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CONSECUTIVE CANCELLATIONS IN FILTERED FREE RESOLUTIONS

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Abstract. Let M be a finitely generated module over a regular local ring $(R, \mathfrak{n})$. We will fix an $\mathfrak{n}$-stable filtration for M and show that the minimal free resolution of M can be obtained from any filtered free resolution of M by zero and negative consecutive cancellations. This result is analogous to [10, Theorem 3.1] in the more general context of filtered free resolutions. Taking advantage of this generality, we will study resolutions obtained by the mapping cone technique and find a sufficient condition for the minimality of such resolutions. Next, we give another application in the graded setting. We show that for a monomial order σ, Betti numbers of I are obtained from those of $\mathrm{LT}_\sigma(I)$ by so-called zero σ-consecutive cancellations. This provides a stronger version of the well-known cancellation "cancellation principle" between the resolution of a graded ideal and that of its leading term ideal, in terms of filtrations defined by monomial orders.

1. Introduction

Consider a regular local ring $(R, \mathfrak{n})$ and let M be a finitely generated R-module. Let $\mathbb{M} = \{M_n\}_{n\geq 0}$ be a stable filtration on M. The associated graded module $\mathrm{gr}_{\mathbb{M}}(M) = \bigoplus_i M_i/M_{i+1}$ has a natural structure as a finitely generated graded P-module in which $P = \mathrm{gr}_{\mathfrak{n}}(R) = \bigoplus \mathfrak{n}^i/\mathfrak{n}^{i+1}$ is the associated graded ring of R with respect to the $\mathfrak{n}$-adic filtration. In the literature, starting from classical results by Northchott, Abhyankar, Matlis, and Sally, several authors have found basic numerical invariants of the module M. Often the Hilbert function of $\mathrm{gr}_{\mathbb{M}}(M)$ (see [11] as an overview) has been the central tool. As it is clear that more information can be achieved from a minimal free resolution of M, this paper is mainly devoted to study free resolutions. Our approach is to find information about minimal free resolution of M through minimizing a filtered free resolution of M.

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Robbiano, by using graded free resolutions of $\mathrm{gr}_{\mathbb{M}}(M)$, has introduced a special filtered free resolution of M (see [7] and also [9]). In [10, Thorem 3.1], it was shown that the Betti numbers of M as an R-module can be obtained from the Betti numbers of $\mathrm{gr}_{\mathbb{M}}(M)$ as a P-module by a sequence of *negative* consecutive cancellations (Definition 3.3). Later, Sammartano [13, Theorem 2] extended this result and showed that for finitely generated modules M and N over a Noetherian local ring R, the bigraded Hilbert series of $\mathrm{gr}(\mathrm{Tor}^R(M,N))$ is obtained from that of $\mathrm{Tor}^{\mathrm{gr}(R)}(\mathrm{gr}(M),\mathrm{gr}(N))$ by negative consecutive cancellations.

In this paper, we first prove a statement similar to [10, Theorem 3.1] in the more general situation of filtered free resolutions, and next, we give two applications of it in the local and the graded setting. For our goal, corresponding to an arbitrary filtered free resolution $(\mathbf{F}_., \delta_.)$ of M, we define the valuation sequence $\{\gamma_i = \sum_{j\in\mathbb{N}} \gamma_{i,j}\}$ (Definition 3.4) and we show that the Betti numbers of M can be obtained from the valuation sequence of $(\mathbf{F}_., \delta_.)$ by *zero and negative* consecutive cancellations (see Theorem 3.6). Briefly, starting from given sequences $\{c_{ij}\}$ and $\{c_i = \sum_{j\in\mathbb{N}} c_{i,j}\}$, we fix an index i, and choose j and j' such that $j \le j'$ and $c_{ij}, c_{i-1,j'} > 0$. Then replace c_{ij} by $c_{ij}-1$ and $c_{i-1,j'}$ by $c_{i-1,j'}-1$, and accordingly, replace in the second sequence c_i by c_i-1 and c_{i-1} by $c_{i-1}-1$. This substitution is called an *i negative consecutive cancellation* when $j < j'$ and an *i zero consecutive cancellation* when $j = j'$.

Theorem 3.6 can help to study the minimal free resolution of some modules through suitable filtered free resolutions. For example, in Section 4, we introduce a filtered free resolution for a given submodule of a free R-module by applying the mapping cone technique. This resolution is different from the special filtered free resolution defined by Robbiano, and in some cases it is more convenient to apply Theorem 3.6 for this resolution instead of getting information via [10, Theorem 3.1] (see Example 4.8 and Corollary 4.9).

Let us explain this idea in more details. Assume that M is a submodule of a filtered free R-module $\bigoplus_{i=1}^s R(-\nu_i)$ and $f \in \bigoplus_{i=1}^s R(-\nu_i) \setminus M$ is of valuation d. Then we have the following short exact sequence of filtered modules:

$$0 \to R/(M:(f))(-d) \to \bigoplus_{i=1}^s R(-\nu_i)/M \to \bigoplus_{i=1}^s R(-\nu_i)/(M+(f)) \to 0.$$

We prove that the mapping cone technique gives us a filtered free resolution of $\bigoplus_{i=1}^s R(-\nu_i)/(M+(f))$. So, we can apply Theorem 3.6 and describe the admissible consecutive cancellations in the filtered free resolution of $\bigoplus_{i=1}^s R(-\nu_i)/(M+(f))$ obtained by the mapping cone technique (see Corollary 4.4).

It is clear that in general, the result of the mapping cone is not a minimal free resolution even if we initially consider the minimal free resolutions of $R/(M:(f))(-d)$ and $\bigoplus_{i=1}^s R(-\nu_i)/M$. In Theorem 4.7, we present a sufficient condition for minimality of such resolution. The importance of this result is distinguished when we apply it for classes of modules in which Theorem 4.7 can compute the Betti numbers while [10, Theorem 3.1] determines some possible consecutive cancellations that actually do not happen. This situation will occur in Example 4.8 and Corollary 4.9. In Corollary 4.9, using Theorem 4.7, we compute Betti numbers of stretched Artinian local rings of almost maximal Cohen-Macauly type.

In the last section, we give another application of our approach in the graded setting. Let σ be a term ordering on the set of monomials of P. It is well-known that the graded Betti numbers of a graded ideal I of P can be obtained from those of $\mathrm{LT}_\sigma(I)$ by zero consecutive cancellations (see [3, Corollary 1.21]). We present a stronger version of this theorem that predicts the place of possible zero consecutive cancellations more precisely.

First, we equip the polynomial ring P with a $(\mathbf{Z^n}, \sigma)$-filtration on P which is called σ-Gröbner filtration and we discuss standard basis theory in this setting. We consider P as a $\mathbb{N}^n$-graded (or multigraded) ring and each monomial ideal as a homogeneous (multigraded) ideal. So we can talk about multigraded Betti numbers of monomial ideals. By a natural modification in the proof of Theorem 3.6 and its prerequisites, we show that the graded Betti numbers of I can be gotten from the multigraded Betti numbers of $\mathrm{LT}_\sigma(I)$ by a sequence of so-called σ-consecutive cancellations (see Definition 5.4 and Theorem 5.5). Since I is a graded ideal, each σ-consecutive cancellation should be also a zero consecutive cancellation. So, we can say that the graded Betti numbers of I can be obtained from those of $\mathrm{LT}_\sigma(I)$ by a sequence of *zero* σ-consecutive cancellations (see Corollary 5.6). Note that often, the number of possible zero σ-consecutive cancellation is less than the number of possible zero consecutive cancellations. For instance, Examples 5.7 and 5.8 both contain cancellations (e.g. $P(-5), P(-5)$) that are a priori admissible by [3, Corolllary 1.21], but are not in fact admissible by Corollary 5.6.

2. Preliminaries on filtered modules

In this section we collect necessary notations, definitions and some known results which will be used in this paper.

Throughout the paper $(R, \mathfrak{n})$ is a regular local ring with the residue field $\mathbf{k}$. If $\dim(R) = n$, then the associated graded ring $\mathrm{gr}_\mathfrak{n}(R)$ with respect to the $\mathfrak{n}$-adic filtration is the polynomial ring $P = \mathbf{k}[x_1, \dots, x_n]$. Let M be a finitely generated R-module. We say, according to the notation in [11], that a filtration of submodules $\mathbb{M} = \{M_n\}_{n\geq 0}$ on M is an $\mathfrak{n}$-filtration if $\mathfrak{n}M_n \subseteq M_{n+1}$ for every $n \geq 0$, and a stable $\mathfrak{n}$-filtration if $\mathfrak{n}M_n = M_{n+1}$ for all sufficiently large n. In the following a *filtered module* M will be always an R-module equipped with a stable $\mathfrak{n}$-filtration $\mathbb{M}$.

