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TOTAL DOMINATION NUMBER OF CENTRAL GRAPHS

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Abstract. Let G be a graph with no isolated vertex. *A total dominating set*, abbreviated TDS of G is a subset S of vertices of G such that every vertex of G is adjacent to a vertex in S. *The total domination number* of G is the minimum cardinality of a TDS of G. In this paper, we study the total domination number of central graphs. Indeed, we obtain some tight bounds for the total domination number of a central graph $C(G)$ in terms of some invariants of the graph G. Also we characterize the total domination number of the central graph of some families of graphs such as path graphs, cycle graphs, wheel graphs, complete graphs and complete multipartite graphs, explicitly. Moreover, some Nordhaus-Gaddum-like relations are presented for the total domination number of central graphs.

Introduction

The concept of total domination in graphs was first introduced by Cockayne, Dawes and Hedetniemi [2] and has been studied extensively by many researchers in the last years. The literature on this subject has been surveyed and detailed in the recent book [3]. In this paper, we study the total domination number of central graphs. In the sequel we remind some concepts and terminology which are used in this paper. Let G be a graph with the vertex set $V(G)$ of *order* n and the edge set $E(G)$ of *size* m. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V(G)$ are $N_G(v) = \{u \in V(G) : \ uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is defined as $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a vertex in G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write K_n, C_n and P_n for a *complete* graph, a *cycle* graph and a *path* graph of order n, respectively, while $G[S]$, W_n and $K_{n_1,n_2,\ldots,n_p}$ denote the subgraph of G induced on the vertex set S, a *wheel* graph of order $n+1$, and a *complete p-partite* graph, respectively. The *complement* of a graph G, denoted by $\overline{G}$, is a graph with the vertex set $V(G)$ such that for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. A *vertex cover* of the graph G is a set $D \subseteq V(G)$ such

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that every edge of G is incident to at least one element of D. The *vertex cover number* of G, denoted by $\tau(G)$, is the minimum cardinality of a vertex cover of G. Moreover, an *edge cover* of G is a set $S \subseteq E(G)$ such that every vertex of G is incident to at least one edge in S. The *edge cover number* of G, denoted by $\rho(G)$, is the minimum cardinality of an edge cover of G. An *independent set* of G is a subset of vertices of G, no two of which are adjacent. Also a *maximum independent set* is an independent set of the largest cardinality in G. This cardinality is called the *independence number* of G, and is denoted by $\alpha(G)$. The *clique number* of G is the maximum cardinality of the vertex set of a clique in G. For a tree graph G, any vertex of degree one is called a *leaf* and the neighbour of a leaf is called a *support vertex* of G.

Vernold et al., in [5] by doing an operation on a given graph G obtained the central graph of G as follows.

Definition 0.1 ([5])**.** The central graph $C(G)$ of a graph G of order n and size m is a graph of order $n+m$ and size $\binom{n}{2}+m$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

We fix a notation for the vertex set and the edge set of the central graph $C(G)$ to work with throughout the paper. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$. We set $V(C(G))=V(G)\cup\mathcal{C}$, where $\mathcal{C}=\{c_{ij} : v_iv_j \in E(G)\}$ and $E(C(G))=\{v_ic_{ij}, v_jc_{ij} : v_iv_j \in E(G)\}\cup\{v_iv_j :\ v_iv_j \notin E(G)\}$.

Definition 0.2. A *total dominating set*, briefly TDS, of a graph G is a set $S \subseteq V(G)$ such that $N_G(v)\cap S \neq \emptyset$, for any vertex $v \in V(G)$. The *total domination number of G is the minimum cardinality of a TDS of G and is denoted by* $\gamma_t(G)$. Moreover, a total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t-set of G.

For the standard graph theory terminology not given here we refer to [6]. Throughout this paper, G is a non-empty, finite, undirected and simple graph with the vertex set $V(G)$ and the edge set $E(G)$.

The paper proceeds as follows. In Section 2, first we present some upper and lower bounds for $\gamma_t(C(G))$ in terms of $\tau(G)$, $\rho(G)$ and the clique number of G (Theorems 1.1 and 1.2). Then it is shown that the only graph with n vertices for which the upper bound $n+\lceil n/2\rceil-1$ is gained for $\gamma_t(C(G))$, is the complete graph K_n. Moreover, among other results we give some nice bounds for $\gamma_t(C(G))$, when G is a tree. In Section 3, we determine $\gamma_t(C(G))$ explicitly, when G is P_n, C_n, W_n, K_n or a complete multipartite graph. Finally, in Section 4 we present some Nordhaus-Gaddum-like relations for the total domination number of central graphs.

