

LEFT QUASI-ABUNDANT SEMIGROUPS

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ABSTRACT. A semigroup S is called a weakly abundant semigroup if its every $\tilde{\mathcal{L}}$ -class and every $\tilde{\mathcal{R}}$ -class contains an idempotent. Our purpose is to study an analogue of orthodox semigroups in the class of weakly abundant semigroups. Such an analogue is called a left quasi-abundant semigroup, which is a weakly abundant semigroup with a left quasi-normal band of idempotents and having the congruence condition (C). To build our main structure theorem for left quasi-abundant semigroups, we first give a sufficient and necessary condition of the idempotent set $E(S)$ of a weakly abundant semigroup S being a left quasi-normal band. And then we construct a left quasi-abundant semigroup in terms of weak spined products. Such a result is a generalisation of that of Guo and Shum for left semi-perfect abundant semigroups. In addition, we consider a type Q semigroup which is a left quasi-abundant semigroup having the PC condition.

1. Introduction

Green relations and Green $*$ -relations play an important role to study regular semigroups and abundant semigroups, respectively. To extend the class of regular semigroups and abundant semigroups, two relations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ on a semigroup S are introduced in the following way. Let S be a semigroup and $E(S)$ the set of all idempotents of S . For any $a, b \in S$ a relation $\tilde{\mathcal{L}}$ on S is defined as follows:

$$\tilde{\mathcal{L}} = \{(a, b) \in S \times S \mid (\forall e \in E(S)) ae = a \Leftrightarrow be = b\}.$$

Dually,

$$\tilde{\mathcal{R}} = \{(a, b) \in S \times S \mid (\forall e \in E(S)) ea = a \Leftrightarrow eb = b\}.$$

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It is easy to verify that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}}$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}$. In particular, if S is an abundant semigroup, then $\tilde{\mathcal{L}} = \mathcal{L}^*$ and $\tilde{\mathcal{R}} = \mathcal{R}^*$. Furthermore, if S is a regular semigroup, then $\mathcal{L} = \mathcal{L}^* = \tilde{\mathcal{L}}$ and $\mathcal{R} = \mathcal{R}^* = \tilde{\mathcal{R}}$.

It is a good place to point out that if U is a subset of $E(S)$, Lawson [5] gave relations $\tilde{\mathcal{L}}^U$ and $\tilde{\mathcal{R}}^U$ by the rule that

$$\tilde{\mathcal{L}}^U = \{(a, b) \in S \times S \mid (\forall e \in U) ae = a \Leftrightarrow be = b\}$$

and

$$\tilde{\mathcal{R}}^U = \{(a, b) \in S \times S \mid (\forall e \in U) ea = a \Leftrightarrow eb = b\}.$$

Clear, if $U = E(S)$, then $\tilde{\mathcal{L}}^U = \tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}^U = \tilde{\mathcal{R}}$. So all the facts related to $\tilde{\mathcal{L}}^U$ and $\tilde{\mathcal{R}}^U$ hold for $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$.

According to [4], we call a semigroup S a *weakly abundant semigroup* if each $\tilde{\mathcal{L}}$ -class and each $\tilde{\mathcal{R}}$ -class of S contains an idempotent. We know that the relations \mathcal{L} and \mathcal{L}^* are always right congruences on a semigroup S , but the same need not be true for $\tilde{\mathcal{L}}$. Following the notion of Fountain, Gomes and Gould in [4], we say that a weakly abundant semigroup S satisfies the *congruence condition (C)* if $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ are a right congruence and a left congruence on S , respectively. A weakly abundant semigroup is called an *E-abundant semigroup* if it has the congruence condition (C). Clearly, regular semigroups and abundant semigroups are all *E-abundant semigroups*. *E-abundant semigroups* have been widely studied by Fountain, Petrich, Lawson, Gould, Gomes, Shum and Ren in [1], [4], [8], [9, 11–13] and [14–20].

In the studying of abundant semigroups, El-Qallali and Fountain in [2] call an abundant semigroup S a *quasi-adequate semigroup* if its set of idempotents forms a subsemigroup of S . Such a semigroup is an analogue of orthodox semigroups. It is well known that homomorphisms play a key role in studying the structure of semigroups. Notice that a homomorphism image of an abundant semigroup need not be abundant, it is necessary to introduce a kind of homomorphism in the class of abundant semigroups to preserve Green $*$ -relations. Such a homomorphism between abundant semigroups is called a *good homomorphism*. To establish a structure of quasi-adequate semigroups, besides the notion of good homomorphisms, a condition, which controls the position of idempotents in the product of elements of a semigroup, is needed. This condition is named *idempotent-connected condition* (for brevity, IC). In [2] El-Qallali and Fountain built a structure theorem of type- W semigroups, which are quasi-adequate semigroups satisfying the IC condition. Later on, Guo and Shum [5] first defined and studied left semi-perfect abundant semigroups which are quasi-adequate semigroups whose set of idempotents forms a left quasi-normal band. In this paper, we consider a class of weakly abundant semigroups, namely, *left quasi-abundant semigroups*, which are generalizations of left semi-perfect abundant semigroups of type W in the class of abundant semigroups.