If $\mathbb{M} = \{M_j\}$ is a stable $\mathfrak{n}$-filtration of M, define

$$\mathrm{gr}_\mathbb{M}(M) = \bigoplus_{j\geq 0}(M_j/M_{j+1})$$

which is a graded $P = \mathrm{gr}_{\mathfrak{n}}(R)$-module in a natural way. It is called the *associated graded module* to the filtration $\mathbb{M}$.

To avoid triviality, we assume that $\mathrm{gr}_{\mathbb{M}}(M)$ is not zero or equivalently $M \neq 0$. If N is a submodule of M, by Artin-Rees Lemma, the sequence $\{N \cap M_j \mid j \geq 0\}$ is a stable $\mathfrak{n}$-filtration of N. Since

$$(1) \qquad (N \cap M_j)/(N \cap M_{j+1}) \simeq (N \cap M_j + M_{j+1})/M_{j+1},$$

$\mathrm{gr}_{\mathbb{M}}(N)$ is a graded submodule of $\mathrm{gr}_{\mathbb{M}}(M)$.

If $m \in M \setminus \{0\}$, we denote by $\nu_{\mathbb{M}}(m)$ the largest integer p such that $m \in M_p$ (the so-called valuation of m with respect to $\mathbb{M}$) and we denote by $\mathrm{gr}_{\mathbb{M}}(m)$ the residue class of m in M_p/M_{p+1} where $p = \nu_{\mathbb{M}}(m)$. If $m = 0$, we set $\nu_{\mathbb{M}}(m) = +\infty$. It is clear that if $f \in R$ is a unit element, then $\nu_{\mathbb{M}}(fm) = \nu_{\mathbb{M}}(m)$. An element $m \in M$ is a *lifting* of an element $x \in \mathrm{gr}_{\mathbb{M}}(M)$ if $\mathrm{gr}_{\mathbb{M}}(m) = x$.

Using (1), it is clear that $\mathrm{gr}_{\mathbb{M}}(N)$ is generated by the elements $\mathrm{gr}_{\mathbb{M}}(x)$ with $x \in N$. When the filtration is clear from the context we denote by N^* the associated graded module $\mathrm{gr}_{\mathbb{M}}(N)$. So,

$$N^* := \mathrm{gr}_{\mathbb{M}}(N) = \langle \mathrm{gr}_{\mathbb{M}}(x) : x \in N \rangle.$$

On the other hand it is clear that $\{(N + M_j)/N \mid j \geq 0\}$ is a stable $\mathfrak{n}$-filtration of M/N which we denote by $\mathbb{M}/N$. These graded modules are related by the graded isomorphism

$$\mathrm{gr}_{\mathbb{M}/N}(M/N) \simeq \mathrm{gr}_{\mathbb{M}}(M)/\mathrm{gr}_{\mathbb{M}}(N).$$

Let I be an ideal of R and $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$. If we consider the $\mathfrak{n}$-adic filtration on R, then $\mathrm{gr}_{\mathfrak{n}/I}(R/I) = P/I^*$ is the associated graded ring of A with respect to the $\mathfrak{m}$-adic filtration.

Given R-modues M and N, let $\mathbb{M}$ and $\mathbb{N}$ be stable $\mathfrak{n}$-filtrations of M and N respectively. We define a new filtration as follows

$$\mathbb{M} \oplus \mathbb{N} : \ M \oplus N \supseteq M_1 \oplus N_1 \supseteq \cdots \supseteq M_n \oplus N_n \supseteq \cdots .$$

It is easy to see that $\mathbb{M} \oplus \mathbb{N}$ is a stable $\mathfrak{n}$-filration on the R-module $M \oplus N$. Moreover, for any $(m, n) \in M \oplus N$ we have

$$\nu_{\mathbb{M}\oplus\mathbb{N}}(m, n) = \min\{\nu_{\mathbb{M}}(m), \nu_{\mathbb{N}}(n)\}.$$

So for filtered free R-modules we have:

Let $\mathbb{F} = \{F_n\}_{n \geq 0}$ be a filtration on a free R-module $F = \bigoplus_{i=1}^{s} Re_i$ (($e_1, \ldots, e_s$) is the canonical basis of F). If each F_n is a direct sum of ideals, then

$$(2) \qquad \nu_{\mathbb{F}}(\sum_{i=1}^{s} g_i e_i) = \min\{\nu_{\mathbb{F}}(g_i e_i) \mid 1 \leq i \leq s, g_i \neq 0\}.$$

Note that working with canonical basis is a crucial point to have the equality (2). Let us explain this fact by the following example:

Example 2.1. Let $R = \mathbf{k}[[x]]$ and $F = R \oplus R$ with the filtration $\mathbb{F} = \{F_n\}_{n \geq 0}$ where

$$F_0 = R \oplus R,\ F_1 = R \oplus R,\ F_2 = R \oplus \mathfrak{n}R,\ F_3 = \mathfrak{n}R \oplus \mathfrak{n}R,\ F_4 = \mathfrak{n}R \oplus \mathfrak{n}^2 R$$

and

$$\forall i \geq 5,\ F_i = \mathfrak{n}F_{i-1}.$$

If we let $\xi_1 = (1, 1)$ and $\xi_2 = (0, 1)$, then (ξ_1, ξ_2) is a basis of F and we have $\nu_{\mathbb{F}}(x\xi_1) = \nu_{\mathbb{F}}(-x\xi_2) = 3$ while $\nu_{\mathbb{F}}((x, 0)) = 4$ and $(x, 0) = x\xi_1 - x\xi_2$.

For completeness we collect, in this section, a part of the well known results concerning the standard basis of filtered modules.

Definition 2.2. If M and N are filtered R-modules and $f : M \to N$ is an R-homomorphism, f is said to be a *homomorphism of filtered modules* if $f(M_p) \subseteq N_p$ for every $p \geq 0$ and f is said *strict* if $f(M_p) = f(M) \cap N_p$ for every $p \geq 0$.

The morphism of filtered modules $f : M \to N$ clearly induces a morphism of graded P-modules

$$\mathrm{gr}(f) : \mathrm{gr}_{\mathbb{M}}(M) \to \mathrm{gr}_{\mathbb{N}}(N).$$

It is clear that $\mathrm{gr}(\cdot)$ is a functor from the category of the filtered R-modules into the category of the graded P-modules. Furthermore we have a canonical embedding $(\ker f)^* \to \ker(\mathrm{gr}(f))$.

Definition 2.3. Let $F = \bigoplus_{i=1}^{s} Re_i$ be a free R-module of rank s and $\nu_1, \dots, \nu_s$ be integers. We define the filtration $\mathbb{F} = \{F_p : p \in \mathbf{Z}\}$ on F as follows

$$F_p := \bigoplus_{i=1}^{s} \mathfrak{n}^{p-\nu_i} e_i = \{(a_1, \dots, a_s) : a_i \in \mathfrak{n}^{p-\nu_i}\}.$$

We denote the filtered free R-module F by $\bigoplus_{i=1}^{s} R(-\nu_i)$ and we call it a *special filtration* on F.

So when we write $F = \bigoplus_{i=1}^{s} R(-\nu_i)$ it means that we consider the free module F of rank s with the special filtration defined above. It is clear that $\mathbb{F}$ is an $\mathfrak{n}$-stable filtration. For the canonical basis $(e_1, \dots, e_s)$ of F, $\nu_{\mathbb{F}}(e_i) = \nu_i$. It is obvious that $\mathrm{gr}_{\mathbb{F}}(F) = \bigoplus_p F_p/F_{p+1}$ is isomorphic as a $P = \mathrm{gr}_{\mathfrak{n}}(R)$-module to $\bigoplus_{i=1}^{s} \mathrm{gr}_{\mathfrak{n}}(R)(-\nu_i) = \bigoplus_{i=1}^{s} P(-\nu_i)$.

The canonical basis $(\mathrm{gr}_{\mathbb{F}}(e_1), \dots, \mathrm{gr}_{\mathbb{F}}(e_s))$ of $\mathrm{gr}_{\mathbb{F}}(F)$ will be simply denoted by $(e_1, \dots, e_s)$. Note that R with the $\mathfrak{n}$-adic filtration is the filtered module $R(0)$.

Let $c = (c_1, \dots, c_s)$ be an element of F. By the definition of the filtration $\mathbb{F}$ on F, we have

$$\nu_{\mathbb{F}}(c) = \min\{\nu_R(c_i) + \nu_i : 1 \leq i \leq s\}.$$

Set $\mathrm{gr}_{\mathbb{F}}(c) = (c'_1, \dots, c'_s)$ and $\nu = \nu_{\mathbb{F}}(c)$, then

$$c'_i = \begin{cases} \mathrm{gr}_{\mathfrak{n}}(c_i) & \text{if } \nu_R(c_i) + \nu_i = \nu, \\ 0 & \text{if } \nu_R(c_i) + \nu_i > \nu. \end{cases}$$

Let M be a finitely generated filtered R-module and $S = \{f_1, \ldots, f_s\}$ be a system of elements of M and let $\nu_{\mathbb{M}}(f_i)$ be the corresponding valuations. As in Definition 2.3, let $F = \bigoplus_{i=1}^{s} Re_i$ be a free R-module of rank s equipped with the filtration $\mathbb{F}$ where $\nu_i = \nu_{\mathbb{M}}(f_i)$. Then we denote the filtered free R-module F by $\bigoplus_{i=1}^{s} R(-\nu_{\mathbb{M}}(f_i))$, hence $\nu_{\mathbb{F}}(e_i) = \nu_{\mathbb{M}}(f_i)$.

