda_2$ because

$$g\cdot\Lambda_2 = g^3\cdot\Lambda_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Lambda_2,$$

and $g^2\cdot\Lambda_2 = g^4\cdot\Lambda_2 = \Lambda_2$. This action on the corresponding real toric space $M^{\mathbb{R}}(K,\Lambda_2) \simeq T^2$ is seen to be $g\colon T^2 \to T^2$, $g(x,y) = (-y,x)$. On the homology, $H_*(M^{\mathbb{R}}(K,\Lambda_2))$ is the trivial representations on degree 0 and 2, and the two dimensional irreducible representation (over $\mathbb{Q}$) on degree 1. On the other hand by $(*)$, we have $H_1(M^{\mathbb{R}}(K,\Lambda_2)) \simeq \tilde{H}_0(\{1,3\}) \oplus \tilde{H}_0(\{2,4\})$. Since $g\{1,3\} = \{2,4\}$ and $g\{2,4\} = \{3,1\}$, taking account the sign of the induced map g_* on the reduced homology, we can also see that it is the two dimensional irreducible representation.

3. Real toric varieties associated to Weyl chambers

An illustrative family of examples of Theorem 2.1 are obtained by the real toric varieties associated to Weyl chambers. The complex toric varieties associated to Weyl chambers are studied by [14,22,27,28], and the real toric variety associated to the type A_n Weyl chambers is studied by [18, Corollary 1.2], [29] and [10]. Here, we carry out explicit computations of the representation of the Weyl groups on the homology, which contrast sharply with the complex toric case.

We fix some notations as follows;

- Φ: an irreducible root system of type R and rank n.
- $\Delta = \{\alpha_1, \dots, \alpha_n\}$: a fixed set of simple roots of Φ.
- $\Omega = \{\omega_1, \dots, \omega_n\}$: the set of fundamental co-weights. That is, $(\omega_i, \alpha_j) = \delta_{ij}$ with respect to the inner product in the ambient space.
- $W_R = \langle s_1, \dots, s_n \rangle$: the Weyl group generated by the simple reflections acting on $\mathbb{R}\langle\Omega\rangle$.
- $V_R = W_R \cdot \Omega = \{v_1, \dots, v_N\}$: the set of rays spanning the chambers.
- $K_R \subset 2^{V_R}$: the corresponding simplicial complex. It is called the *Coxeter complex* of type R [20, §1.15].
- $\Lambda_R = (v_1, v_2, \dots, v_N)$: the mod 2 characteristic matrix, where the columns are the mod 2 coordinates of the rays. We set $v_1 = \omega_1, \dots, v_n = \omega_n$ so that the first $n \times n$-submatrix is the identity.
- $X_R^{\mathbb{R}}$: the real toric space $M^{\mathbb{R}}(K_R, \Lambda_R)$; it is indeed the real toric variety associated to the Weyl chambers.

We recall the combinatorial structure of the Coxeter complex K_R. See [5] for more details.

- The set of 0-simplices is the W_R-orbit of the fundamental co-weights $W_R \cdot \Omega$. The orbits $W_R\omega_i$ and $W_R\omega_j$ are disjoint if $i \neq j$. The stabilizer of ω_i is the parabolic subgroup W_i generated by $\{s_j \mid j \neq i\}$. Hence, the set of vertices is described by
$$V_R = \{w\omega_i \mid w \in W_R/W_i, \omega_i \in \Omega\}.$$
- Two vertices u and v form a 1-simplex if and only if there exist $w \in W_R$ and $\omega_i, \omega_j \in \Omega$ such that $wu = \omega_i, wv = \omega_j$. Similarly, the set of $(k-1)$-simplices is
$$\{w \cdot L \mid L \subset \Omega, |L| = k, w \in W_R/W_L\},$$
where W_L is the stabilizer of $L = \{\omega_{i_1}, \dots, \omega_{i_k}\}$, which is the parabolic subgroup of W_R generated by $\{s_i \mid i \notin L\}$. In particular, maximal simplices are obtained by the free and transitive action of W_R on the fundamental chamber.

By definition, the Weyl group W_R acts on K_R. We compute the action of W_R on the rows of Λ_R explicitly.

Lemma 3.1. *The Weyl group W_R acts on K_R and preserves* $\ker \Lambda_R$. *More precisely, let $\Lambda_R^j \in \mathbb{Z}_2^m$ be the jth row of Λ_R, which corresponds the ω_j coordinates of the rays. Then, we have*

$$(s_i(\Lambda_R))^j = \Lambda_R^j - c_{ij}\Lambda_R^i,$$

where $c_{ij} = (\alpha_i^\vee, \alpha_j)$ are the entries of the Cartan matrix of R.

Proof. This is a special instance of Theorem 2.1 since W_R acts on $\mathbb{R}\langle\Omega\rangle$ linearly. Recall that the action of the simple reflections on $\mathbb{R}\langle\Omega\rangle$ is given by

$$\begin{aligned} s_i(\omega_j) &= \omega_j - \frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)}\alpha_i \\ &= \begin{cases} \omega_j & (i \neq j) \\ \omega_i - \alpha_i^\vee & (i = j), \end{cases} \end{aligned}$$

where $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$ is the simple co-root. Therefore, if we write $\alpha_i^\vee = \sum_j c_{ij}\omega_j$, we have

$$s_i(\sum_j d_j\omega_j) = \sum_j (d_j - d_i c_{ij})\omega_j$$

and $(s_i(\Lambda_R))^j = \Lambda_R^j - c_{ij}\Lambda_R^i \in \mathrm{Row}(\Lambda_R)$. □

Now we state a theorem which gives a combinatorial description of the W_R module structure of the (co)homology of $X_R^{\mathbb{R}}$.