The structure of this paper is as follows. In Section 2, we recall basic definitions and make some elementary observations concerning weakly abundant semigroups. In Section 3 we study properties of left quasi-abundant semigroups which are E -abundant semigroups whose set of idempotents forms a left quasi-normal band. Let S be a weakly abundant semigroup with set of idempotents $E(S)$. We give a sufficient and necessary condition of $E(S)$ being a left quasi-normal band. Section 4 presents our main theorem which build a structure of left quasi-abundant semigroups in terms of weak spined products. Such a result is a generalisation of that of Guo and Shum for left semi-perfect abundant semigroups. In Section 5, we consider a special kind of left quasi-abundant semigroups, namely, type Q -semigroup.

For notations and terminologies not given in this paper, the reader is referred to [2, 3], [6], [7] and [10].

2. Preliminaries

In this section we provide the notions and basic results on weakly abundant semigroups necessary for the rest of the paper. For the proofs and for more details, the reader is referred to [14].

Suppose that S is a weakly abundant semigroup and $a \in S$. As usual, we use \tilde{L}_a and \tilde{R}_a , respectively, to denote the $\tilde{\mathcal{L}}$ -class and $\tilde{\mathcal{R}}$ -class of S containing a , and also we denote idempotents in the \tilde{L}_a -class and the \tilde{R}_a -class by a^* and a^+ , respectively. Note that there need not be a unique choice of a^* and a^+ unless $E(S)$ is a semilattice.

From the definition of relations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$, it is easy to obtain the following lemmas.

Lemma 2.1. *Let S be a weakly abundant semigroup with a set of idempotents $E(S)$. If $a \in S$ and $e \in E(S)$, then the following statements hold on S :*

- (i) $a\tilde{\mathcal{L}}e$ if and only if $ae = a$, and for all $f \in E(S)$, $af = a$ implies that $ef = e$;
- (ii) $a\tilde{\mathcal{R}}e$ if and only if $ea = a$, and for all $f \in E(S)$, $fa = a$ implies that $fe = e$.

Lemma 2.2. *Suppose that S is a semigroup. Let $Reg(S) = \{a \in S \mid (\exists a' \in S) aa'a = a\}$. Then*

- (i) $\tilde{\mathcal{L}} \cap (Reg(S) \times Reg(S)) = \mathcal{L} \cap (Reg(S) \times Reg(S))$;
- (ii) $\tilde{\mathcal{R}} \cap (Reg(S) \times Reg(S)) = \mathcal{R} \cap (Reg(S) \times Reg(S))$.

Now we remark that homomorphisms between semigroups preserve Green relations, but they need not preserve $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$. So the notion of admissible homomorphisms is introduced in [14]. For ease to refer, we state it and some useful results in the following.

A semigroup homomorphism $\varphi : S \rightarrow T$ is *admissible* if $a\tilde{\mathcal{L}}b$ implies $\varphi(a)\tilde{\mathcal{L}}\varphi(b)$, and $a\tilde{\mathcal{R}}b$ implies $\varphi(a)\tilde{\mathcal{R}}\varphi(b)$ for all $a, b \in S$. Naturally, a congruence ρ on S is *admissible* if the natural homomorphism $\rho^\natural : S \rightarrow S/\rho$ is admissible [14].

Lemma 2.3 ([14]). *Suppose that S is an E -abundant semigroup and ρ is an admissible congruence on S . Then S/ρ is an E -abundant semigroup.*

Lemma 2.4 ([14]). *Suppose that S is a weakly abundant semigroups with a set of idempotents $E(S)$ and ρ is a congruence on S . Then the following statements are equivalent:*

- (i) ρ is an admissible congruence;
- (ii) For every $a \in S$ there exist $f \in \tilde{L}_a \cap E(S)$ and $e \in \tilde{R}_a \cap E(S)$ such that for any $g \in E(S)$,
 - (a) $(ag, a) \in \rho$ implies that $(fg, f) \in \rho$;
 - (b) $(ga, a) \in \rho$ implies that $(ge, e) \in \rho$.

Let S be a weakly abundant semigroup and $E(S)$ be the set of its idempotents. For any $e \in E(S)$, $\omega(e) = \{x \in E(S) : xe = ex = x\}$. If K is the semiband generated by $E(S)$, then for any $e \in E(S)$, we use $\langle e \rangle$ to denote the semigroup generated by $eKe \cap E(S)$. In particular, if $E(S)$ forms a semigroup, that is, a band, then $K = E(S)$ and $\langle e \rangle = \omega(e)$.

A weakly abundant semigroup S is said to satisfy the *projection-connected* (for brevity, *PC*) condition if for every $a \in S$ and any $a^\dagger \in \tilde{R}_a, a^* \in \tilde{L}_a$, there exists an isomorphism $\alpha : \langle a^\dagger \rangle \rightarrow \langle a^* \rangle$ such that $xa = a(x\alpha)$ for any $x \in \langle a^\dagger \rangle$. Such an isomorphism α is called a *projection-connected isomorphism* in [14].

Lemma 2.5 ([14]). *Suppose that S is a weakly abundant semigroup with a band of idempotents $E(S)$ and satisfying the PC condition. Then for each element a of S , the following statements hold:*

- (i) for any a^\dagger and $h \in \omega(a^\dagger)$, there exists $g \in \omega(a^*)$ such that $ha = ag$;
- (ii) for any a^* and $e \in \omega(a^*)$, there exists $f \in \omega(a^\dagger)$ such that $ae = fa$.

Our main purpose of this paper is to consider a weakly abundant semigroup whose set of idempotents forms a band. So at the end of this section we recall some results on bands and give some notions of weakly abundant semigroups.