1. General bounds

In this section, we establish some bounds on the total domination number of a central graph. At the first step we consider connected graphs.

Theorem 1.1. *For any connected graph G of order $n \geq 2$,*

$$\tau(G) \leq \gamma_t(C(G)) \leq \tau(G) + \rho(G).$$

Also these bounds are tight.

Proof. Let G be a connected graph of order $n \geq 2$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Then $V(C(G)) = V \cup \mathcal{C}$, where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$. Let D be a minimal vertex cover of G such that $\tau(G) = |D|$ and S be a minimal edge cover of G such that $\rho(G) = |S|$. Then we show that $W = D \cup \{c_{ij} : v_iv_j \in S\}$ is a TDS of $C(G)$. For any $v_i \in V(G)$, there exists a vertex $v_j \in N_G(v_i)$ such that $v_iv_j \in S$. Thus $c_{ij} \in W \cap N_{C(G)}(v_i) \neq \emptyset$, and we are done. Now for any arbitrary vertex $c_{ij} \in V(C(G))$, we show that $N_{C(G)}(c_{ij}) \cap W \neq \emptyset$. Since D is a vertex cover of G, we have either $v_i \in D \subseteq W$ or $v_j \in D \subseteq W$. So either $v_i \in N_{C(G)}(c_{ij}) \cap W$ or $v_j \in N_{C(G)}(c_{ij}) \cap W$. Hence W is a TDS of $C(G)$ and $\gamma_t(C(G)) \leq |W| = |D| + |S| = \tau(G) + \rho(G)$. To show that $\tau(G) \leq \gamma_t(C(G))$, it is enough to note that for any γ_t-set of $C(G)$ say A, we have $\emptyset \neq N_{C(G)}(c_{ij}) \cap A \subseteq \{v_i, v_j\}$ for any $v_iv_j \in E(G)$. In other words, for any edge $v_iv_j \in E(G)$, we have either $v_i \in A$ or $v_j \in A$. Hence $A \cap V(G)$ is a vertex cover of G. We have $\tau(G) \leq |A| = \gamma_t(C(G))$.

The lower bound is tight. Because if $G = P_n$ for $n \geq 6$, then $\gamma_t(C(G)) = \tau(G)$ by Proposition 2.1. Also by Theorem 1.3 the upper bound is tight. □

Theorem 1.2. *For any connected graph G of order $n \geq 3$ with clique number ω, $3 \leq \gamma_t(C(G)) \leq n + \lceil\frac{\omega}{2}\rceil - 1$. Also these bounds are tight.*

Proof. Let G be a connected graph of order $n \geq 3$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Then $V(C(G)) = V \cup \mathcal{C}$, where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$. If $n = 3$, then G is isomorphic to P_3 or K_3, and so $C(G)$ is isomorphic to cycles C_5 or C_6, and $\gamma_t(C(G)) = 3$ or 4, respectively. So we assume $n \geq 4$. Let $S = S_{\mathcal{C}} \cup S_V$ be a TDS of $C(G)$, where $S_{\mathcal{C}} = S \cap \mathcal{C}$ and $S_V = S \cap V$. By contradiction assume that $|S| = 2$. Since S is a total dominating set, $S_V \neq \emptyset$. If $S_{\mathcal{C}} = \emptyset$, then $|S_V| = |S| = 2$. Let $S = \{v_i, v_j\}$. Since G is connected of order at least 3, without loss of generality, we may assume $P_3 : v_iv_tv_j$ be a path of order 3 as a subgraph of G. This implies that $C(G)$ contains the path $P_5 : v_ic_{it}v_tc_{tj}v_j$ as a subgraph. Obviously $N(v_t) \cap S = \emptyset$, is a contradiction. Now, let $S = \{v_i, c_{ij}\}$. Then there exist $c_{i'j'} \in \mathcal{C}$, such that $i \neq i', j'$ and $N(c_{i'j'}) \cap S = \emptyset$, is a contradiction. Hence $\gamma_t(C(G)) \geq 3$. Let $G[\{v_1, v_2, \ldots, v_\omega\}]$ be a complete graph of order $\omega \leq n$ in G. Since $S = \{v_i : 1 \leq i \leq n-1\} \cup \{c_{(2i-1)(2i)} : 1 \leq i \leq \lceil\omega/2\rceil\}$ is a TDS of $C(G)$ with cardinality $n + \lceil\omega/2\rceil - 1$, we have $\gamma_t(C(G)) \leq n + \lceil\omega/2\rceil - 1$. The lower bound is tight. Because if $G = K_{1,n}$ $(n \geq 2)$, then $\gamma_t(C(G)) = 3$ by Proposition 2.6. Also the upper bound is tight by Theorem 1.3. □