Theorem 3.2. *There is a W_R-module isomorphism*

$$H^*(X_R^{\mathbb{R}}) \cong H_*(X_R^{\mathbb{R}}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda_R)} \widetilde{H}_{*-1}(K_S).$$

Proof. Since irreducible characters of W_R-modules are real-valued for any Weyl group W_R, W_R-modules are self-dual. As $H^*(X_R^{\mathbb{R}})$ and $H_*(X_R^{\mathbb{R}})$ are dual to each other as W_R-modules, we obtain the first isomorphism. The second isomorphism follows by Lemma 3.1 and Theorem 2.1. □

In the following subsections, by using Lemma 3.1 and Theorem 3.2, we compute the Betti numbers of $X_R^{\mathbb{R}}$ for some exceptional cases, and describe explicitly the W_R-module structures of $H_*(X_R^{\mathbb{R}})$ for $R = A_n$ and B_n. In particular, $H_*(X_{A_n}^{\mathbb{R}})$ is isomorphic to the famous Foulkes representation.

3.1. Computation of Betti numbers

As an interesting application of this explicit description, we compute the Betti numbers of $X_R^{\mathbb{R}}$ for some exceptional cases. The (rational) Betti number of $X_R^{\mathbb{R}}$ has been computed when $R = A_n$ (in [10], [18], and [29]), $R = B_n$ (in [9]), and when $R = C_n$ and D_n (in [7]). However, for exceptional types the Betti numbers have not been known except for $R = G_2$. It is known that $X_{G_2}^{\mathbb{R}}$ is homeomorphic to the connected sum of ten copies of the real projective

plane. In this subsection, we successfully apply our result to compute the Betti numbers for the cases $R = F_4$ and E_6.

In order to compute the Betti number of $X_R^{\mathbb{R}}$, it is enough to compute the rank of $H_*(K_S)$ for all $S \in \mathrm{Row}(\Lambda_R)$ due to $(*)$. By Lemma 3.1, $K_S \simeq K_{gS}$ for $S \in \mathrm{Row}(\Lambda_R)$ and $g \in W_R$. Therefore, we only have to consider representatives K_S of the W_R-orbits in $\mathrm{Row}(\Lambda_R)$, which can be obtained by the explicit description of the W_R-action on the rows of Λ given in Lemma 3.1. The reduced Betti number $\widetilde{\beta}_*(K_S)$ of K_S is computable by a computer program (for example, CHomP [12]) within reasonable time for R of small rank. By exploiting this symmetry, we obtained the following result with the aid of computer (the Maple code used to produce the result is available at https://github.com/shizuo-kaji/WeylGroup).

Proposition 3.3. *The Betti numbers of $X_R^{\mathbb{R}}$ for $R = G_2, F_4, E_6$ are given as follows*:

$$\beta_i(X_{G_2}^{\mathbb{R}}) = \begin{cases} 1, & i = 0; \\ 9, & i = 1; \\ 0, & \textit{otherwise,} \end{cases}$$

$$\beta_i(X_{F_4}^{\mathbb{R}}) = \begin{cases} 1, & i = 0; \\ 57, & i = 1; \\ 264, & i = 2; \\ 0, & \textit{otherwise,} \end{cases} \qquad \textit{and}$$

$$\beta_i(X_{E_6}^{\mathbb{R}}) = \begin{cases} 1, & i = 0; \\ 36, & i = 1; \\ 1323, & i = 2; \\ 4392, & i = 3; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For type G_2, there are three non-zero elements in $\mathrm{Row}(\Lambda_{G_2})$. These are all in the same orbit, and, for the first row S of Λ_{G_2}, K_S consists of 4 distinct points, and hence, $\widetilde{\beta}_0(K_S) = 3$. Therefore, the first reduced Betti number $\widetilde{\beta}_1(X_{G_2}^{\mathbb{R}})$ of $X_{G_2}^{\mathbb{R}}$ is $3 \times 3 = 9$.

For type F_4 there are 15 non-zero elements in $\mathrm{Row}(\Lambda_{F_4})$ and they are partitioned into two orbit types; 12 elements are in the orbit generated by the first row of Λ_{F_4} ($\widetilde{\beta}_0(K_S) = 1, \widetilde{\beta}_1(K_S) = 22$) and 3 elements are in the orbit generated by the third row ($\widetilde{\beta}_0(K_S) = 15$).

For type E_6 there are 63 non-zero elements in $\mathrm{Row}(\Lambda_{E_6})$. They are partitioned into two orbit types; 27 elements are in the orbit generated by the first row ($\widetilde{\beta}_1(K_S) = 49$) and 36 elements are in the orbit generated by the sum of the first and the fourth row ($\widetilde{\beta}_0(K_S) = 1, \widetilde{\beta}_2(K_S) = 122$). □

Remark 3.4. It is well-known the ith $\mathbb{Z}_2$-Betti number of a real toric variety $X^{\mathbb{R}}$ is equal to the ith component of the h-vector of its underlying complex K (see [13]). We denote by $h_q(K)$ the h-polynomial of a simplicial complex K,

and by $\chi(X^{\mathbb{R}})$ the Euler characteristic number of $X^{\mathbb{R}}$. Here is the list of the h-polynomials of K_R for exceptional types R:

$$\begin{aligned}
h_q(K_{G_2}) &= q^2 + 10q + 1,\\
h_q(K_{F_4}) &= q^4 + 236q^3 + 678q^2 + 236q + 1,\\
h_q(K_{E_6}) &= q^6 + 1272q^5 + 12183q^4 + 24928q^3 + 12183q^2 + 1272q + 1,\\
h_q(K_{E_7}) &= q^7 + 17635q^6 + 309969q^5 + 1123915q^4 + 1123915q^3 + 309969q^2\\
&\quad + 17635q + 1,\\
h_q(K_{E_8}) &= q^8 + 881752q^7 + 28336348q^6 + 169022824q^5 + 300247750q^4\\
&\quad + 169022824q^3 + 28336348q^2 + 881752q + 1.
\end{aligned}$$

We recall that the Euler characteristic does not depend on the field with respect to which cohomology is taken. Therefore, by using above, we can compute the Euler characteristic numbers as

$$\begin{aligned}
&\chi(X^{\mathbb{R}}_{G_2}) = -8,\ \chi(X^{\mathbb{R}}_{F_4}) = 208,\ \chi(X^{\mathbb{R}}_{E_6}) = -3104,\\
&\chi(X^{\mathbb{R}}_{E_7}) = 0,\ \text{and } \chi(X^{\mathbb{R}}_{E_8}) = 17111296.
\end{aligned}$$

The Euler characteristic numbers of $X^{\mathbb{R}}_{G_2}, X^{\mathbb{R}}_{F_4}$, and $X^{\mathbb{R}}_{E_6}$ can be also computed from Proposition 3.3.