It is well-known that every band is a semilattice of rectangular bands. Let B be a band which is a semilattice Y of rectangular bands E_α on Y for all $\alpha \in Y$. If $E_\alpha E_\beta \subseteq E_\alpha$, then we write $E_\alpha \leq E_\beta$. If $e \in E_\alpha$, then we sometimes denote E_α by $E(e)$.

The following result is obtained from [14] directly, so we omit its proof.

Lemma 2.6. *Let S be an E -abundant semigroup with a band of idempotents $E(S)$ and assume that $b = eaf$ with $e \in E(a^\dagger)$ and $f \in E(a^*)$ for $a, b \in S$. Then for any $b^* \in \tilde{L}_b$ and $b^\dagger \in \tilde{R}_b$, we have $E(b^*) \leq E(f)$ and $E(b^\dagger) \leq E(e)$.*

We next list the notation for some of varieties of bands.

SL	semilattices	$xy = yx,$
\mathcal{LNB}	left normal bands	$axy = ayx,$
\mathcal{RNB}	right normal bands	$xya = yxa,$
\mathcal{LRB}	left regular bands	$ax = axa,$
\mathcal{RLB}	right regular bands	$xa = axa,$
\mathcal{LQNB}	left quasi-normal bands	$axy = axay,$
\mathcal{RQNB}	right quasi-normal bands	$yxa = yaxa.$

It is easy to see that left (resp. right) normal bands are left (resp. right) regular bands, and left (resp. right) regular bands are left (resp. right) quasi-normal bands. For a band, we have:

Lemma 2.7 ([10]). *Let B be a band. Then the following statements are equivalent:*

- (i) B is a left quasi-normal band;
- (ii) B is the spined product of a left regular band and a right normal band;
- (iii) \mathcal{R} is a left regular band congruence and \mathcal{L} is a right normal band congruence.

Finally, we give some special kinds of weakly abundant semigroups.

Definition 2.8. An E -abundant (resp. weakly abundant) semigroup S is said to be (resp. weakly) left quasi-abundant if the set of idempotents of S forms a left quasi-normal band.

Definition 2.9. An E -abundant (resp. weakly abundant) semigroup S whose set of idempotents forms a band is said to be (resp. weakly) $\tilde{\mathcal{R}}$ -unipotent if every $\tilde{\mathcal{R}}$ -class of S contains exactly one idempotent.

3. Left quasi-abundant semigroups

In this section, we focus on left quasi-abundant semigroups.

Theorem 3.1. *Let S be a weakly left quasi-abundant semigroup with a band of idempotents $E(S)$. Then the following statements hold:*

- (i) for any $e \in E(S)$, eSe is a weakly abundant subsemigroup of S and its set of idempotents $E(eSe)$ is a left regular band;
- (ii) for all $e, f \in E(S)$, $efSfe$ is a weakly abundant subsemigroup of S and its set of idempotents $E(efSfe)$ is a left regular band.

Proof. (i) It is easy to see that eSe is a subsemigroup of S . We first show that every $\tilde{\mathcal{L}}$ -class and every $\tilde{\mathcal{R}}$ -class of eSe contains an idempotent. Suppose that $a\tilde{\mathcal{L}}f$ for $a \in eSe$ and $f \in E(S)$. By Lemma 2.1, we have $af = a$. Clearly, $aef = a$. Again, if $ag = a$ for any $g \in E(eSe)$, then $fg = f$ by Lemma 2.1. Hence, $efg = ef$. Since $ae = a$ and $a\tilde{\mathcal{L}}f$, it follows that $fe = f$ and $ef = efe \in E(eSe)$. Thus, by Lemma 2.1(i), $a\tilde{\mathcal{L}}(eSe)ef$, that is, every

$\tilde{\mathcal{L}}$ -class of eSe contains an idempotent. Similarly, we can prove that every $\tilde{\mathcal{R}}$ -class of eSe contains an idempotent belonging to $E(eSe)$. It follows that eSe is a weakly abundant subsemigroup of S .

To show that $E(eSe)$ is a left regular band, let $x, y \in E(eSe)$. Since $E(S)$ is a left quasi-normal band, we have $xy = xye = xyxe = xyx$ and so $E(eSe)$ is a left regular band.

(ii) It is easy to show that $efSfe$ is a subsemigroup of S and $E(efSfe)$ is a left regular band for all $e, f \in E(S)$. We now show that every $\tilde{\mathcal{L}}$ -class and every $\tilde{\mathcal{R}}$ -class of $efSfe$ contains an idempotent belonging to $E(efSfe)$. Suppose that $a \in efSfe$, $g \in E(S)$ and $a\tilde{\mathcal{L}}g$. On the one hand, by Lemma 2.1, $aefeg = a$. Since $E(S)$ is a left quasi-normal band, it follows that $aefg = a$. On the other hand, suppose that $ah = a$ for any $h \in E(efSfe)$. From $a\tilde{\mathcal{L}}g$, we have that $gfe = g$, $gh = g$ and $efgh = efg$. Clearly, $efg = efgfe \in E(efSfe)$. Hence, $efg\mathcal{L}(efSfe)a$. This shows that every $\tilde{\mathcal{L}}$ -class of $efSfe$ contains an idempotent. Similarly, we can prove that every $\tilde{\mathcal{R}}$ -class of $efSfe$ contains an idempotent in $E(efSfe)$. Hence $efSfe$ is a weakly abundant subsemigroup of S where $E(efSfe)$ is a left regular band. \square

Theorem 3.2. *Let S be a left quasi-abundant semigroup. Then the following statements hold:*

(i) *a relation*

$$\eta = \{(x, y) \in S \times S : (\exists f \in E(y^*))x = yf\}$$

is the minimum admissible congruence on S such that S/η is an $\tilde{\mathcal{R}}$ -unipotent semigroup;

(ii) $\eta \cap \tilde{\mathcal{L}} = \iota_S$.