Let $\phi : F \to M$ be a morphism defined by

$$\phi(e_i) = f_i.$$

It is clear that ϕ is a morphism of filtered modules and $\mathrm{gr}_{\mathbb{F}}(F)$ is isomorphic to the graded free P-module $\bigoplus_{i=1}^{s} P(-\nu_{\mathbb{M}}(f_i))$ with a basis $(e_1, \ldots, e_s)$ where $\deg(e_i) = \nu_{\mathbb{M}}(f_i)$. In particular ϕ induces a natural graded morphism (of degree zero) $\mathrm{gr}(\phi) : \mathrm{gr}_{\mathbb{F}}(F) \to \mathrm{gr}_{\mathbb{M}}(M)$ sending e_i to $\mathrm{gr}_{\mathbb{M}}(f_i) = f_i^*$.

Definition 2.4. Let M be a filtered R-module. A subset $S = \{f_1, \ldots, f_s\}$ of M is called a *standard basis* of M if

$$\mathrm{gr}_{\mathbb{M}}(M) = \langle \mathrm{gr}_{\mathbb{M}}(f_1), \ldots, \mathrm{gr}_{\mathbb{M}}(f_s) \rangle.$$

Recall that we denote by $\mathrm{Syz}(f_1, \ldots, f_s)$ the submodule of F generated by the first syzygies of $f_1, \ldots, f_s,$. Likewise let $\mathrm{Syz}(\mathrm{gr}_{\mathbb{M}}(f_1), \ldots, \mathrm{gr}_{\mathbb{M}}(f_s))$ be the module generated by the first syzygies of $\mathrm{gr}_{\mathbb{M}}(f_1), \ldots, \mathrm{gr}_{\mathbb{M}}(f_s)$. By following the basic idea of Robbiano and Valla in [8], Shibuta in [15] (see also [10]) characterized the standard basis of filtered modules as follows.

Theorem 2.5. *Let M be a filtered R-module and $f_1, \ldots, f_s \in M$. The following facts are equivalent*:

(1) $\{f_1, \ldots, f_s\}$ *is a standard basis of* M.
(2) $\{f_1, \ldots, f_s\}$ *generates M and every element of* $\mathrm{Syz}(\mathrm{gr}_{\mathbb{M}}(f_1), \ldots, \mathrm{gr}_{\mathbb{M}}(f_s))$ *can be lifted to an element in $Syz(f_1, \ldots, f_s)$.*
(3) $\{f_1, \ldots, f_s\}$ *generates M and* $\mathrm{Syz}(\mathrm{gr}_{\mathbb{M}}(f_1), \ldots, \mathrm{gr}_{\mathbb{M}}(f_s)) = \mathrm{gr}_{\mathbb{F}}(\mathrm{Syz}(f_1, \ldots, f_s))$.

Since computing the minimal free resolution of M is the central goal of our work, here we discuss *minimizing* of a given free resolution. We try to clarify the method explained in [4, Pages 127 and 167].

Let $\{f_1, \ldots, f_s\}$ be a system of generators for M and $\mathcal{M}$ be a matrix whose columns generate $\mathrm{Syz}(f_1, \ldots, f_s)$. Then $\{f_1, \ldots, f_s\}$ is a *minimal* system of generators of M if and only if all the entries of $\mathcal{M}$ are non-units. If it is not the case, we can perform a sequence of elementary column operations (i.e., adding a multiple of one column to another and multiplying any column by a unit element) to produce a *reduced column form matrix* $\widetilde{\mathcal{M}}$. Where by reduced column form matrix, we mean that in each column of $\widetilde{\mathcal{M}}$ if we have some unit entries, then the leading unit entry is 1 and it is the only non-zero entry in its row. Of course the columns of $\widetilde{\mathcal{M}}$ again generate $\mathrm{Syz}(f_1, \ldots, f_s)$ and if $i_1, \ldots, i_r$ (resp. $j_1, \ldots, j_r$) are the numbers of the rows (resp. columns) of the leading 1s, then $\{f_1, \ldots, f_s\} \setminus \{f_{i_1}, \ldots, f_{i_r}\}$ is a minimal system of generators for M.

Moreover, by deleting the rows $i_1, \dots, i_r$ and the columns $j_1, \dots, j_r$ of $\widetilde{\mathcal{M}}$, we get a matrix whose columns generate the syzygy module of the remaining f_is.

Let $(\mathbf{F}_., \delta_.)$ be an R-free resolution of M. For each $j \in \mathbb{N} \cup \{0\}$, the differential δ_j is given by a matrix $\mathcal{M}_j$. These matrices are called the differential matrices (note that they depend on the chosen basis of F_js). Following the above discussion, we can construct a minimal free resolution of M starting from $(\mathbf{F}_., \delta_.)$. Let j be the least integer that $\mathcal{M}_j$ has a unit entry and $\widetilde{\mathcal{M}}_j$ be the reduced column form of $\mathcal{M}_j$. Assume that the leading 1s of $\widetilde{\mathcal{M}}_j$ occur in the rows $i_1, \dots, i_r$ and in the columns $j_1, \dots, j_r$. We delete the columns $i_1, \dots, i_r$ of $\mathcal{M}_{j-1}$ and the rows $j_1, \dots, j_r$ of $\mathcal{M}_{j+1}$. Also we replace $\mathcal{M}_j$ with the matrix obtained from $\widetilde{\mathcal{M}}_j$ by deleting the rows $i_1, \dots, i_r$ and the columns $j_1, \dots, j_r$ and we do not change the other $\mathcal{M}_i$s. Corresponding to this new matrices, we obtain a free resolution of M that we show it again by $(\mathbf{F}_., \delta_.)$. Note that now for each $i \leq j$ the differential matrix $\mathcal{M}_i$ does not have unit entries. Starting from this new resolution, we repeat the above procedure. Continuing in this way, we get a minimal free resolution of M.

3. Filtered free resolutions

Let M be a filtered module. The aim of this section is to study the minimal R-free resolutions of M starting from a filtered free resolution of M. In the rest of the paper, a filtration on a free R-module $\oplus_{i=1}^{s} Re_i$ is as $\mathbb{H} = \{H_n\}_{n \geq 0}$ where each H_n is a direct sum of ideals of R. In the following, we first recall some definitions.

Definition 3.1. Let $(\mathbf{F}_., \delta_.)$ be a complex of R-modules. We say $(\mathbf{F}_., \delta_.)$ is a *complex of filtered modules* if each F_i is a filtered module and each δ_i is a homomorphism of filtered modules.

Definition 3.2. Let M be a filtered module. A *filtered* free resolution of M is a free resolution $(\mathbf{F}_., \delta_.)$ of M:

$$\cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M$$

which is a complex of finitely generated filtered modules.

Definition 3.3 ([10])**.** Let $\{c_{ij}\}_{i,j \in \mathbb{N}}$ be a sequence of integers and for each i let $c_i = \sum_{j \in \mathbb{N}} c_{ij}$. Starting from the sequences $\{c_{ij}\}$ and $\{c_i\}$ we obtain new sequences by a *consecutive cancellation* as follows: fix an index i, and choose j and j' such that $j \leq j'$ and $c_{ij}, c_{i-1,j'} > 0$; then replace c_{ij} by $c_{ij} - 1$ and $c_{i-1,j'}$ by $c_{i-1,j'} - 1$, and accordingly, replace in the second sequence c_i by $c_i - 1$ and c_{i-1} by $c_{i-1} - 1$. If $j < j'$, we call it an *i negative consecutive cancellation* and if $j = j'$, an *i zero consecutive cancellation.*

A *sequence of consecutive cancellations* will mean a finite number of consecutive cancellations performed on given sequences $\{c_{ij}\}$ and $\{c_i = \sum_{j \in \mathbb{N}} c_{ij}\}$.

Finally, we define the valuation sequence and the valuation matrices of a filtered free resolution. Note that the valuation sequence often plays the same role as the graded Betti table of a graded module, but we want to point out that there are important differences because the valuation sequence is not independent of the chosen filtered free resolution.