For type E_7, there are three orbit types; 63 elements are in the orbit generated by the first row of Λ_{E_7}, 63 elements are in the orbit generated by the sum of the fifth and the seventh rows, and one element in the orbit generated by the sum of the first, the fifth and the seventh rows. For type E_8, there are two orbit types; 120 elements are in the orbit generated by the first row of Λ_{E_8}, 135 elements are in the orbit generated by the sum of the sixth and the eighth rows. However, we are currently unable to compute the reduced Betti numbers of K_S since it is too big.

Remark 3.5. It should be noted that K_S also determines the torsions in (co)homology except for 2-torsion. More precisely, by [11] and [6], we know that, for $k > 1$ and odd number q,

$$\begin{aligned}
H^*(M(K,\Lambda);\mathbb{Z}_{2^k}) &\cong \bigoplus_{S\in \mathrm{Row}(\Lambda)} \widetilde{H}^{*-1}(K_S;\mathbb{Z}_{2^{k-1}}), \quad \text{and}\\
H^*(M(K,\Lambda);\mathbb{Z}_q) &\cong \bigoplus_{S\in \mathrm{Row}(\Lambda)} \widetilde{H}^{*-1}(K_S;\mathbb{Z}_q).
\end{aligned}$$

With the help of computer program, one can easily check that each K_S in Proposition 3.3 is torsion-free, so $X^{\mathbb{R}}_R$ for $R = G_2, F_4, E_6$ has no torsion in (co)homology other than 2-torsion. Therefore, by Proposition 3.3 and Remark 3.4, one can apply the universal coefficient theorem to compute the integral (co)homology group for $X^{\mathbb{R}}_R$ for $R = G_2, F_4, E_6$.

3.2. Type A_n representation

When $R = A_n$, the Weyl group W_{A_n} is the symmetric group $\mathfrak{S}_{n+1}$ and K_{A_n} has a nice description as the dual of the permutohedron of order $n+1$. The map $V_{A_n} \to 2^{[n+1]} \setminus \{[n+1], \emptyset\}$ sending $w\omega_i \to \{w(1), w(2), \ldots, w(i)\}$ induces an $\mathfrak{S}_{n+1}$-equivariant bijection. Under this identification, $J_1, \ldots, J_k \subset [n+1]$ form a simplex if and only if they form a nested chain of subsets up to permutation. To represent Λ_{A_n}, we use the basis consisting of $\mathbf{e}_k := (1,k)\omega_1$ for $1 \leq k \leq n$, where $(1,k) \in \mathfrak{S}_{n+1}$ is the transposition. The coordinates of $J \in V_{A_n}$ with this basis are given by $\sum_{k \in J} \mathbf{e}_k$, where we set $\mathbf{e}_{n+1} = \sum_{i=1}^{n} \mathbf{e}_i$.

Note that Theorem 3.2 specializes in this case to giving an $\mathfrak{S}_{n+1}$-module isomorphism $H_*(X^{\mathbb{R}}_{A_n}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda_{A_n})} \widetilde{H}_{*-1}(K_S)$. Thus, in order to compute $H_*(X^{\mathbb{R}}_{A_n})$, we have to investigate each K_S. We establish a correspondence between $\mathrm{Row}(\Lambda_{A_n})$ and the even cardinality subsets of $[n+1]$ in the following way. Any element $S \in \mathrm{Row}(\Lambda_{A_n})$ is a linear sum of rows of Λ_{A_n}. If it is a sum of an even number of rows, say $i_1, i_2, \ldots, i_{2r}$th rows, then we associate $I_S = \{i_1, i_2, \ldots, i_{2r}\} \subset [n+1]$. If it is a sum of an odd number of rows, say $i_1, i_2, \ldots, i_{2r-1}$th rows, then we associate $I_S = \{i_1, i_2, \ldots, i_{2r-1}, n+1\} \subset [n+1]$.

One can see that K_S is the full-subcomplex of K_{A_n} consisting of the vertices J such that $|J \cap I_S|$ is odd. By [10, Lemma 5.3], one can inductively remove $J \not\subset I_S$ from K_S without changing the homotopy type of K_S. This gives a homotopy equivalence between K_S and the poset complex associated to the odd rank-selected Boolean algebra $\mathcal{B}^{odd}_{I_S}$ over I_S, which induces an $\mathfrak{S}_{n+1}$-module isomorphism

$$H_*(X^{\mathbb{R}}_{A_n}) \cong \bigoplus_{I \in \binom{[n+1]}{2r}} \widetilde{H}_{*-1}(\mathcal{B}^{odd}_I). \tag{1}$$

Before we describe the $\mathfrak{S}_{n+1}$-module structure of $H_*(X^{\mathbb{R}}_{A_n})$ in detail, we review some basic facts on the representations of symmetric groups. A reference for basics of the representation theory of symmetric groups is [24]. A *partition* λ of a positive integer k, denoted by $\lambda \vdash k$, is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers such that $\sum_i \lambda_i = k$ and we let $|\lambda| = k$. The *diagram* of a partition λ is the left-justified array of λ_i boxes in the ith row, and a *standard tableau* of shape λ is a filling of boxes in the diagram of λ with numbers $1, 2, \ldots, |\lambda|$ whose rows are increasing from left to right and columns are increasing from top to bottom. The number of standard tableaux of shape λ is denoted by f^λ. For two partitions λ, μ such that $\mu_i \leq \lambda_i$ for all i, $\mu \subseteq \lambda$ in notation, the diagram of λ/μ is the diagram of λ with the first μ_i boxes deleted in the ith row for all i. An entry i of a standard tableau is a *descent* if $i+1$ is located in a lower row than the row where i is located. The set of descents of a standard tableau T is called the *descent set of* T.