Proof. (i) We first show that η is an equivalence relation on S . Suppose that S is a left quasi-abundant semigroup and $x\eta y$ for $x, y \in S$. Then, by the definition of η , there exists $e \in E(y^*)$ such that $x = ye$. From Lemma 2.6, we have that $E(x^*) \leq E(e) = E(y^*)$. Again, by $x = ye$, we deduce that $y = yy^*ey^* = yey^* = xy^* = xx^*y^*$. This implies that $E(y^*) \leq E(x^*y^*) \leq E(x^*)$. Thus, $E(x^*) = E(y^*)$. Therefore, $y\eta x$, that is, η is symmetric. If $y\eta z$, then there is $f \in E(z^*)$ such that $y = zf$. By using the same arguments as above, we have $E(y^*) = E(z^*)$. Thus $fe \in E(z^*)E(y^*) \subseteq E(z^*)$. But $x = z \cdot fe$. This implies that η is transitive. Since S is a weakly abundant semigroup, it is clear that η is reflexive. So η is an equivalence relation on S .

To show that η is a congruence, let $x, y, z \in S$ and $x\eta y$. Then $x = ye$ for some $e \in E(y^*)$. Since $E(S)$ is a left quasi-normal band, it follows that

$$xz = yez = yy^*ez^\dagger z = yy^*ey^*z^\dagger z = yz.$$

Hence $xz\eta yz$. This shows that η is a right congruence on S . One the other hand, notice that

$$zx = zye = zy \cdot (zy)^*e.$$

Again by Lemma 2.6, we have $E((zx)^*) \leq E(e) = E(y^*)$. Thus, $(zx)^*y^* \in E((zx)^*)$. However, we have

$$zy = zyy^*ey^* = zye y^* = zxy^* = zx \cdot (zx)^*y^*.$$

This implies that $(zx)\eta(zy)$ and so η is a left congruence on S . Hence, η is a congruence on S .

It is easy to see that the restriction of η to $E(S)$ is just \mathcal{R} and that η is idempotent-pure. We observe that $E(S/\eta) = E(S)/\mathcal{R}$ and thereby $E(S/\eta)$ is a left regular band. Next we show that every $\tilde{\mathcal{L}}$ -class of S/η contains an idempotent. It is clear that $(a\eta)(a^*\eta) = a\eta$ for all $a\eta \in S/\eta$. If $(a\eta)(e\eta) = (a\eta)$ for any $e\eta \in E(S/\eta)$, then we have

$$\begin{aligned} (a\eta)(e\eta) = a\eta &\Rightarrow (ae)\eta = a\eta \\ &\Rightarrow ae = ag && (\exists g \in E(a^*)) \\ &\Rightarrow aea^* = aa^*ga^* && (\exists g \in E(a^*)) \\ &\Rightarrow aea^* = a \\ &\Rightarrow a^*ea^* = a^* && (a\tilde{\mathcal{L}}a^*) \\ &\Rightarrow (a^*\eta)(e\eta)(a^*\eta) = a^*\eta \\ &\Rightarrow (a^*\eta)(e\eta) = a^*\eta && (E(S/\eta) \text{ is a left regular band}). \end{aligned}$$

Hence, by Lemma 2.1, $(a\eta)\tilde{\mathcal{L}}(S/\eta)(a^*\eta)$. This shows that every $\tilde{\mathcal{L}}$ -class of S/η contains an idempotent. Similarly, we can prove that every $\tilde{\mathcal{R}}$ -class of S/η contains an idempotent.

We now prove that η is an admissible congruence on S . It is clear that $a^*\tilde{\mathcal{L}}a\tilde{\mathcal{R}}a^\dagger$ for any $a \in S$. Firstly we show that for any $g \in E(S)$, $(ag, a) \in \eta$ implies that $(a^*g, a^*) \in \eta$. Let $(ag, a) \in \eta$ for any $g \in E(S)$. Then, there exists $h \in E((ag)^*)$ such that $a = agh$. Since $\tilde{\mathcal{L}}$ is a right congruence on S , it is clear that $ag\tilde{\mathcal{L}}a^*g$ and so $E((ag)^*) = E(a^*g)$. Hence, using $a^*\tilde{\mathcal{L}}a$, we obtain that $a^* = a^*gh$ which implies that $(a^*g, a^*) \in \eta$. Now we prove that $(ga, a) \in \eta$ implies that $(ga^\dagger, a^\dagger) \in \eta$ for any $g \in E(S)$. Then, there exists $f \in E(a^*)$ such that $ga = af$. Hence, $ga = aa^*fa^* = a$ gives that $ga^\dagger = a^\dagger$. This implies that $(ga^\dagger, a^\dagger) \in \eta$. By Lemma 2.4, we have already proved that η is an admissible congruence on S .

Since $E(S/\eta)$ is a left regular band, we deduce that every $\tilde{\mathcal{R}}$ -class of S/η contains exactly one idempotent. Because η is an admissible congruence on S , it follows from Lemma 2.3 that S/η is an E -abundant semigroup and so S/η is an $\tilde{\mathcal{R}}$ -unipotent semigroup.