Definition 3.4. Assume that $(\mathbf{F}_., \delta_.)$ is a filtered free resolution of M and each F_i is of rank γ_i. For each i fix a basis $B_i = (\xi_{i1}, \ldots, \xi_{i\gamma_i})$ of F_i (as a permutation of the canonical basis $(e_1, \ldots, e_{\gamma_i})$) in such a way that $\nu_{\mathbb{F}_i}(\xi_{i1}) \leq \cdots \leq \nu_{\mathbb{F}_i}(\xi_{i\gamma_i})$. For example, if F is the filtered free module defined in Example 2.1, we consider the basis (ξ_1, ξ_2) where $\xi_1 = e_2$ and $\xi_2 = e_1$. For each $j \in \mathbb{N}$, let $\gamma_{i,j} = |\{\xi_{ir} \,|\, 1 \leq r \leq \gamma_i,\ \nu_{\mathbb{F}_i}(\xi_{ir}) = j\}|$. It is clear that $\gamma_i = \sum_{j \in \mathbb{N}} \gamma_{i,j}$. We say that $\mathcal{B} = \{B_i\}$ is an *ordered basis* of $(\mathbf{F}_., \delta_.)$ and $\{\gamma_{i,j}\}$ is the valuation sequence of $(\mathbf{F}_., \delta_.)$.

Let $1 \leq s \leq \gamma_i$, $1 \leq r \leq \gamma_{i-1}$ and set $u_{rs} = \nu_{\mathbb{F}_i}(\xi_{is}) - \nu_{\mathbb{F}_{i-1}}(\xi_{i-1r})$. Then the matrix $U_i = (u_{rs})$ is called *the i-th valuation matrix* of $(\mathbf{F}_., \delta_.)$ with respect to $\mathcal{B}$.

Let us compute the valuation sequence and matrices for the following example.

Example 3.5. Let $\mathbf{C}$ be the field of complex numbers and $I = \langle x^2 + xy^3, xy + z^3, xz^3 - xy^4 + y^2z^4 \rangle$ be an ideal of $R = \mathbf{C}[[x, y, z]]$. Considering the $\mathfrak{n}$-adic filtration on R, R/I has the following filtered free resolution.

$$\begin{aligned}\mathbf{F}. : \mathbf{0} \to & R(-6) \oplus R(-9) \oplus R(-11) \to R(-3) \oplus R^2(-5) \\ & \oplus R^2(-7) \oplus R(-8) \oplus R^2(-10) \\ \to & R^2(-2) \oplus R(-4) \oplus R^2(-6) \oplus R(-9) \to R,\end{aligned}$$

where each F_i considered with a special filtration and the differential matrices are respectively:

$$\mathcal{M}_1 = \begin{pmatrix} xy + z^3 & x^2 + xy^3 & xz^3 - xy^4 + y^2z^4 & y^2z^4 & -z^6 + y^4z^3 & y^6z^3 \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} x & z^3 - y^4 & -xy^3 - y^2z^3 + y^6 & 0 & y^5z & 0 & 0 & y^9 \\ -y & 0 & z^3 & 0 & 0 & 0 & 0 & 0 \\ -1 & -y & -x + y^3 & z^3 & y^2z & 0 & 0 & y^6 \\ 1 & y & x + z^2 - y^3 & -z^3 & -x - y^3 - y^2z & z^2 & y^4 & -y^6 \\ 0 & 1 & 0 & x & 0 & y^2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -z & -x - y^3 \end{pmatrix} \text{ and}$$

$$\mathcal{M}_3 = \begin{pmatrix} z^3 & 0 & 0 \\ -x + y^3 & y^5 & 0 \\ y & 0 & 0 \\ 1 & y^2 & 0 \\ 0 & -z^2 & y^4 \\ -y & -x - y^3 & 0 \\ 0 & 0 & x + y^3 \\ 0 & 1 & -z \end{pmatrix}.$$

Here for each i, B_i is the canonical basis, $\nu_{\mathbb{F}_0}(\xi_{01}) = 0$, $\nu_{\mathbb{F}_1}(\xi_{11}) = \nu_{\mathbb{F}_1}(\xi_{12}) = 2$, $\nu_{\mathbb{F}_1}(\xi_{13}) = 4$, $\nu_{\mathbb{F}_1}(\xi_{14}) = \nu_{\mathbb{F}_1}(\xi_{15}) = 6$, $\nu_{\mathbb{F}_1}(\xi_{16}) = 9$, $\nu_{\mathbb{F}_2}(\xi_{21}) = 3, \nu_{\mathbb{F}_2}(\xi_{22}) = \nu_{\mathbb{F}_2}(\xi_{23}) = 5$. Similarly, the valuations of other entries of B_2 and B_3 can be read. The non-zero terms of the valuation sequence of $(\mathbf{F}., \delta.)$ are $\gamma_{0,0} = 1,\ \gamma_{1,2} = \gamma_{1,6} = 2,\ \gamma_{1,4} = \gamma_{1,9} = 1,\ \gamma_{2,3} = \gamma_{2,8} = 1,\ \gamma_{2,5} = \gamma_{2,7} = \gamma_{2,10} = 2,\ \gamma_{3,6} = \gamma_{3,9} = \gamma_{3,11} = 1$. The valuation matrices of $(\mathbf{F}., \delta.)$ are

$$U_1 = \begin{pmatrix} 2 & 2 & 4 & 6 & 6 & 9 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 3 & 3 & 5 & 5 & 6 & 8 & 8 \\ 1 & 3 & 3 & 5 & 5 & 6 & 8 & 8 \\ -1 & 1 & 1 & 3 & 3 & 4 & 6 & 6 \\ -3 & -1 & -1 & 1 & 1 & 2 & 4 & 4 \\ -3 & -1 & -1 & 1 & 1 & 2 & 4 & 4 \\ -6 & -4 & -4 & -2 & -2 & -1 & 1 & 1 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 3 & 6 & 8 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \\ -1 & 2 & 4 \\ -1 & 2 & 4 \\ -2 & 1 & 3 \\ -4 & -1 & 1 \\ -4 & -1 & 1 \end{pmatrix}.$$

As we can see above, for the (r, s)th entry of U_i, say u_{rs}, and the (r, s)th entry of $\mathcal{M}_i$, say m^i_{rs}, we have

$$u_{rs} \leq \nu_{\mathfrak{n}}(m^i_{rs}) \quad \text{and} \quad \forall s\ \exists j;\ u_{js} = \nu_{\mathfrak{n}}(m^i_{js}).$$

Theorem 3.6. *Let M be a filtered R-module, $(\mathbf{F}_., \delta_.)$ be a filtered free resolution of M with the ordered basis $\mathcal{B}$ and the sequence $\{\gamma_i = \sum_{j\in\mathbb{N}} \gamma_{i,j}\}$ be the valuation sequence of $(\mathbf{F}_., \delta_.)$. Then the Betti numbers of M can be obtained from $\{\gamma_i\}$ by a sequence of zero and negative consecutive cancellations.*

Proof. For each i let $\mathcal{M}_i = (m^i_{rs})$ be the matrix of δ_i with respect to $\mathcal{B} = \{B_i\}$ where $B_i = \{\xi_{i1}, \ldots, \xi_{i\gamma_i}\}$ and $U_i = (u_{rs})$ be the i-th valuation matrix of $(\mathbf{F}_., \delta_.)$ with respect to $\mathcal{B}$. Then for each $1 \leq s \leq \gamma_i$, $\delta_i(\xi_{is}) = \sum m^i_{rs}\xi_{i-1r}$ and since δ_i is a morphism of filtered modules, $\nu_{\mathbb{F}_i}(\xi_{is}) \leq \nu_{\mathbb{F}_{i-1}}(\delta_i(\xi_{is}))$. Note that by Equation (2),

$$\nu_{\mathbb{F}_{i-1}}(\delta_i(\xi_{is})) = \min\{\nu_{\mathbb{F}_{i-1}}(m^i_{rs}\xi_{i-1r}) \,|\, 1 \leq r \leq \gamma_{i-1},\ m^i_{rs} \neq 0\}.$$

So for each $1 \leq r \leq \gamma_{i-1}$, if $m^i_{rs} \neq 0$, then $\nu_{\mathbb{F}_{i-1}}(m^i_{rs}\xi_{i-1r}) \geq \nu_{\mathbb{F}_i}(\xi_{is})$. Moreover if m^i_{rs} is a unit element, then $\nu_{\mathbb{F}_{i-1}}(m^i_{rs}\xi_{i-1r}) = \nu_{\mathbb{F}_{i-1}}(\xi_{i-1r})$ and consequently $\nu_{\mathbb{F}_{i-1}}(\xi_{i-1r}) \geq \nu_{\mathbb{F}_i}(\xi_{is})$. This shows that if m^i_{rs} is a unit, then u_{rs} is not positive.

Now, assume that j is the least integer that $\mathcal{M}_j$ has unit entries. Following the procedure described in Section 2 we can perform a sequence of elementary column operations on $\mathcal{M}_j$ to produce a reduced column form $\widetilde{\mathcal{M}}_j$ such that the entries of U_j corresponding to the leading 1s of $\widetilde{\mathcal{M}_j}$ are non-positive. Assume that $u_{i_1j_1}, \ldots, u_{i_rj_r}$ is the mentioned entries. The sequences $\{\gamma_{i,j}\}$ and $\{\gamma_i = \sum_{j\in\mathbb{N}} \gamma_{i,j}\}$ admit some i-negative and zero consecutive cancellations corresponding to $u_{i_1j_1}, \ldots, u_{i_rj_r}$.