Irreducible representations of the symmetric group $\mathfrak{S}_k$ are indexed by the partitions of k. For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of k, corresponding irreducible $\mathfrak{S}_k$-module S^λ is called the *Specht module*, and it is known that $\dim S^\lambda = f^\lambda$.

Each direct summand on the right hand side of (1) was computed by Solomon and also by Stanley:

Theorem 3.6 ([25, 26])**.** *Let $Q \subset [m-1]$. Then the homology of the Q-rank-selected poset $\mathcal{B}_{[m]}^{Q}$ is given, as an $\mathfrak{S}_m$-module, by*

$$\widetilde{H}_*(\mathcal{B}_{[m]}^{Q}) \cong \begin{cases} \bigoplus_{\nu} c_{Q,\nu} S^{\nu} & (* = |Q| - 1) \\ 0 & (* \neq |Q| - 1), \end{cases}$$

where $c_{Q,\nu}$ is the number of standard tableaux of shape ν with descent set Q.

In particular, by setting $Q = \{1, 3, \dots, 2r-1\}$, we have an $\mathfrak{S}_{[2r]}$-module isomorphism

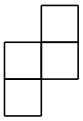
$$\widetilde{H}_*(\mathcal{B}_{[2r]}^{odd}) \cong \begin{cases} \bigoplus_{\nu} c_{Q,\nu} S^{\nu} & (* = r - 1) \\ 0 & (* \neq r - 1). \end{cases}$$

Remark 3.7 (See [31, Theorem 3.4.4]). The module $\bigoplus_{\nu} c_{Q,\nu} S^{\nu}$ that appears in Theorem 3.6 can be understood as a *skew Specht module* $S^{\lambda/\mu}$ for some skew hook λ/μ. A skew hook is an edge-wise connected skew Young diagram which contains no (2×2)-subdiagram. A skew hook with k boxes is determined by its *descent set* $S \subset [k-1]$; when one fills a skew hook with numbers 1 through k in order from southwest to northeast, i is a *descent* of the skew hook if $i+1$ is placed above i. A skew Specht module corresponding to a skew hook is often referred to as the *Foulkes representation*. Its decomposition into irreducible modules (Specht modules of ordinary shape) follows from the Littlewood-Richardson rule, and $c_{Q,\nu}$ is seen to be the Littlewood-Richardson number $c_{\mu\nu}^{\lambda}$. See [24] for the description of Littlewood-Richardson numbers.

In particular,

$$\widetilde{H}_{r-1}(\mathcal{B}_{[2r]}^{odd}) \cong S^{\lambda_r/\mu_r}$$

when $\lambda_r = (r, r, r-1, r-2, \dots, 2, 1)$, and $\mu_r = (r-1, r-2, \dots, 2, 1)$. For example, the diagram of the skew hook λ_2/μ_2 is



whose descent set is $\{1, 3\}$.

Let $\mathfrak{S}_I$ be the stabilizer subgroup of $\mathfrak{S}_{n+1}$ fixing all $i \notin I$. Then $\mathfrak{S}_I$ preserves $\widetilde{H}_*(\mathcal{B}_I^{odd})$ and $\mathfrak{S}_{\bar{I}}$ acts on $\widetilde{H}_*(\mathcal{B}_I^{odd})$ trivially, where $\bar{I} = [n+1] \setminus I$. Moreover, each $I \in \binom{[n+1]}{2r}$ is in one to one correspondence with a coset $\sigma(\mathfrak{S}_{\{1,\dots,2r\}} \times \mathfrak{S}_{\{2r+1,\dots,n+1\}})$ in $\mathfrak{S}_{n+1}$ via $I = \sigma\{1, \dots, 2r\}$. This proves the following.

Theorem 3.8. *The rth homology $H_r(X^{\mathbb{R}}_{A_n})$ of $X^{\mathbb{R}}_{A_n}$ with the natural action of $\mathfrak{S}_{n+1}$, is isomorphic to the induced representation*

$$\mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_{\{1,\dots,2r\}}\times\mathfrak{S}_{\{2r+1,\dots,n+1\}}}(\widetilde{H}_{r-1}(\mathcal{B}^{odd}_{[2r]})\otimes S^{(n-2r)})$$

of $\mathfrak{S}_{n+1}$, where $S^{(n-2r)}$ is the trivial representation of $\mathfrak{S}_{\{2r+1,\dots,n+1\}}$.

The irreducible decomposition is given by a standard argument:

Corollary 3.9. *Let $Q=\{1,3,\dots,2r-1\}$ and let $c_{Q,\nu}$ be the number of standard tableaux of shape ν with descent set Q. Then, we have*

$$H_r(X^{\mathbb{R}}_{A_n})\cong\bigoplus_{\eta\vdash(n+1)}\left(\sum_{\nu}c_{Q,\nu}\right)S^{\eta},$$

where ν runs over all partitions of $2r$ that are contained in η, and η/ν has at most one box in each column.