Now let ρ be an admissible congruence on S such that S/ρ is an $\tilde{\mathcal{R}}$ -unipotent semigroup. If $x, y \in S$ and $x\eta y$, then there exists $f \in E(y^*)$ such that $x = yf$. Since ρ is an admissible congruence on S , we obtain that $(y\rho)\tilde{\mathcal{L}}(y^*\rho)$. Because $E(S/\rho)$ is a left regular band, we have

$$x\rho = y\rho \cdot y^*\rho \cdot f\rho = y\rho \cdot y^*\rho \cdot f\rho \cdot y^*\rho = y\rho \cdot y^*\rho = y\rho$$

and so $\eta \subseteq \rho$. This implies that η is the minimum admissible congruence on S such that S/η is an $\tilde{\mathcal{R}}$ -unipotent semigroup.

(ii) If $(x, y) \in \tilde{\mathcal{L}} \cap \eta$, then there exists $f \in E(y^*)$ such that $x = yf$. This implies that $x = xf$. Now, from $x\tilde{\mathcal{L}}y$, we have $y = yf$. Thus $x = y$ and whence $\eta \cap \tilde{\mathcal{L}} = \iota_S$. □

Next we give an equivalent description of weakly left quasi-abundant semigroups. To do this we need a concept of E -isomorphism. A surjective homomorphism φ of a semigroup S onto another semigroup T is an E -isomorphism if the restriction of φ to eSf is injective for all $e, f \in E(S)$.

Theorem 3.3. *Let S be a weakly abundant semigroup with set of idempotents $E(S)$. Then $E(S)$ is a left quasi-normal band if and only if there exists a weakly $\tilde{\mathcal{R}}$ -unipotent semigroup T and an E -isomorphism φ of S onto T .*

Proof. We begin with that S is a weakly abundant semigroup and $E(S)$ is a left quasi-normal band. By Theorem 3.2, η is the minimum admissible congruence on S such that S/η is a weakly $\tilde{\mathcal{R}}$ -unipotent semigroup. Denote by φ the natural homomorphism of S onto S/η . Now let $e, f \in E(S)$ and $x, y \in eSf$. Suppose that $x\varphi = y\varphi$. Then $x = yg$ for some $g \in E(y^*)$. Hence, we have that

$$x = x f = y g f = y y^* g f = y y^* g y^* f = y y^* f = y.$$

This implies that the restriction of φ to eSf is injective. Therefore φ is an E -admissible isomorphism of S onto S/η .

Conversely, suppose that there exist a weakly $\tilde{\mathcal{R}}$ -unipotent semigroup T and an E -isomorphism of S onto T . Also, let φ be the E -isomorphism of S onto T . We first prove that $E(S)$ is a band. Let $x, y \in E(S)$. Clearly, $x\varphi, y\varphi \in E(T)$ and

$$(xy)\varphi = x\varphi \cdot y\varphi = x\varphi \cdot y\varphi \cdot x\varphi = (xyx)\varphi = (xyxy)\varphi.$$

Hence, by our hypothesis, $xy = (xy)^2$. This shows that $E(S)$ is a band.

Now let $x, y, z \in E(S)$. Since T is a weakly $\tilde{\mathcal{R}}$ -unipotent semigroup, it follows that $E(T)$ is a left regular band and so $(xyz)\varphi = (xyxz)\varphi$. Notice that φ is the E -admissible isomorphism of S onto T . We obtain that $xyz = xyxz$. Thus $E(S)$ is a left quasi-normal band and S is a weakly left quasi-abundant semigroup. □

4. A structure theorem

In this section, we establish a structure theorem of left quasi-abundant semigroups.

Let Y be a semilattice. Let T be an $\tilde{\mathcal{R}}$ -unipotent semigroup whose set of idempotents forms a left regular band E such that $E = \bigcup_{\alpha \in Y} E_\alpha$ where Y is the semilattice and E_α is a left zero band. Let B be a right normal band such that $B = \mathcal{S}(Y; B_\alpha; \varphi_{\alpha, \beta})$, i.e., B is a strong semilattice of right zero semigroups B_α . For each $a \in T$, if $a^* \in E_\alpha$, then we write α as a^\diamond . Consider a

set $M = \{(a, x) \in T \times B \mid x \in B_{a^\circ}\}$ and define a multiplication on the set M by the rule that

$$(4.1) \quad (a, x) \cdot (b, y) = (ab, y\varphi_{b^\circ, (ab)^\circ})$$

for any $(a, x), (b, y) \in M$.

It is routine to check that (M, \cdot) is a semigroup and its set of idempotents is the set $\{(a, x) \in M \mid a \in E(T)\}$ which forms a subsemigroup. The semigroup (M, \cdot) constructed above is called a *weak spined product* of an E -abundant semigroup T having a left regular band of idempotents, and a right normal band B . We denote it by $W(T, B)$.