As we explained in Section 2, we can obtain a free resolution $(\widetilde{\mathbf{F}_.}, \widetilde{\delta_.})$ of M just by changing the matrices $\mathcal{M}_{j-1}, \mathcal{M}_j$ and $\mathcal{M}_{j+1}$. It is enough to delete the columns $i_1, \ldots, i_r$ of $\mathcal{M}_{j-1}$, delete the rows $j_1, \ldots, j_r$ of $\mathcal{M}_{j+1}$ and replace $\mathcal{M}_j$ with the matrix obtained from $\widetilde{\mathcal{M}_j}$ by deleting the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$. It is clear that the matrix of $\widetilde{\delta_i}$ has no unit entries for each $i \leq j$ and for each $i > j$ the eventually remaining unit entries of the matrix of $\widetilde{\delta_i}$ still correspond to the non-positive entries of the valuation matrix U_i.

We can repeat the procedure on $(\widetilde{\mathbf{F}_.}, \widetilde{\delta_.})$ until we get a minimal free resolution of M. □

If $(\mathbf{F}., \delta.)$ is a complex of finitely generated free R-modules, a special filtration on $\mathbf{F.}$ is a special filtration on each F_i that makes $(\mathbf{F}., \delta.)$ a complex of filtered modules. Next goal is to consider special filtration on an R-free resolution of a filtered module M.

The following result presented in [7] (and also [9, Theorem 1.8]) gives a comparison between an R-free resolution of M and a P-free resolution of $\mathrm{gr}_{\mathbb{M}}(M)$. The result will be an important tool in the rest of the paper.

Theorem 3.7. *Let M be a filtered R-module and*

$$\mathbf{G}. : \ \cdots \to \bigoplus_{i=1}^{\beta_l} P(-a_{li}) \xrightarrow{d_l} \bigoplus_{i=1}^{\beta_{l-1}} P(-a_{l-1i}) \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} \bigoplus_{i=1}^{\beta_0} P(-a_{0i}) \xrightarrow{d_0} \mathrm{gr}_{\mathbb{M}}(M)$$

be the minimal graded P-free resolution of $\mathrm{gr}_{\mathbb{M}}(M)$. Then we can build up an R-free resolution $(\mathbf{F}., \delta.)$ of M and a special filtration $\mathbb{F}$ on it such that $\mathrm{gr}_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.

In Theorem 3.7, $(\mathbf{F}., \delta.)$ is defined by inductive process focused on Theorem 2.5. For each $j \geq 0$, F_j is the filtered free R-module $F_j = \bigoplus_{i=1}^{\beta_j} R(-a_{ji})$ and each δ_j is a strict morphism. So, each finitely generated filtered module M has a filtered free resolution $(\mathbf{F}., \delta.)$ whose valuation sequence is the sequence of the graded Betti numbers of $\mathrm{gr}_{\mathbb{M}}(M)$.

Note that the R-free resolution of M, given in Theorem 3.7, is not necessarily minimal. A filtered module M is said to be *of homogeneous type* with respect to the given filtration $\mathbb{M}$ if $\beta_i(\mathrm{gr}_{\mathbb{M}}(M)) = \beta_i(M)$ for every $i \geq 0$.

The following theorem is one of the main results of [10]. Note that there is an obvious difference between Theorem 3.6 and the next theorem. Actually, thanks to good properties of the resolution of Theorem 3.7, we can say that only negative consecutive cancellations happen when the valuation sequence is the sequence of the graded Betti numbers of $\mathrm{gr}_{\mathbb{M}}(M)$.

Theorem 3.8 ([10, Theorem 3.1]). *Let M be a filtered R-module and $\{\beta_i = \sum_{j\in\mathbb{N}} \beta_{i,j}\}$ be the sequence of the Betti numbers of* $\mathrm{gr}_{\mathbb{M}}(M)$*. The Betti numbers of M as an R-module can be obtained from $\{\beta_i\}$ by a sequence of negative consecutive cancellations.*

4. Resolutions by mapping cones

In this section, we consider filtered free resolutions of a module that are obtained by the mapping cone technique and we apply Theorem 3.6 for these resolutions in order to get some information about the minimal free resolution of the underlying module.

Definition 4.1. Let $(\mathbf{F}_., \delta_.)$ and $(\mathbf{G}_., d_.)$ be two complexes of filtered modules. Then a *filtered chain map* $\mathbf{f} : (\mathbf{G}_., d_.) \to (\mathbf{F}_., \delta_.)$ is a sequence of homomorphism of filtered modules $f_n : G_n \to F_n$ such that for each n, $f_n d_{n+1} = \delta_{n+1} f_{n+1}$.

Theorem 4.2 (Comparison Theorem). *Let M and M' be two filtered R-modules and $f : M \to M'$ be a filtered homomorphism. Consider the diagram*

$$\begin{array}{c} \cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0 \\ \qquad\qquad\qquad\qquad\quad f\downarrow \\ \cdots \to P'_2 \overset{\delta_2}{\to} P'_1 \overset{\delta_1}{\to} P'_0 \overset{\delta_0}{\to} M' \to 0 \end{array}$$

where the rows are complexes of finitely generated filtered modules. If each P_n in the top row is projective and if the bottom row is exact, then there exists a filtered chain map $\mathbf{f}$ *making the completed diagram commute, provided that each δ_i is strict.*

Proof. By [12, Theorem 6.9] we have a chain map $\mathbf{f}$ making the completed diagram commute. Since each δ_i is a strict morphism, if $m \in P_n$ has valuation d, then we can define $f_n(m)$ with $\nu_{\mathbb{P}'_n}(f_n(m)) \geq d$. So the conclusion follows. □

Definition 4.3. Given any filtered R-module M, we can form a new filtered R-module by shifting the filtration on M as follows: if n is any integer, define $M(n)$ to be equal to M as an R-module but its filtration defined by $M(n)_k = M_{n+k}$. It is clear that the corresponding associated graded module is $\mathrm{gr}_{\mathbb{M}}(M)(n)$.

In the following, assume that M is a submodule of the filtered free R-module $\bigoplus_{i=1}^s R(-\nu_i)$ and $f \in \bigoplus_{i=1}^s R(-\nu_i) \setminus M$ is an element of valuation d. We equip R with the usual $\mathfrak{n}$-adic filtration we have the following short exact sequence of filtered modules:

$$0 \to R/(M:(f))(-d) \to \bigoplus_{i=1}^{s} R(-\nu_i)/M \to \bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f)) \to 0.$$

Assume that $(\mathbf{G}., d.)$ is a filtered free resolution of $R/(M:(f))(-d)$ and $(\mathbf{F}., \delta.)$ is a filtered free resolution of $\bigoplus_{i=1}^s R(-\nu_i)/M$. If each δ_i is a strict morphism, then, by the Comparison Theorem there exists a filtered chain map $\mathbf{f} : (G., d.) \to (F., \delta.)$. Now, the mapping cone $\mathbf{M}(\mathbf{f})$ is the complex such that

$$\mathbf{M}(\mathbf{f})_i = F_i \oplus G_{i-1}$$

with the differential map

$$\phi_i(x, y) = (f_{i-1}(y) + \delta_i(x), -d_{i-1}(y)).$$

Since each f_i, δ_i and d_i is a homomorphism of filtered modules, it is clear that δ_is are homomorphisms of filtered modules. So, the mapping cone is a filtered free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$. Next, we apply Theorem 3.6 for the result of the mapping cone.

Corollary 4.4. *With the notations as above,*

(1) $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ *has a filtered free resolution constructed by the mapping cone technique.*

(2) *If for each i, j we let*

$$(3)\quad \gamma_{i,j} = \beta_{i,j}(\bigoplus_{i=1}^{s} P(-\nu_i)/M^*) + \beta_{i-1,j-d}(P/(M:(f))^*) \text{ and } \{\gamma_i = \sum_{j\in\mathbb{N}} \gamma_{i,j}\}$$

then the Betti numbers of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ can be obtained from $\{\gamma_i\}$ by a sequence of zero and negative consecutive cancellations.

Proof. (1) By Theorem 3.7, we can build up a filtered free resolution $(\mathbf{G}., d_.)$ of $R/(M:f)$ with the valuation sequence

$$\{\beta_i(P/(M:f)^*) = \sum_{j\in\mathbb{N}} \beta_{i,j}(P/(M:f)^*)\}.$$

The shifted complex $(\mathbf{G}.(-d), d_.)$ is a filtered free resolution of $R/(M:f)(-d)$. Again by Theorem 3.7, we can get a filtered free resolution $(\mathbf{F}., \delta.)$ of

$$\bigoplus_{i=1}^{s} R(-\nu_i)/M$$

such that each δ_i is a strict morphism and the corresponding valuation sequence is

$$\{\beta_i(\bigoplus_{i=1}^{s} P(-\nu_i)/M^*) = \sum_{j\in\mathbb{N}} \beta_{i,j}(\bigoplus_{i=1}^{s} P(-\nu_i)/M^*)\}.$$

Now, we can apply the mapping cone technique as described just before the theorem and find a filtered free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ whose valuation sequence is given by the Equation (3).