Proof. By Theorem 3.8, we have following $\mathfrak{S}_{n+1}$ module isomorphisms.

$$\begin{aligned}H_r(X^{\mathbb{R}}_{A_n})&\cong\mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_{\{1,\dots,2r\}}\times\mathfrak{S}_{\{2r+1,\dots,n+1\}}}\left(\bigoplus_{\nu}c_{Q,\nu}S^{\nu}\otimes S^{(n-2r)}\right)\\&\cong\bigoplus_{\nu}c_{Q,\nu}\left(\mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_{\{1,\dots,2r\}}\times\mathfrak{S}_{\{2r+1,\dots,n+1\}}}\left(S^{\nu}\otimes S^{(n-2r)}\right)\right)\\&\cong\bigoplus_{\nu}c_{Q,\nu}\left(\bigoplus_{\nu\rightsquigarrow\eta}S^{\eta}\right)\cong\bigoplus_{\eta\vdash(n+1)}\left(\sum_{\nu\rightsquigarrow\eta}c_{Q,\nu}\right)S^{\eta}\end{aligned}$$

where $\nu\rightsquigarrow\eta$ means that we can obtain η from ν by adding $n+1-2r$ boxes so that no two new boxes are in one column. In the third isomorphism, the well known Pieri rule (a special case of the Littlewood-Richardson rule) is used. □

Example 3.10. If $n=5$ and $r=3$, we let $Q=\{1,3,5\}$. Then, the homology $H_3(X^{\mathbb{R}}_{A_5})$ is decomposed into irreducible $\mathfrak{S}_6$ modules in the following way due to Corollary 3.9.

$$H_3(X^{\mathbb{R}}_{A_5})\cong S^{(3,3)}\oplus 2S^{(3,2,1)}\oplus S^{(3,1,1,1)}\oplus S^{(2,2,2)}\oplus S^{(2,2,1,1)}.$$

For example, when $\eta=(2,2,2)\vdash 6$, the only ν we have to consider is $(2,2,2)$ and $c_{Q,\nu}=1$ hence $S^{(2,2,2)}$ is a summand in the decomposition of $H_3(X^{\mathbb{R}}_{A_5})$ with multiplicity 1. We list all standard tableaux of shape ν with descent set Q for all possible $\nu\vdash 6$.

1	3	5
2	4	6

1	3	5
2	4	
6		

1	3	5
2	6	
4		

1	3	5
2		
4		
6		

1	3
2	5
4	6

1	3
2	5
4	
6	

It can be checked that

$$\dim(\mathrm{S}^{(3,3)}) + 2\dim(\mathrm{S}^{(3,2,1)}) + \dim(\mathrm{S}^{(3,1,1,1)}) + \dim(\mathrm{S}^{(2,2,2)}) + \dim(\mathrm{S}^{(2,2,1,1)})$$
$$= 5 + 2 \times 16 + 10 + 5 + 9 = 61,$$

which is the number of alternating permutations of length 6 (times $\binom{n+1}{2r}$), and it is known to be the dimension of $H_3(X^{\mathbb{R}}_{A_5})$ (see [9, 18]).

Example 3.11. When $n = 5$ and $r = 2$, we have the decomposition of $H_2(X^{\mathbb{R}}_{A_5})$ as in the following;

$$H_2(X^{\mathbb{R}}_{A_5}) \cong S^{(4,2)} \oplus S^{(4,1,1)} \oplus 2S^{(3,2,1)} \oplus S^{(3,1,1,1)} \oplus S^{(2,2,2)} \oplus S^{(2,2,1,1)}\,.$$

To compute the multiplicity of $S^{(3,2,1)}$, we consider partitions of $2r = 4$, which is contained in $(3, 2, 1)$ and can be obtained from $(3, 2, 1)$ by deleting two boxes from different columns. There are three such partitions $(2, 1, 1), (2, 2)$ and $(3, 1)$, and the values $c_{Q,\nu}$ for each of them are $1, 1, 0$ respectively where $Q = \{1, 3\}$. Hence $S^{(3,2,1)}$ appears twice in the decomposition of $H_2(X^{\mathbb{R}}_{A_5})$. By computing the dimension of each Specht module, we have

$$\dim(H_2(X^{\mathbb{R}}_{A_5})) = 9 + 10 + 2 \times 16 + 10 + 5 + 9 = 75\,.$$

Note that $75 = 5 \times 15$, where 5 is the number of alternating permutations of length 4, and 15 is the index of the parabolic subgroup $\mathfrak{S}_4 \times \mathfrak{S}_2$ in $\mathfrak{S}_6$.

Remark 3.12. We, in Corollary 3.9, show how to write down the rational homology $H_r(X^{\mathbb{R}}_{A_n})$ as a sum of irreducible $\mathfrak{S}_{n+1}$-modules in an explicit way, while Henderson in his 2012 paper showed that the $\mathfrak{S}_{n+1}$-module structure of the rational cohomology $H^r(X^{\mathbb{R}}_{A_n})$ is isomorphic to a signed sum of induced representations of the tensor product of a trivial representation and a sign representation of some parabolic subgroups of $\mathfrak{S}_{n+1}$; see Corollary 1.2 in [18].

3.3. Type B_n representation

When $R = B_n$, the Weyl group W_{B_n} is the group of signed permutations (also called the hyperoctahedral group) and K_{B_n} has a nice description as the dual of the type B_n permutohedron. We write $[\pm n] = \{\pm 1, \dots, \pm n\}$, $-J = \{-j \mid j \in J\}$ for $J \subset [\pm n]$, and $T_n = \{J \in 2^{[\pm n]} \mid J \cap -J = \emptyset, J \neq \emptyset\}$. For each $J \in T_n$, we write $J^{\pm} = (J \cup -J) \cap [n]$.