Lemma 4.1. *The semigroup $W(T, B)$ is a weakly abundant semigroup.*

Proof. Suppose that $(a, x) \in W(T, B)$. Take $(a^*, x) \in E(W(T, B))$. Then we have

$$(a, x) \cdot (a^*, x) = (aa^*, x\varphi_{(a^*)^\circ, (aa^*)^\circ}) = (a, x\varphi_{a^\circ, a^\circ}) = (a, x).$$

Let $(e, y) \in E(W(T, B))$ be such that $(a, x) \cdot (e, y) = (a, x)$. This implies that $ae = a$ and $y\varphi_{e^\circ, (ae)^\circ} = x$. Hence,

$$\begin{aligned} (a^*, x) \cdot (e, y) &= (a^*e, y\varphi_{e^\circ, (a^*e)^\circ}) \\ &= (a^*, y\varphi_{e^\circ, a^\circ}) \\ &= (a^*, x). \end{aligned}$$

By Lemma 2.1, we have that $(a, x)\tilde{\mathcal{L}}(a^*, x)$ in $W(T, B)$. To see that each $\tilde{\mathcal{R}}$ -class of $W(T, B)$ contains an idempotent, we suppose $(a, x) \in W(T, B)$ and put $(a^\dagger, y) \in E(W(T, B))$ where $y \in B_{(a^\dagger)^\circ}$. It is clear that $(a^\dagger, y) \cdot (a, x) = (a^\dagger a, x\varphi_{a^\circ, (a^\dagger a)^\circ}) = (a, x\varphi_{a^\circ, a^\circ}) = (a, x)$. For any $(f, z) \in E(M)$, let $(f, z) \cdot (a, x) = (a, x)$. This implies that $fa = a$. From $a\tilde{\mathcal{R}}a^\dagger$, it follows that

$$\begin{aligned} (f, z) \cdot (a^\dagger, y) &= (fa^\dagger, y\varphi_{(a^\dagger)^\circ, (fa^\dagger)^\circ}) \\ &= (a^\dagger, y\varphi_{(a^\dagger)^\circ, (a^\dagger)^\circ}) \\ &= (a^\dagger, y). \end{aligned}$$

This shows that $(a, x)\tilde{\mathcal{R}}(a^\dagger, y)$ in $W(T, B)$ and so $W(T, B)$ is a weakly abundant semigroup. \square

Lemma 4.2. *The semigroup $W(T, B)$ satisfies the congruence conditions (C).*

Proof. To see that $\tilde{\mathcal{L}}$ is a right congruence on $W(T, B)$, we suppose that $(a, x)\tilde{\mathcal{L}}(b, y)$ for any $(a, x), (b, y) \in W(T, B)$. By the proof of Lemma 4.1, we have $(a, x)\tilde{\mathcal{L}}(a^*, x)$ and $(b, y)\tilde{\mathcal{L}}(b^*, y)$ in $W(T, B)$.

Now let $(c, u) \in W(T, B)$. We first show that

$$(a, x)(c, u)\tilde{\mathcal{L}}(a^*, x)(c, u).$$

Suppose that $(e, v) \in E(W(T, B))$ and $(a, x)(c, u)(e, v) = (a, x)(c, u)$. This implies that $ace = ac$ and $v\varphi_{e^\circ, (ac)^\circ} = u\varphi_{c^\circ, (ac)^\circ}$. Since T is an E -abundant semigroup, it follows that $ac\tilde{\mathcal{L}}a^*c$ in T . Hence, by definition, we have $a^*ce = a^*c$. Consequently, we obtain that $(a^*, x)(c, u)(e, v) = (a^*, x)(c, u)$. Conversely, it is easy to verify that if $(a^*, x)(c, u)(e, v) = (a^*, x)(c, u)$, then $(a, x)(c, u)(e, v) = (a, x)(c, u)$. This shows that $(a, x)(c, u)\tilde{\mathcal{L}}(a^*, x)(c, u)$.

Similarly, we have $(b, y)(c, u)\tilde{\mathcal{L}}(b^*, y)(c, u)$. Clearly, $(a^*, x)\tilde{\mathcal{L}}(b^*, y)$. Hence, by Lemma 2.2, $(a^*, x)\mathcal{L}(b^*, y)$ and so $(a^*, x)(c, u)\mathcal{L}(b^*, y)(c, u)$ because \mathcal{L} is a right congruence on $W(T, B)$. Thus, we have $(a, x)(c, u)\tilde{\mathcal{L}}(b, y)(c, u)$, i.e., $\tilde{\mathcal{L}}$ is a right congruence on $W(T, B)$. Similarly, we can prove that $\tilde{\mathcal{R}}$ is a left congruence on $W(T, B)$ and so $W(T, B)$ satisfies the congruence conditions (C). \square

Theorem 4.3. *Let T be an $\tilde{\mathcal{R}}$ -unipotent semigroup and B a right normal band. Then $W(T, B)$ is a left quasi-abundant semigroup.*

Conversely, every left quasi-abundant semigroup can be constructed as a weak spined product mentioned above.

Proof. Suppose that T is an $\tilde{\mathcal{R}}$ -unipotent semigroup, B is a right normal band and $M = W(T, B)$ is the weak spined product of T and B . Then it is easy to see that $E(T)$ is a left regular band. By the formula (4.1), we have known that $E(W(T, B))$ is exactly a spined product of a left regular band $E(T)$ and a right normal band B . It follows from Lemma 2.7 that $E(W(T, B))$ is a left quasi-normal band. Again using Lemma 4.1 and Lemma 4.2, we obtain that $M = W(T, B)$ is a left quasi-abundant semigroup.