Theorem 1.3. *Let G be a connected graph of order $n \geq 2$. Then*

$$\gamma_t(C(G)) = n + \lceil n/2\rceil - 1 \text{ if and only if } G \cong K_n.$$

Proof. Let $G = K_n$. Then obviously $\tau(G) = n - 1$ and $\rho(G) = \lceil n/2 \rceil$. So $\gamma_t(C(G)) \leq n + \lceil n/2 \rceil - 1$ by Theorem 1.1. Now let S be a γ_t-set of $C(G)$. Then $S_1 = S \cap V(G)$ is a vertex cover of G, since $\emptyset \neq N_{C(G)}(c_{ij}) \cap S \subseteq \{v_i, v_j\}$ for every $v_iv_j \in E(G)$. Also $S_2 = S \cap \{c_{ij} : v_iv_j \in E(G)\}$ is in a bijection with an edge cover of G. Indeed, set $S_2^{'} = \{v_iv_j : c_{ij} \in S_2\} = \{v_iv_j : c_{ij} \in S\}$. For any $v_i \in V(G)$, $\emptyset \neq N_{C(G)}(v_i) \cap S \subseteq \{c_{ij} : v_j \in N_G(v_i)\}$ because G is a complete graph. So there exists $v_j \in N_G(v_i)$ such that $c_{ij} \in S$. Thus $v_iv_j \in S_2^{'}$. This implies that $S_2^{'}$ is an edge cover of G. We have $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Thus $\gamma_t(C(G)) = |S| = |S_1| + |S_2| \geq \tau(G) + \rho(G) = n + \lceil n/2 \rceil - 1$. Hence the equality holds. Now let $\gamma_t(C(G)) = n + \lceil n/2 \rceil - 1$. By Theorem 1.2 $\gamma_t(C(G)) \leq n + \lceil \omega/2 \rceil - 1$ where ω is clique number of G. Thus $n + \lceil \omega/2 \rceil - 1 \geq n + \lceil n/2 \rceil - 1$. Hence $\lceil \omega/2 \rceil = \lceil n/2 \rceil$. So $n - 1 \leq \omega \leq n$. We show that $\omega = n$. Note that if n is odd, then $\lceil (n-1)/2 \rceil \neq \lceil n/2 \rceil$. Hence $\omega = n$. So we assume that n is even. Let $\omega = n - 1$. Without loss of generality let $G[\{v_1, \ldots, v_{n-1}\}] \cong K_{n-1}$ and $v_{n-1}v_n \notin E(G)$. Then the set $\{v_1, \ldots, v_{n-2}\} \cup \{c_{1n}\} \cup \{c_{(2i)(2i+1)} : 1 \leq i \leq \lceil (n-2)/2 \rceil = \lceil n/2 \rceil - 1\}$ is a TDS of $C(G)$ of cardinality $n + \lceil n/2 \rceil - 2$, which contradicts to $\gamma_t(C(G)) = n + \lceil n/2 \rceil - 1$. So $\omega = n$ and $G \cong K_n$. □

Since for any connected graph G of order $n \geq 3$ with $\Delta(G) \leq n - 2$, the set $S = V(G) = \{v_1, \ldots, v_n\}$ is a TDS of $C(G)$, the upper bound $n + \lceil \omega/2 \rceil - 1$ in Theorem 1.2 can be improved to n, as it is stated in Theorem 1.4.

Theorem 1.4. *For any connected graph G of order $n \geq 3$ with $\Delta(G) \leq n - 2$,*

$$3 \leq \gamma_t(C(G)) \leq n.$$

The next theorem shows that the upper bound in Theorem 1.4 is sharp.