(2) follows by (1) and Theorem 3.6. □

In the next example, we give two different filtered free resolutions for an ideal and compare these resolutions with the minimal free resolution of the ideal. Note that these resolutions are constructed by the mapping cone technique.

Example 4.5. Let

$$I = \langle z^2 + x^3, zx^2 + y^3, zxy^2, x^2yz\rangle \subset R = \mathbf{C}[[x, y, z]].$$

Considering the $\mathfrak{n}$-adic filtration and let

$$J_1 = \langle z^2 + x^3, zx^2 + y^3, zxy^2\rangle$$

and

$$J_2 = \langle x^2yz, zx^2 + y^3, zxy^2\rangle.$$

Then computations show that

$$J_1^* = \langle z^2, x^2z + y^3, xy^2z, x^6, x^4y^2\rangle, \qquad J_2^* = J_2,$$

$$L_1 = J_1 : \langle x^2yz\rangle = \langle y, xz, x^3 + z^2\rangle, \qquad L_1^* = \langle y, xz, z^2, x^4\rangle$$

and

$$L_2 = J_2 : \langle z^2 + x^3\rangle = J_2.$$

The graded minimal free resolution of P/L_1^* is

$$\begin{aligned} 0 \to & P(-4) \oplus P(-6) \to P^3(-3) \oplus P^2(-5) \\ \to & P(-1) \oplus P^2(-2) \oplus P(-4) \to P \to P/L_1^* \end{aligned}$$

and the graded minimal free resolution of P/J_1^* is

$$\begin{aligned} 0 \to & P(-6) \oplus P(-10) \to P^3(-5) \oplus P^2(-7) \oplus P(-8) \\ \to & P(-2) \oplus P(-3) \oplus P(-4) \oplus P^2(-6) \to P \to P/J_1^*. \end{aligned}$$

So, by the mapping cone technique, R/I has the following filtered free resolution:

$$\begin{aligned} \mathbf{F_1}. : \mathbf{0} \to & R(-8) \oplus R(-10) \to R(-6) \oplus R^3(-7) \oplus R^2(-9) \oplus R(-10) \\ \to & R^4(-5) \oplus R^2(-6) \oplus R^2(-7) \oplus R^2(-8) \\ \to & R(-2) \oplus R(-3) \oplus R^2(-4) \oplus R^2(-6) \to R \to R/I. \end{aligned}$$

On the other hand, the graded minimal free resolution of P/J_2^* is

$$0 \to P(-5) \oplus P(-6) \to P(-3) \oplus P^2(-4) \to P \to P/J_2^*.$$

Since $J_2^* = L_2^*$, by the mapping cone technique R/I has the following filtered free resolution:

$$\begin{aligned} \mathbf{F_2}. : 0 \to & R(-7) \oplus R(-8) \to R^2(-5) \oplus R^3(-6) \\ \to & R(-2) \oplus R(-3) \oplus R^2(-4) \to R \to R/I. \end{aligned}$$

Note that $\mathbf{F_2}.$ is itself a minimal free resolution and we don't need any cancellation while $\mathbf{F_1}.$ is not minimal and we need 7 consecutive cancellations and one of them is $(R(-10), R(-10), 0, 0)$ which means that at least one zero cancellation happens.

If $(\mathbf{G}_., d_.)$ and $(\mathbf{F}_., \delta_.)$ are arbitrary free resolutions of $R/(M : f)$ and $\bigoplus_{i=1}^{s} R(-\nu_i)/M$ respectively, then we can apply the mapping cone technique and get a free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$. It is clear that in general, the result is not a minimal free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ even if both $(\mathbf{G}_., d_.)$ and $(\mathbf{F}_., \delta_.)$ are minimal. Below, we find a sufficient condition for the minimality of $(\mathbf{M(f)}, \phi_.)$. We first need the following definition.

Definition 4.6. Assume that M and M' are two filtered R-modules and $f : M \to M'$ be a filtered homomorphism. Let $(\mathbf{G}., d.)$ and $(\mathbf{F}., \delta_.)$ be the minimal free resolutions of $\mathrm{gr}_{\mathbb{M}}(M)$ and $\mathrm{gr}_{\mathbb{M}'}(M')$ respectively. Assume that

$$G_j = \bigoplus_{i=1}^{\beta_j} P(-a_{ji})$$

and

$$F_j = \bigoplus_{i=1}^{\beta'_j} P(-b_{ji})$$

where $a_{j1} \leq \cdots \leq a_{j\beta_j}$ and $b_{j1} \leq \cdots \leq b_{j\beta'_j}$.

Let $1 \leq s \leq \beta_j$, $1 \leq r \leq \beta'_j$ and $u_{rs} := a_{js} - b_{jr}$. Then the matrix $U_j(M, M') = (u_{rs})$ is called the j-th degree matrix of the pair (M, M'). The matrix $U_j(M, M')$ is called a positive matrix when all entries of it are positive.

Theorem 4.7. *If M is a submodule of $\bigoplus_{i=1}^{s} R(-\nu_i)$ and $f \in \bigoplus_{i=1}^{s} R(-\nu_i) \backslash M$, then the free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ constructed by the mapping cone technique is minimal provided that for each $j \leq \mathrm{pd}(P/(M : f)^*)$, $U_j(R/(M : (f))(-d), \bigoplus_{i=1}^{s} R(-\nu_i)/M)$ is a positive matrix.*

Proof. In order to prove theorem, it is enough to show that for each $j \geq 0$,

$$\beta_j(\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))) = \beta_j(\bigoplus_{i=1}^{s} R(-\nu_i)/M) + \beta_{j-1}(R/(M : f)).$$

Let $(\mathbf{G}., d.)$ and $(\mathbf{F}., \delta.)$ be filtered free resolutions of $R/(M : (f))(-d)$ and $\bigoplus_{i=1}^{s} R(-\nu_i)/M$ coming from the minimal free resolutions of $P/(M : (f))^*(-d)$ and $\bigoplus_{i=1}^{s} P(-\nu_i)/M^*$ respectively. Then $(\mathbf{M(f)}, \phi.)$ is a filtered free resolution of $\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f))$ with the valuation sequence as (3).

Let for each r, $\mathcal{M}_r$ (resp. $\mathcal{N}_r$) be the matrix of δ_r (resp. d_r) with respect to the ordered basis of F_r and F_{r-1} (resp. G_r and G_{r-1}). Also assume that for each r, $\mathcal{O}_r$ is the matrix of $f_r : G_r \to F_r$. Then, by the mapping cone construction, the matrix of ϕ_r, with respect to the corresponding basis of $F_r \oplus G_{r-1}$ and $F_{r-1} \oplus G_{r-2}$, is denoted by $\mathcal{M}'_r$, has the following shape:

$$\mathcal{M}'_r = \left(\begin{array}{c|c} \mathcal{M}_r & \mathcal{O}_{r-1} \\ \hline 0 & -\mathcal{N}_{r-1} \end{array} \right).$$

Since for each r, $U_r(R/(M : (f))(-d), \bigoplus_{i=1}^{s} R(-\nu_i)/M)$ is a positive matrix, all the unit entries of $\mathcal{M}'_r$ lie in the submatrix $\mathcal{M}_r$ or $\mathcal{N}_{r-1}$ and correspond to the negative entries of the rth valuation matrix of $(\mathbf{F}.,\delta.)$ or $r-1$th valuation matrix of $(\mathbf{G}.,d.)$.

Now, let j be the least integer that $\mathcal{M}'_j$ has some unit entries. Then one can do the elementary column operations to produce a reduced column form matrix $\widetilde{\mathcal{M}'_j}$ in such a way that the following conditions hold:

1) The entries of the jth valuation matrix of $(\mathbf{M}(\mathbf{f}),\phi.)$ corresponding to the leading 1s of $\widetilde{\mathcal{M}'_j}$ are negative and belong to one of the submatrices coincide with the jth valuation matrix of $(\mathbf{F}.,\delta.)$ or the $j-1$th valuation matrix of $(\mathbf{G}.,d.)$.

2) The submatrix of $\widetilde{\mathcal{M}'_j}$ correspond to $\mathcal{M}_j$ is the reduced column form of $\mathcal{M}_j$ and the submatrix of $\widetilde{\mathcal{M}'_j}$ correspond to $-\mathcal{N}_{j-1}$ is the reduced column form of $-\mathcal{N}_{j-1}$.

The above discussion shows that in the same time we are doing the minimizing method for $(\mathbf{F}_.,\delta.)$ and $(\mathbf{G}.,d.)$ too.