The map $V_{B_n} \to T_n$ sending $w\omega_i \to \{w(1), w(2), \dots, w(i)\}$ induces a W_{B_n}-equivariant bijection. Under this identification, $J_1, \dots, J_k \in T_n$ form a simplex if and only if they form a nested chain of subsets up to permutation. To represent Λ_{B_n}, we use the basis consisting of $\mathbf{e}_k := (1, k)\omega_1$ for $1 \le k \le n$, where $(1, k) \in W_{B_n}$ is the transposition. The coordinates of $J \in V_{B_n}$ with this basis are given by $\sum_{k \in J^{\pm}} \mathbf{e}_k$. Note that, by Theorem 2.1, $H_*(X^{\mathbb{R}}_{B_n}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda_{B_n})} \widetilde{H}_{*-1}(K_S)$ as W_{B_n}-modules. We establish a correspondence between $\mathrm{Row}(\Lambda_{B_n})$ and the power set of $[n]$ in the following way. If $S \in \mathrm{Row}(\Lambda_{B_n})$ is a sum of r rows, say $i_1, \dots, i_r$th rows, then we associate $I_S = \{i_1, \dots, i_r\} \subset [n]$ to it.

One can see that K_S is the full-subcomplex of K_{B_n} consisting of the vertices J such that $|J^{\pm} \cap I_S|$ is odd. By Lemmas 3.7 and 3.8 in [9], all vertices J such that $J^{\pm} \not\subset I_S$ are removable without changing the homotopy type of K_S. Therefore, K_S is homotopy equivalent to the poset complex associated to the odd rank-selected lattice $\mathcal{C}_{I_S}^{odd}$ of faces of the cross-polytope over I_S.

We review some basic facts on the representations of hyperoctahedral groups. (See [16]). A *double partition* (λ, μ) of n, denoted by $(\lambda, \mu) \vdash n$, is an ordered pair of partitions such that $|\lambda| + |\mu| = n$.

Irreducible representations of the hyperoctahedral group W_{B_n} are indexed by the double partitions of n and we let $S^{(\lambda,\mu)}$ be an irreducible representation corresponding to the double partition (λ, μ). Then the dimension of $S^{(\lambda,\mu)}$ is given by the number of *double standard tableaux* of shape (λ, μ), that is the ordered pair of standard tableaux of shape λ and μ respectively such that each number from 1 to n appears exactly once in two tableaux. Hence $\dim(S^{(\lambda,\mu)}) = \binom{n}{k} f^{\lambda} f^{\mu}$, where $\lambda \vdash k$. For a double standard tableau (T_1, T_2), an entry i is a *descent* if

(1) i and $i+1$ are in the same tableau and $i+1$ appears in a lower row than i, or
(2) i is in T_1, $i+1$ is in T_2, or
(3) $i = n$ appears in T_1.

Stanley computed the W_{B_r}-module $\widetilde{H}_*(\mathcal{C}_{I_S}^{odd})$:

Theorem 3.13 ([26])**.** *When $|I_S| = r$,*

$$\widetilde{H}_*(\mathcal{C}_{I_S}^{odd}) \cong \begin{cases} \bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) S^{(\lambda,\mu)} & (* = \lfloor \frac{r-1}{2} \rfloor), \\ 0 & (* \neq \lfloor \frac{r-1}{2} \rfloor), \end{cases}$$

where $b(\lambda, \mu)$ is the number of double standard Young tableaux of shape (λ, μ) whose descent set is the set of odd numbers less than or equal to $r = |\lambda| + |\mu|$.

Similarly to Theorem 3.8, we can prove the following theorem:

Theorem 3.14. *The kth homology $H_k(X_{B_n}^{\mathbb{R}})$ of $X_{B_n}^{\mathbb{R}}$ with the natural action of W_{B_n} is isomorphic to the sum of two induced representations*

$$\bigoplus_{r \in \{2k-1, 2k\}} \left(\mathrm{Ind}_{W_{B_r} \times W_{B_{n-r}}}^{W_{B_n}} \left(\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) S^{(\lambda,\mu)} \otimes S^{(\emptyset;(n-r))} \right) \right)$$

of W_{B_n}, where $S^{(\emptyset;(n-r))}$ is the trivial representation of $W_{B_{n-r}}$ and the embedding $W_{B_r} \times W_{B_{n-r}} \subseteq W_{B_n}$ comes from the diagonal embedding $GL_r(\mathbb{R}) \times GL_{n-r}(\mathbb{R}) \subseteq GL_n(\mathbb{R})$.

The irreducible decomposition is given by a standard argument:

Corollary 3.15. *We have*

$$H_k(X^{\mathbb{R}}_{B_n}) \cong \bigoplus_{(\lambda,\nu)\vdash n} \left(\sum_{r\in\{2k-1,2k\}} \sum_{\mu} b(\lambda,\mu) \right) S^{(\lambda,\nu)},$$

where μ in the inside summation, runs over all partitions that are contained in ν, and ν/μ has $(n-r)$ boxes with at most one box in each column.

Proof. We compute a summand of the decomposition given in Theorem 3.14 using (type B) Littlewood-Richardson rule. Then the result is a direct consequence.

We have the following W_{B_n}-module isomorphisms:

$$\begin{aligned}
&\mathrm{Ind}_{W_{B_r}\times W_{B_{n-r}}}^{W_{B_n}} \left(\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) S^{(\lambda,\mu)} \otimes S^{(\emptyset;(n-r))} \right) \\
\cong &\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) \left(\mathrm{Ind}_{W_{B_r}\times W_{B_{n-r}}}^{W_{B_n}} (S^{(\lambda,\mu)} \otimes S^{(\emptyset;(n-r))}) \right) \\
\cong &\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) \left(\bigoplus_{(\nu_1,\nu_2)} c^{\nu_1}_{\lambda,\emptyset} c^{\nu_2}_{\mu,(n-r)} S^{(\nu_1,\nu_2)} \right) \\
\cong &\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) \left(\bigoplus_{\mu\rightsquigarrow\nu} S^{(\lambda,\nu)} \right) \cong \bigoplus_{(\lambda,\nu)\vdash n} \left(\sum_{\mu\rightsquigarrow\nu} b(\lambda,\mu) \right) S^{(\lambda,\nu)},
\end{aligned}$$

where $\mu \rightsquigarrow \nu$ means that we can obtain ν from μ by adding $n-r$ boxes so that no two new boxes are in one column. In the second isomorphism, we used the Littlewood-Richardson rule for hyperoctahedral group representations (see Chapter 6 of [15]), where $c^{\gamma}_{\alpha,\beta}$ is the well known Littlewood-Richardson number. □