Conversely, suppose that S is a left quasi-abundant semigroup. Clearly, $E(S)$ is a left quasi-normal band. Let $Y = E(S)/\mathcal{J}$, where \mathcal{J} is the usual Green relation. Then Y is a semilattice and by Lemma 2.7, $E(S)$ is a spined product of a left regular band $E(S)/\mathcal{R}$ and a right normal band $E(S)/\mathcal{L}$ with respect to Y . By Theorem 3.2, we see that S/η is an $\tilde{\mathcal{R}}$ -unipotent semigroup. Again using Lemma 2.7, we immediately have that $E(S)/\mathcal{L}$ is a right normal band. Consequently, we obtain a weak spined product of the semigroups S/η and $E(S)/\mathcal{L}$, denoted by $W(S/\eta, E(S)/\mathcal{L})$.

It remains to prove that the following mapping

$$\theta : S \rightarrow W(S/\eta, E(S)/\mathcal{L}), \quad s \mapsto (s\eta, s^*\mathcal{L})$$

is a semigroup isomorphism. Clearly, θ is well-defined. It follows from Theorem 3.2(ii) that θ is injective.

Now, let $E(S)/\mathcal{L} = (Y; B_\alpha)$, which is the semilattice decomposition of $E(S)/\mathcal{L}$ over the rectangular bands B_α . If $(a, x) \in W(S/\eta, E(S)/\mathcal{L})$, then there exists $t \in S$ such that $a = t\eta$ and $x \in B_{(t\eta)^\circ}$. We observe that $B_{(t\eta)^\circ} = E(t^*)/\mathcal{L}$ and so $f \in E(t^*)$ such that $f\mathcal{L} = x$. Since $\tilde{\mathcal{L}}$ is a right congruence on T and $f \in E(t^*)$, we have that $t\tilde{\mathcal{L}}t^*f$. Also from $f \cdot t^*f = f$, it follows that $t\tilde{\mathcal{L}}t^*f\tilde{\mathcal{L}}f$ in T and $t\tilde{\mathcal{L}}f = t \cdot f$ for $f \in E(t^*)$. Hence, by definition, $(t\tilde{\mathcal{L}}f)\eta = t\eta$. Thus, $(t\tilde{\mathcal{L}}f)\theta = (a, x)$ and θ is surjective.

Next we will prove that θ is a homomorphism. For any $s, t \in S$, since $st = st \cdot t^*$, it follows that $(st)^* = (st)^* \cdot t^*$ and so $(st)^* \mathcal{L} = (st)^* \mathcal{L} \cdot t^* \mathcal{L}$. Consequently, by Theorem 3.2, we deduce that

$$\begin{aligned}(st)\theta &= ((st)\eta, (st)^* \mathcal{L}) \\ &= (s\eta \cdot t\eta, (st)^* \mathcal{L} \cdot t^* \mathcal{L}),\end{aligned}$$

and

$$\begin{aligned}(s\theta)(t\theta) &= (s\eta, s^* \mathcal{L})(t\eta, t^* \mathcal{L}) \\ &= (s\eta \cdot t\eta, (t^* \mathcal{L})\varphi_{(t\eta)^\circ, (s\eta \cdot t\eta)^\circ}) \\ &= (s\eta \cdot t\eta, (t^* \mathcal{L})\varphi_{(t\eta)^\circ, ((st)\eta)^\circ}) \\ &= (s\eta \cdot t\eta, (st)^* \mathcal{L} \cdot t^* \mathcal{L}).\end{aligned}$$

Hence, θ is a semigroup homomorphism. This shows that θ is a semigroup isomorphism. \square

At the end of this section we should mention that if we replace the whole set of idempotents by the distinguished subset of idempotents U and use relations $\tilde{\mathcal{L}}^U$ and $\tilde{\mathcal{R}}^U$ instead of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ in the definition of left quasi-abundant semigroups, we obtain a general left quasi-abundant semigroup with respect to U , where U forms a left quasi-normal band. Let U take the role of the whole set of idempotents $E(S)$ in the proof of Theorem 4.3. It is routine to obtain a general structure theorem of left quasi-abundant semigroups with respect to U as Theorem 4.3. Here we should stress that U -orthodox semigroups mentioned in [14] meet the PC conditions, while left quasi-abundant semigroups with respect to U do not need satisfy the PC conditions.

5. Type Q semigroups

In this section, we consider a special kind of left quasi-abundant semigroups, namely, type Q semigroup. We first give its definition.

Definition 5.1. A left quasi-abundant semigroup S is said to be a type Q semigroup if S satisfies the PC conditions.

Now we give a description of type Q semigroups.

Theorem 5.2. *Suppose that an E -abundant semigroup S satisfies the PC condition. Then S is a type Q semigroup if and only if the following conditions hold:*

- (i) η is the admissible congruence on S such that S/η is an $\tilde{\mathcal{R}}$ -unipotent semigroup, where η is given in Theorem 3.2;
- (ii) a relation

$$\xi = \{(x, y) \in S \times S : (\exists e \in E(y^\dagger))x = ey\}$$

is an admissible congruence on S such that S/ξ is an $\tilde{\mathcal{L}}$ -unipotent semigroup.

Proof. Necessity. Suppose that S is a type Q semigroup. It follows from Theorem 3.2 that Condition (i) holds. It is necessary to prove that Condition (ii) holds. Firstly, we prove that ξ is an admissible congruence on S . By Lemma 2.6 and by using similar arguments to those for the relation η , we can easily check that ξ is an equivalence relation on S .