Doing the corresponding cancellations, we see that

$$\forall i<j,\quad \beta_i(\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f)))=\beta_i(\bigoplus_{i=1}^{s} R(-\nu_i)/M)+\beta_{i-1}(R/(M:f)).$$

Continuing in this way, we can see that for each i, the necessary cancellations we need to perform on the sequence (3) in order to find $\beta_i(\bigoplus_{i=1}^{s} R(-\nu_i)/(M+(f)))$ correspond to the necessary cancellations in $\{\beta_i(\bigoplus_{i=1}^{s} P(-\nu_i)/M^*)\}$ for finding $\beta_i(\bigoplus_{i=1}^{s} R(-\nu_i)/M)$ or correspond to the necessary cancellations in $\{\beta_i(P/(M:f)^*\}$ for finding $\beta_{i-1}(R/M:f)$. So the conclusion follows. □

Example 4.8. Consider the ideals $I=\langle x_1x_2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_3x_4,x_1^2-x_2^3-x_3^3\rangle$ and $J=\langle x_1x_2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_3x_4\rangle$ of the power series ring $\mathbf{C}[[x_1,x_2,x_3,x_4]]$. Then $I=J+\langle x_1^2-x_2^3-x_3^3\rangle$ and $L=J:\langle x_1^2-x_2^3-x_3^3\rangle=\langle x_1x_2,x_2x_3,x_1x_3,x_4\rangle$. Consider the $\mathfrak{n}$-adic filtration, the graded minimal free resolution of $P/L^*(-2)$ is

$$0\to P^2(-6)\to P^5(-5)\to P(-3)\oplus P^3(-4)\to P(-2)\to P/L^*(-2)$$

and the graded minimal free resolution of P/J^* is

$$0\to P^3(-4)\to P^8(-3)\to P^6(-2)\to P\to P/J.$$

So,

$$U_0(R/L(-2),R/J)=\begin{pmatrix}2\end{pmatrix},$$

$$U_1(R/L(-2),R/J)=\begin{pmatrix}1&2&2&2\\1&2&2&2\\1&2&2&2\\1&2&2&2\\1&2&2&2\\1&2&2&2\end{pmatrix},$$

$$U_2(R/L(-2), R/J) = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ .2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix},$$

$$U_3(R/L(-2), R/J) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

So, by Theorem 4.7, for each $j \geq 0$,

$$\beta_j(R/I) = \beta_j(R/J) + \beta_{j-1}(R/J : \langle x_1^2 - x_2^3 - x_3^3 \rangle).$$

Now, since both R/J and $R/J : \langle x_1^2 - x_2^3 - x_3^3 \rangle$ are of homogeneous type, the minimal free resolution of R/I is

$$0 \to R^2 \to R^8 \to R^{12} \to R^7 \to R \to R/I.$$

In fact the filtered free resolution constructed by the mapping cone technique is the minimal free resolution of R/I. Note that

$$I^* = \langle x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1^2, x_2^4, x_3^4 \rangle$$

and the graded minimal free resolution of P/I^* is

$$\begin{aligned} 0 \to & P(-5) \oplus P^2(-7) \to P^6(-4) \oplus P^6(-6) \to P^{11}(-3) \oplus P^6(-5) \\ \to & P^7(-2) \oplus P^2(-4) \to P \to P/I^*. \end{aligned}$$

By Theorem 3.7, we can build up a filtered free resolution $(\mathbf{F}., \delta.)$ of P/I whose valuation sequence is $\{\beta_i(P/I^*) = \sum_{j \in \mathbb{N}} \beta_{i,j}(P/I^*)\}$. By Theorem 3.8, the Betti numbers of R/I can be obtained from this sequence by some negative consecutive cancellations. Actually the necessary negative consecutive cancellations on (F_4, F_3, F_2, F_1) are $(0, 0, R(-3), R(-4))$, $(0, 0, R(-3), R(-4))$, $(0, R(-4), R(-5), 0)$, $(0, R(-4), R(-5), 0)$, $(0, R(-4), R(-5), 0)$ and $(R(-5), R(-6), 0, 0)$.

As we see, in this example the result of the mapping cone is much better than the filtered free resolution described in Theorem 3.7. One reason may be the fact that both R/J and $R/(J : \langle x_1^2 - x_2^3 - x_3^3 \rangle)$ are of homogeneous type while R/I is not.

We end this section by an application of Theorem 4.7, in finding Betti numbers of a class of Artinian local rings. Recall that an Artinian local ring $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ is called *stretched* when $\mathfrak{m}^2$ is minimally generated by one element.

Corollary 4.9. *Let $A = R/I$ be a stretched Artinian local ring with the Cohen-Macaulay type $\tau(A) = n-1$ and socle degree $s \geq 2$. Then*

$$\beta_i(R/I) = i\binom{n}{i+1} + (n-1)\binom{n-1}{i-1}.$$

Proof. By [2, Theorem 3.1] one can find a set of generators $\{X_1, \dots, X_n\}$ of $\mathfrak{n}$ such that the ideal I is minimally generated by

$$\{X_iX_j\}_{1\leq i<j\leq n}, \quad \{X_j^2\}_{2\leq j\leq n-1}, \quad \{X_n^2 - X_1^s\}.$$

If J is the ideal generated by all generators of I except $X_n^2 - X_1^s$, then $I = J + \langle X_n^2 - X_1^s\rangle$ and $L = J : \langle X_n^2 - X_1^s\rangle = \langle X_1X_n, X_2, \dots, X_{n-1}\rangle$. Consider the $\mathfrak{n}$-adic filtration, L^* is generated by the regular sequence $x_1x_n, x_2, \dots, x_{n-1}$. So its graded minimal free resolution is given by the Koszul complex and we have

$$\beta_{i,j}(P/L^*) = \begin{cases} \binom{n-2}{i}, & j = i; \\ \binom{n-2}{i-1}, & j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$J_1^* = \langle x_ix_j, 1 \leq i < j \leq n\rangle,$$

then

$$J^* = J_1^* + \langle x_j^2, 2 \leq j \leq n-1\rangle.$$

By [14, Theorem 4.1] the minimal free resolution of P/J^* is computed by iterated mapping cone starting from the minimal free resolution of P/J_1^* and we have

$$\beta_{i,i+1}(P/J^*) = \begin{cases} i\binom{n}{i+1} + (n-2)\binom{n-1}{i-1}, & j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

So for each i, all the entries of $U_i(R/L(-2), R/L)$ are 2 or 1. Now, by Theorem 4.7, the minimal free resolution of R/I is given by the mapping cone technique. In particular we have

$$\begin{aligned} \beta_i(R/I) &= i\binom{n}{i+1} + (n-2)\binom{n-1}{i-1} + \binom{n-2}{i-1} + \binom{n-2}{i-2} \\ &= i\binom{n}{i+1} + (n-1)\binom{n-1}{i-1}. \end{aligned}$$

□

5. Filtered free resolutions of graded ideals

In this section, we are going to apply the main result of Section 3 in the graded setting. Our approach is to equip the polynomial ring $P = \mathbf{k}[x_1, \dots, x_n]$ with a suitable $(\mathbf{Z^n}, \sigma)$ *filtration* (where σ is a monoid ordering on $\mathbf{Z^n}$) and define the notion of consecutive cancellations to find graded Betti numbers of a graded ideal by a sequence of consecutive cancellations from the multigraded Betti numbers of corresponding associated graded ideal.

Recall that one may consider the polynomial ring as a $\mathbf{N}^n$-graded (or multigraded) ring and construct *minimal multigraded free resolution* of every multigraded module. It is clear that every multigraded free resolution is also a graded resolution. Note that every monomial ideal is homogeneous with respect to the $\mathbf{N}^n$-grading. So, we can consider *multigraded Betti numbers* of monomial ideals (see [6, Chapter 1, Section 26]).

We refer to [5] for background on $(\mathbf{Z^n}, \sigma)$ filtrations. Note that $(\mathbf{Z^n}, \sigma)$ filtrations have a difference with the usual filtrations that we considered in the previous sections. Because $P_\gamma \subseteq P_{\gamma'}$ for all $\gamma, \gamma' \in \mathbf{Z^n}$ with $\gamma \leq_\sigma \gamma'$.

Let σ be a term ordering on $\mathbf{T^n}$ where by $\mathbf{T^n}$ we mean the set of all monomials of P. Using the isomorphism of monoids $\log : \mathbf{T^n} \to \mathbb{N}^\mathbf{n}$, we can view σ as a monoid ordering on $\mathbb{N}^n$, and by [5, Proposition 1.4.14], we can extend σ uniquely to a monoid ordering on $\mathbf{Z^n}$ which we denote by σ again. For every $\gamma \in \mathbf{Z^n}$, we define the vector space

$$P_\gamma = \{f \in P \setminus \{0\} \mid \log(\mathrm{LT}_\sigma(f)) \leq_\sigma \gamma\} \cup \{0\}.$$

Then it is easy to see that $\mathbb{P} = \{P_\gamma \mid \gamma \in \mathbf{Z^n}\}$ is a $(\mathbf{Z^n}, \sigma)$-filtration on P. It is called the σ-Gröbner filtration on P. Let $I \subset P$ be an ideal. Then, by [5, Remark 6.5.14], $I^* = \mathrm{LT}_\sigma(I)$ and an standard basis of I is nothing but a Gröbner basis.