Theorem 1.5. *For any $n \geq 4$, there exists a connected graph G of order n with $\gamma_t(C(G)) = n$.*

Proof. Set

$$G = K_n \setminus \{v_{2i-1}v_{2i} : 1 \leq i \leq \lfloor n/2 \rfloor\} \text{ for even } n,$$

and

$$G = K_n \setminus (\{v_1v_n\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq \lfloor n/2 \rfloor\}) \text{ for odd } n.$$

Let S be a TDS of $C(G)$. We claim that $|S \cap V(G)| \geq n - 2$. Otherwise, there exist at least two vertices $v_i, v_j \in V(G)$ such that $v_iv_j \in E(G)$ and $v_i, v_j \notin S$. We conclude that $N(c_{ij}) \cap S = \emptyset$, which is a contradiction. So without loss of generality, we can assume that $V \setminus \{v_3, v_4\} \subseteq S$, because $v_3v_4 \notin E(G)$. Now since $\emptyset \neq N_{C(G)}(v_3) \cap S \subseteq \{c_{3j} : v_3v_j \in E(G)\}$ and $\emptyset \neq N_{C(G)}(v_4) \cap S \subseteq \{v_3, c_{4j} : v_4v_j \in E(G)\}$, so $|S| \geq n$. Thus $\gamma_t(C(G)) \geq n$. Now, by Theorem 1.4, the equality holds. □

In the next two theorems we consider tree graphs.

Theorem 1.6. *Let $\mathbb{T}$ be a tree of order $n \geq 3$ such that $\Delta(\mathbb{T}) \geq n - 3$. Then $\gamma_t(C(\mathbb{T})) = 3$.*

Proof. Let $\mathbb{T}$ be a tree of order $n \geq 3$ with the vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and set $\Delta = \Delta(\mathbb{T})$. Then $V(C(\mathbb{T})) = V \cup \mathcal{C}$, where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(\mathbb{T})\}$. By Theorem 1.2 it is enough to show that $\gamma_t(C(\mathbb{T})) \leq 3$ or equivalently $C(\mathbb{T})$ has a TDS with 3 elements. Let $\Delta = n-1$ and $\deg(v_0) = n-1$. Then $\mathbb{T} \cong K_{1,n-1}$ and $S = \{v_0, v_1, c_{01}\}$ is a TDS of $C(\mathbb{T})$. Let $\Delta = n-2$ and $\deg(v_0) = n-2$. This implies that there exists a vertex $v_i \in N_{\mathbb{T}}(v_0)$ such that v_i is a support vertex of $\mathbb{T}$. Then $S = \{v_0, v_i, c_{0i}\}$ is a TDS of $C(\mathbb{T})$. Let $\Delta = n-3$ and $\deg(v_0) = n-3$. Then $\mathbb{T}$ has either two or three support vertices. In the following we show that in any case, $C(\mathbb{T})$ has a TDS of cardinality 3.

Case 1. Assume that $\mathbb{T}$ has two support vertices say v_i and v_j. If $i, j \neq 0$, then $S = \{v_0, w, z\}$ is a TDS of $C(\mathbb{T})$ where $w \in N_{\mathbb{T}}(v_i)$ and $z \in N_{\mathbb{T}}(v_j)$. Now, let $0 \in \{i, j\}$ and v_0 be a support vertex of $\mathbb{T}$. Let v_k be a leaf of $\mathbb{T}$ such that $d(v_0, v_k) > 1$. If $d(v_0, v_k) = 2$ and v_0, v_t, v_k is a path in $\mathbb{T}$, then $S = \{v_0, v_t, c_{0t}\}$ is a TDS of $C(\mathbb{T})$. Also if $d(v_0, v_k) = 3$ and v_0, v_t, v_ℓ, v_k is a path in $\mathbb{T}$, then $S = \{v_0, v_t, v_k\}$ is a TDS of $C(\mathbb{T})$.

Case 2. Assume that $\mathbb{T}$ has three support vertices and v_0, v_i and v_j be three support vertices of $\mathbb{T}$. Then $S = \{v_0, v, w\}$ is a TDS of $C(\mathbb{T})$ where v and w are two leaves of $\mathbb{T}$ such that $v \in N_{\mathbb{T}}(v_i)$ and $w \in N_{\mathbb{T}}(v_j)$. □

As an immediate consequence of Theorem 1.6 and Propositions 2.1 and 2.6, we have the following result.

Corollary 1.7. *Let $\mathbb{T}$ be a tree of order $3 \leq n \leq 6$. Then $\gamma_t(C(\mathbb{T})) = 3$.*

The next theorem improves the upper bounds given in Theorems 1.2 and 1.4 for a tree graph $\mathbb{T}$ of order $n \geq 7$ with $\Delta(\mathbb{T}) \leq n-4$.