Example 3.16. We let $n = 3$ and $k = 2$, then since $2k = 4 > n$ we need take care of the case when $r = 3$ only. Among 10 possible double partitions $(\lambda,\mu) \vdash 3$, only four of them have nonzero $b(\lambda,\mu)$; $b((1),(1,1)) = b((2),(1)) = b((1,1),(1)) = b((2,1),\emptyset) = 1$. For example, $\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline 2 \\ \hline \end{array} \right)$ is the double standard tableau of shape $((1,1),(1))$ with descent set $\{1,3\}$. We hence have the following decomposition of $H_2(X^{\mathbb{R}}_{B_3})$ into irreducible W_{B_3} modules;

$$H_2(X^{\mathbb{R}}_{B_3}) \cong S^{((1),(1,1))} \oplus S^{((2),(1))} \oplus S^{((1,1),(1))} \oplus S^{((2,1),\emptyset)}.$$

Since $\dim(S^{((1),(1,1))}) = \dim(S^{((2),(1))}) = \dim(S^{((1,1),(1))}) = 3$ and $\dim(S^{((2,1),\emptyset)}) = 2$, $\dim(H_2(X^{\mathbb{R}}_{B_3})) = 11$, which is the number of alternating signed permutations in W_{B_3} (see [9]).

Example 3.17. Due to Corollary 3.15, we have the following decomposition;

$$H_1(X^{\mathbb{R}}_{B_3}) \cong S^{(\emptyset,(2,1))} \oplus S^{(\emptyset,(1,1,1))} \oplus 2S^{((1),(2))} \oplus S^{((1),(1,1))}.$$

Since $\dim(S^{(\emptyset,(2,1))}) = 2$, $\dim(S^{(\emptyset,(1,1,1))}) = 1$, $\dim(S^{((1),(2))}) = \dim(S^{((1),(1,1))}) = 3$, we have

$$\dim(H_1(X^{\mathbb{R}}_{B_3})) = 12 = \binom{3}{2} b_2 + \binom{3}{1} b_1,$$

where b_i is the number of alternating signed permutations in W_{B_i}, as it was shown in [9].

4. Real toric varieties associated to nestohedra

In this section, we deal with real toric varieties corresponding to nestohedra as a generalization of $X^{\mathbb{R}}_{A_n}$ introduced in Section 3. Then, we realize some Foulkes representations as the top homology of those varieties.

A *building set* $\mathfrak{B}$ on a finite set S is a collection of nonempty subsets of S such that

(1) $\mathfrak{B}$ contains all singletons $\{i\}$, $i \in S$,
(2) if $I, J \in \mathfrak{B}$ and $I \cap J \neq \varnothing$, then $I \cup J \in \mathfrak{B}$.

Let $\mathfrak{B}$ be a building set on $[n+1] = \{1, \ldots, n+1\}$. If $[n+1] \in \mathfrak{B}$, then $\mathfrak{B}$ is said to be *connected*.

Definition 4.1. For a connected building set $\mathfrak{B}$ on $[n+1]$, a subset $N \subset \mathfrak{B} \setminus \{[n+1]\}$ is called a *nested set* if both of the following conditions hold:

(N1) For any $I, J \in N$, one has $I \subset J$, $J \subset I$, or $I \cap J = \emptyset$.
(N2) For any collection of $k \geq 2$ disjoint subsets $J_1, \ldots, J_k \in N$, their union $J_1 \cup \cdots \cup J_k$ is not in $\mathfrak{B}$.

Let $V_{\mathfrak{B}} = \mathfrak{B} \setminus \{[n+1]\}$. A simplicial complex $K_{\mathfrak{B}}$ over $V_{\mathfrak{B}}$ called the *nested set complex* is defined to be the set of all nested sets for $\mathfrak{B}$.

It should be noted that $K_{\mathfrak{B}}$ is realizable as the boundary complex of a Delzant polytope $P_{\mathfrak{B}}$. Refer to [10, Section 3] for details. The normal vectors of facets of $P_{\mathfrak{B}}$ define a characteristic matrix $\Lambda_{\mathfrak{B}}$ over $K_{\mathfrak{B}}$: the column of $\Lambda_{\mathfrak{B}}$ indexed by $I \in \mathfrak{B} \setminus \{[n+1]\}$ is $\sum_{k \in I} \mathbf{e}_k$, where $\mathbf{e}_{n+1} = \sum_{i=1}^{n} \mathbf{e}_i$. Then we obtain the real toric variety $X^{\mathbb{R}}_{\mathfrak{B}} := M^{\mathbb{R}}(K_{\mathfrak{B}}, \Lambda_{\mathfrak{B}})$ associated to a connected building set $\mathfrak{B}$. Notice that $V_{\mathfrak{B}}$ is a subset of V_{A_n}, and the corresponding column of $\Lambda_{\mathfrak{B}}$ is equal to that of Λ_{A_n}.

We consider the automorphism group $Aut(\mathfrak{B})$ of $\mathfrak{B}$ consisting of all elements of $\mathfrak{S}_{n+1}$ which preserve $\mathfrak{B}$. Then, $Aut(\mathfrak{B}) \subset \mathfrak{S}_{n+1}$ acts on $\mathfrak{B}$ as well as $K_{\mathfrak{B}}$.