Let $F = \bigoplus_{i=1}^s Pe_i$ be a free P-module of rank s and $\gamma_1, \ldots, \gamma_s \in \mathbf{Z^n}$. We define the σ-Gröbner filtration $\mathbb{F} = \{F_\gamma : \gamma \in \mathbf{Z^n}\}$ on F as follows:

$$F_\gamma = \{(h_1, \ldots, h_s) \in \oplus_{i=1}^s Pe_i \mid \log(\mathrm{LT}_\sigma(h_i)) \leq_\sigma \gamma - \gamma_i\}.$$

We denote the σ-Gröbner filtered free P-module F by $\oplus_{i=1}^s P(\gamma_i)$ and we call it *σ-Gröbner special filtration* on F.

Let M be a finitely generated submodule of $\oplus_{i=1}^s P(\gamma_i)$, Clearly, we can consider σ-Gröbner filtration for M. If $m \in M \setminus \{0\}$, then $\nu_\mathbb{F}(m) = \min\{\gamma \in \mathbf{Z}^n \mid m \in F_\gamma\}$. Let $\{f_1, \ldots, f_t\}$ be a system of elements of M and let $F_1 = \oplus_{i=1}^t P(\nu_\mathbb{F}(f_i))$ be the corresponding σ-Gröbner filtered free P-module. Based on the properties of σ-Gröbner filtration (see [5, Section 6.5.1]) and by similar methods as previous sections we have:

Theorem 5.1. *Let M be a σ-Gröbner filtered submodule of $\oplus_{i=1}^s P(\gamma_i)$ and $f_1, \ldots, f_t \in M$. The following facts are equivalent*:

1. $\{f_1, \ldots, f_t\}$ *is a standard basis of M.*
2. $\{f_1, \ldots, f_t\}$ *generates M and every element of* $\mathrm{Syz}(\mathrm{gr}_\mathbb{F}(f_1), \ldots, \mathrm{gr}_\mathbb{F}(f_t))$ *can be lifted to an element in $Syz(f_1, \ldots, f_t)$.*
3. $\{f_1, \ldots, f_t\}$ *generates M and* $\mathrm{Syz}(\mathrm{gr}_\mathbb{F}(f_1), \ldots, \mathrm{gr}_\mathbb{F}(f_t)) = \mathrm{gr}_{\mathbb{F}_1}(\mathrm{Syz}(f_1, \ldots, f_t))$.

In particular, if M is a graded submodule of $\oplus_{i=1}^s P(-|\gamma_i|)$, then we can assume that $f_1, \ldots, f_t$ are homogeneous elements of M.

Theorem 5.2. *Let M be a σ-Gröbner filtered submodule of $\oplus_{i=1}^{s} P(\gamma_i)$ and $(\mathbf{G}., d.)$ be a P-free multigraded resolution of M^*. Then we can build up a P-free resolution $(\mathbf{F}., \delta.)$ of M and a σ-Gröbner special filtration $\mathbb{F}$ on it such that $\mathrm{gr}_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.*

In particular, if M is a graded submodule of $\oplus_{i=1}^{s} P(-|\gamma_i|)$, then $(\mathbf{F}., \delta.)$ is a graded free resolution of M.

In particular, for the graded ideals of P we have:

Theorem 5.3. *Let I be a graded ideal of P and $(\mathbf{G}., d.)$ be the minimal P-free multigraded resolution of $\mathrm{LT}_\sigma(I) = I^*$. Then we can build up a graded P-free resolution $(\mathbf{F}., \delta.)$ of I and a σ-Gröbner special filtration $\mathbb{F}$ on it such that $\mathrm{gr}_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.*

For this kind of filtered free resolutions of graded ideals, consecutive cancellation has the following meaning:

Definition 5.4. Let $\{c_{ij}\}_{i\in\mathbb{N}, j\in\mathbb{N}^n}$ be a sequence of integers and for each i let $c_i = \sum_{j\in\mathbb{N}^n} c_{ij}$. Starting from the sequences $\{c_{ij}\}$ and $\{c_i\}$ we obtain new sequences by a *consecutive cancellation* as follows: fix an index i, and choose j and j' such that $j' \leq_\sigma j$ and $c_{ij}, c_{i-1,j'} > 0$; then replace c_{ij} by $c_{ij} - 1$ and $c_{i-1,j'}$ by $c_{i-1,j'} - 1$, and accordingly, replace in the second sequence c_i by $c_i - 1$ and c_{i-1} by $c_{i-1} - 1$. We call it an (i, σ)-consecutive cancellation.

A *sequence of consecutive cancellations* will mean a finite number of consecutive cancellations performed on given sequences $\{c_{ij}\}$ and $\{c_i = \sum_{j\in\mathbb{N}^n} c_{ij}\}$.

Theorem 5.5. *Let I be a graded ideal of P and $\{\beta_i = \sum_{j\in\mathbb{N}^n} \beta_{i,j}\}$ be the sequence of multigraded Betti numbers of* $\mathrm{LT}_\sigma(I)$*. Then the graded Betti numbers of I can be obtained from $\{\beta_i\}$ by a sequence of σ-consecutive cancellations.*

As we recall in the introduction, it is well-known that the graded Betti numbers of a graded ideal are obtained from those of its leading term ideal by a sequence of zero consecutive cancellations. It is more convenient to consider both kinds of cancellations together. Actually, since in Theorem 5.3, $(\mathbf{F}., \delta.)$ is a *graded* resolution, one can see that each σ-consecutive cancellation should be also a zero consecutive cancellation. So we have:

Corollary 5.6. *Let I be a graded ideal of P and $\{\beta_i = \sum_{j\in\mathbb{N}^n} \beta_{i,j}\}$ be the sequence of multigraded Betti numbers of* $\mathrm{LT}_\sigma(I)$*. Then the graded Betti numbers of I can be obtained from $\{\beta_i\}$ by a sequence of zero σ-consecutive cancellations.*

We end this section with two examples that show how Corollary 5.6 can help to encode the Betti numbers of an ideal by just looking to the multigraded Betti numbers of $\mathrm{LT}_\sigma(I)$. Both of these examples contain cancellations (e.g. $P(-5), P(-5)$) that are a priori admissible by [3, Corollary 1.21], but are not in fact admissible by Corollary 5.6.

Example 5.7. Let $P = \mathbf{C}[x, y, z]$, $\sigma =$ DegRevLex and $I = \langle z^2 + x^2, yz, y^3 + z^3, xz\rangle$. Then $I^* = \mathrm{LT}_\sigma(I)$ has the following minimal multigraded free resolution:

$$\begin{aligned}\mathbf{G} : 0 \to &P(1,1,3) \oplus P(2,3,1)\\ \to &P(1,1,1) \oplus P(2,0,1) \oplus P(0,1,3) \oplus P(1,0,3) \oplus P(0,3,1) \oplus P(2,0,3)\\ \to &P(2,0,0) \oplus P(0,1,1) \oplus P(1,0,1) \oplus P(0,3,0) \oplus P(0,0,3) \to I^*.\end{aligned}$$

One can see that the only possible zero σ-cancellation on (F_2, F_1, F_0) is corresponded to $(0, P(1,1,1), P(0,0,3))$ or $(0, P(2,0,1), P(0,0,3))$ and exactly one of them occurs. So, the minimal graded free resolution of I is:

$$0 \to P(-5) \oplus P(-6) \to P(-3) \oplus P^3(-4) \oplus P(-5) \to P^3(-2) \oplus P(-3) \to I.$$

Example 5.8. Let $P = \mathbf{C}[x, y, z]$, $\sigma =$ DegRevLex and $I = \langle x^3 + xy^2 + y^3, xyz, y^2z, z^2\rangle$. Then $I^* = \mathrm{LT}_\sigma(I) = \langle x^3, xyz, y^2z, z^2\rangle$ has the following multigraded free resolution:

$$\begin{aligned}0 \to &P(1,2,2) \oplus P(3,1,2)\\ \to &P(3,0,2) \oplus P(3,1,1) \oplus P(0,2,2) \oplus P(1,1,2) \oplus P(1,2,1)\\ \to &P(0,0,2) \oplus P(3,0,0) \oplus P(1,1,1) \oplus P(0,2,1) \to I^*.\end{aligned}$$

As we see there is no zero σ-cancellation. So the minimal graded free resolution of I is

$$0 \to P(-5) \oplus P(-6) \to P^3(-4) \oplus P^2(-5) \to P(-2) \oplus P^3(-3) \to I.$$

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