Now let $x, y, z \in S$ and $x\xi y$. Then $x = ey$ for some $e \in E(y^\dagger)$. Clearly, $xz = e(yz)^\dagger yz$. By Lemma 2.6, we have that $E(xz)^\dagger \leq E(e) = E(y^\dagger)$ and $E((xz)^\dagger) \leq E((yz)^\dagger)$. On the other hand, $x = ey$ implies that $y = y^\dagger ey^\dagger y = y^\dagger x$. Hence, $yz = y^\dagger xz$. Again by Lemma 2.6, we have $E((yz)^\dagger) \leq E((xz)^\dagger)$ and so $E((xz)^\dagger) = E((yz)^\dagger)$. Thus, $y^\dagger(xz)^\dagger \in E((xz)^\dagger)$. But, from $y = y^\dagger x$, we have $yz = y^\dagger(xz)^\dagger(xz)$ which implies that $(xz)\xi(yz)$. This shows that ξ is a right congruence on S .

To see that ξ is also a left congruence on S , we notice that

$$zx = zey = z \cdot z^*ey^\dagger \cdot y = z \cdot z^*ez^*y^\dagger \cdot y = z \cdot z^*ez^* \cdot y$$

and $z^*ez^* \in \omega(z^*)$. By our hypothesis that S satisfies PC condition, it follows from Lemma 2.5 that there exists $f \in \omega(z^\dagger)$ such that $zx = fzy$. It follows from Lemma 2.6 that $E((zx)^\dagger) \leq E(f)$ and $E((zx)^\dagger) \leq E((zy)^\dagger)$. Since S satisfies the PC condition, by Lemma 2.5 there exists $g \in \omega(z^\dagger)$ such that $z \cdot z^*y^\dagger z^* = gz$. Hence, we have

$$zy = zy^\dagger x = z \cdot z^*y^\dagger z^*x^\dagger \cdot x = z \cdot z^*y^\dagger z^* \cdot x = gzx.$$

This implies that $E((zy)^\dagger) \leq E((zx)^\dagger)$ and so $E((zx)^\dagger) = E((zy)^\dagger)$. By the formula $zx = f(zy)^\dagger \circ zy$ and $f(zy)^\dagger \in E((zy)^\dagger)$, we obtain that ξ is a left congruence on S . Consequently, ξ is a congruence on S .

We next show that ξ is an admissible congruence on S . Clearly, $a^* \tilde{\mathcal{L}}a \tilde{\mathcal{R}}a^\dagger$ for any $a \in S$. We first prove that $(ag, a) \in \xi$ implies that $(a^*g, a^*) \in \xi$ for any $g \in E(S)$. Suppose that $(ag, a) \in \xi$ for any $g \in E$. Then, there exists $h \in E(a^\dagger)$ such that $ag = ha$. Hence, $ag = a^\dagger ha^\dagger a = a$. From $a^* \tilde{\mathcal{L}}a$, it follows that $a^*g = a^*$ which implies that $(a^*g, a^*) \in \xi$. Next we prove that $(ga, a) \in \xi$ implies that $(ga^\dagger, a^\dagger) \in \xi$ for any $g \in E(S)$. Since $\tilde{\mathcal{R}}$ is a left congruence on S , it follows that $(ga) \tilde{\mathcal{R}}(ga^\dagger)$ and so $E((ga)^\dagger) = E(ga^\dagger)$. Let $(ga, a) \in \xi$ for any $g \in E$. Then, $a = h \cdot ga$ for some $h \in E((ga)^\dagger)$. By $a \tilde{\mathcal{R}}a^\dagger$, it follows that $a^\dagger = h \cdot ga^\dagger$ which implies that $(a^\dagger, ga^\dagger) \in \xi$. By Lemma 2.4, we have that ξ is an admissible congruence on S .

Since ξ is an admissible congruence on S , we can verify that S/ξ is an E -abundant semigroup. Notice that the restriction of ξ to $E(S)$ is just \mathcal{L} and $E(S)$ is a left quasi-normal band. We obtain that $E(S/\xi) = E(S)/\mathcal{L}$ and so $E(S/\xi)$ is a right normal band. Thus S/ξ is an $\tilde{\mathcal{L}}$ -unipotent semigroup.

Sufficiency. We first show that $E(S)$ is a band. Suppose that $a, b \in E(S)$. Clearly, $a\eta, b\eta \in E(S/\eta)$. Since $E(S/\eta)$ is a left regular band, it follows that $a\eta b\eta = (ab)\eta \in E(S/\eta)$. Hence, we have

$$(ab)\eta \in E(S/\eta) \Rightarrow (ab)^2\eta = (ab)\eta$$

$$\begin{aligned} \Rightarrow (ab)^2 &= ab \cdot e && (\exists e \in E((ab)^*)) \\ \Rightarrow (ab)^2(ab)^* &= ab \cdot (ab)^*e(ab)^* \\ \Rightarrow (ab)^2 &= ab. \end{aligned}$$

This shows that $E(S)$ forms a subsemigroup. Next, we need only prove that $E(S)$ is a left quasi-normal band. In fact, since the restriction of η to $E(S)$ is \mathcal{R} and the restriction of ξ to $E(S)$ is \mathcal{L} , as well as η and ξ are both idempotent-pure, by hypothesis, we can easily see that $E(S/\eta) = E(S)/\mathcal{R}$ is a left regular band and $E(S/\xi) = E(S)/\mathcal{L}$ a right normal band, and hence, by Lemma 2.7, $E(S)$ is a left quasi-normal band. \square

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