Lemma 4.2. *Let $\mathfrak{B}$ be a connected building set. Then, $Aut(\mathfrak{B})$ acts on $X^{\mathbb{R}}_{\mathfrak{B}}$ and we have an $Aut(\mathfrak{B})$-module isomorphism*

$$H_*(X^{\mathbb{R}}_{\mathfrak{B}}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda_{\mathfrak{B}})} \widetilde{H}_{*-1}\left((K_{\mathfrak{B}})_S\right).$$

Proof. For each $g \in Aut(\mathfrak{B}) \subset \mathfrak{S}_{n+1}$, there exists an $n \times n$-matrix A_g such that $g \cdot \Lambda_{A_n} = A_g \Lambda_{A_n}$ by Lemma 3.1. Since $\Lambda_{\mathfrak{B}}$ is obtainable from Λ_{A_n} by removing columns, we have $g \cdot \Lambda_{\mathfrak{B}} = A_g \Lambda_{\mathfrak{B}}$. Therefore, the lemma follows from Theorem 2.1. □

Now, we consider a particular class of connected building sets on $[n+1]$ whose automorphism group is $\mathfrak{S}_{n+1}$. As in Section 3.2, we denote by $\mathcal{B}_I$ the Boolean algebra over $I \subset [n+1]$, and by $\mathcal{B}_I^Q$ the rank selected Boolean algebra over I with respect to $Q \subset [n+1]$.

For $k \geq 1$, we set $Q_k = \{1, k+1, \ldots, n+1\}$. Then, $\mathfrak{B}_{n,k} = \mathcal{B}_{[n+1]}^{Q_k}$ is a connected building set and $Aut(\mathfrak{B}_{n,k}) = \mathfrak{S}_{n+1}$. We remark that this is a generalisation of $X_{A_n}^{\mathbb{R}}$ as $\mathfrak{B}_{n,1}$ for $k = 1$ is the power set of $[n+1]$ and $X_{\mathfrak{B}_{n,1}}^{\mathbb{R}}$ is exactly equal to $X_{A_n}^{\mathbb{R}}$.

Similarly to the case $X_{A_n}^{\mathbb{R}}$, we identify an element $S \in \mathrm{Row}(\Lambda_{\mathfrak{B}_{n,k}})$ with the even cardinality subsets I_S of $[n+1]$, then Lemma 4.2 specializes to giving an $\mathfrak{S}_{n+1}$-module isomorphism $H_*(X_{\mathfrak{B}_{n,k}}^{\mathbb{R}}) \cong \bigoplus_{I_S \in \binom{[n+1]}{2r}} \widetilde{H}_{*-1}(K_S)$.

Lemma 4.3. *Let n be an odd integer, and $S \in \mathrm{Row}(\Lambda_{\mathfrak{B}_{n,k}})$ the sum of all row vectors of $\mathrm{Row}(\Lambda_{\mathfrak{B}_{n,k}})$, that is, $I_S = [n+1]$. Then, K_S is $\mathfrak{S}_{n+1}$-equivariantly homeomorphic to $\mathcal{B}_{[n+1]}^Q$, where $Q = \{1, 2, \ldots, k\} \cup \{1, 3, 5, \ldots, n\}$.*

Proof. We observe that $V_{\mathfrak{B}_{n,k}} = \{J \subset [n+1] \mid |J| = 1 \text{ or } k < |J| \leq n\}$, and K_S is the full-subcomplex of $K_{\mathfrak{B}_{n,k}}$ on the vertices $J \in V_{\mathfrak{B}_{n,k}}$ such that $|J|$ is odd. It should be noted that any set of k singletons in K_S form a $(k-1)$-simplex in K_S. Therefore, we can obtain $\mathcal{B}_{[n+1]}^Q$ by taking the barycentric subdivision of such $(k-1)$-simplices, and hence, K_S' is homeomorphic to $\mathcal{B}_{I_S}^Q$. It is clear that it is compatible with the $\mathfrak{S}_{n+1}$-action. □

By Theorem 3.6 and Remark 3.7, we obtain:

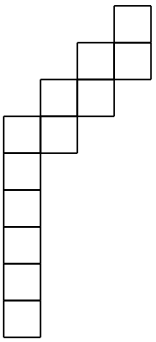
Theorem 4.4. *Let n be an odd integer. The Foulkes representation appears on the top homology of $X_{\mathfrak{B}_{n,k}}^{\mathbb{R}}$. More precisely, we have an $\mathfrak{S}_{n+1}$-module isomorphism*

$$H_*(X_{\mathfrak{B}_{n,k}}^{\mathbb{R}}) \cong \begin{cases} 0 & (* > d) \\ S^{\lambda_Q} & (* = d), \end{cases}$$

where $d = \frac{n+1}{2} + \lfloor \frac{k}{2} \rfloor$, $Q = \{1, 2, \ldots, k\} \cup \{1, 3, 5, \ldots, n\}$, λ_Q is the skew hook with $n+1$ cells whose descent set is Q, and S^{λ_Q} is the skew Specht module.

Proof. For $I_S = [n+1]$, by Lemma 4.3, K_S is homeomorphic to $\mathcal{B}_{[n+1]}^Q$ and $H_{d-1}(K_S) \cong S^Q$ by Theorem 3.6. For $I_S \subsetneq [n+1]$, since I_S has even cardinality, $|I_S| \leq n-1$. Recall that K_S is the full-subcomplex of $K_{\mathfrak{B}_{n,k}}$ on the vertices $J \in V_{\mathfrak{B}_{n,k}}$ such that $|J \cap I_S|$ is odd. Since at most k singletons can be adjacent in K_S, the dimension of K_S is less than $\frac{n-1}{2} + \lfloor \frac{k}{2} \rfloor = d - 1$. □

Example 4.5. Fix $n = 11$ and $k = 4$ or $k = 5$. In this case, $d = 8$, and λ_Q is as follows:



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