Theorem 1.8. *Let $\mathbb{T}$ be a tree of order $n \geq 7$ such that $\Delta(\mathbb{T}) \leq n-4$. Then $\gamma_t(C(\mathbb{T})) \leq \lfloor 2n/3 \rfloor$. Moreover, the upper bound is tight.*

Proof. Let $\mathbb{T}$ be a tree with the vertex set $V = \{v_0, \ldots, v_{n-1}\}$. Then $V(C(\mathbb{T})) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(\mathbb{T})\}$. Choose a leaf v_0 of $\mathbb{T}$ and label each vertex of $\mathbb{T}$ with its distance from v_0 modulo 3. This partitions V to the three independent sets A_0, A_1 and A_2 where $A_i = \{u \in V : d_{\mathbb{T}}(u, v_0) \equiv i \pmod 3\}$ for $0 \leq i \leq 2$. Then by the piegonhole principle at least one of them, say A_i, contains at least one third of the vertices of $\mathbb{T}$, and so $|A_j \cup A_k| \leq \lfloor 2n/3 \rfloor$, where $\{j, k\} = \{0, 1, 2\} \setminus \{i\}$. Moreover, for every $v_sv_t \in E(\mathbb{T})$, either $v_s \in A_j \cup A_k$ or $v_t \in A_j \cup A_k$, because $d_{\mathbb{T}}(v_0, v_s) \not\equiv d_{\mathbb{T}}(v_0, v_t) \pmod 3$, and so $N_{C(\mathbb{T})}(c_{st}) \cap (A_j \cup A_k) \neq \emptyset$. We have $v_0 \in A_0$. If $|A_0| = 1$, then $\mathbb{T} \cong K_{1,n-1}$ which contradicts to $\Delta(\mathbb{T}) \leq n-4$. So $|A_0| \geq 2$. If $|A_1| = |A_2| = 1$, then $\Delta(\mathbb{T}) > n-4$ which is a contradiction. So $|A_1| \geq 2$ or $|A_2| \geq 2$. The following cases may happen, where in each case we present a set S which is a TDS of $C(\mathbb{T})$ with $|S| \leq \lfloor 2n/3 \rfloor$.

Case 1. Let $|A_1| = 1$ and $|A_2| \geq 2$. Assume that $A_1 = \{v_1\}$. Then any element of A_0 is a leaf of $\mathbb{T}$ and any element of A_2 is adjacent to v_1. If $|A_0| \leq |A_2|$, then $S = A_0 \cup A_1$ is a TDS of $C(\mathbb{T})$, since $|A_0| \geq 2$ as was shown above. One can easily see that $|S| \leq \lfloor 2n/3 \rfloor$. If $|A_2| < |A_0|$, then we set $S = \{v_0, v_i\} \cup A_2$, where $v_i \in A_0$ and $v_i \neq v_0$. One can see that S is a TDS of $C(\mathbb{T})$. Since $|A_0| + |A_2| = n - 1$ and $|A_2| < |A_0|$, we have $|A_2| \leq \lfloor n/2 \rfloor - 1$. Thus $|S| = |A_2| + 2 \leq \lfloor n/2 \rfloor + 1 \leq \lfloor 2n/3 \rfloor$, since $n \geq 7$.

Case 2. Let $|A_2| = 1$ and $|A_1| \geq 2$. If $|A_0| < |A_1|$, then we set $S = A_0 \cup A_2$ and otherwise we set $S = A_1 \cup A_2$. Then S is a TDS of $C(\mathbb{T})$ with $|S| \leq \lfloor 2n/3 \rfloor$.

Case 3. Let $|A_i| \geq 2$ for every $0 \leq i \leq 2$. Let $p, q \in \{0, 1, 2\}$ such that $|A_p \cup A_q| \leq \lfloor 2n/3 \rfloor$. Then we set $S = A_p \cup A_q$. For every $v_i \in A_t$ where $t = p, q$, there exists at least a vertex $v_j \in A_t$ such that $v_j \in N_{C(\mathbb{T})}(v_i) \cap S$. If $S = A_0 \cup A_1$, then for every $v_i \in A_2$, $v_0 \in N_{C(\mathbb{T})}(v_i) \cap S$. If $S = A_1 \cup A_2$, then for every $v_i \in A_0$, there exists at least a vertex $v_j \in A_2$ such that $v_j \in N_{C(\mathbb{T})}(v_i) \cap S$. Let $S = A_0 \cup A_2$ and $v_i \in A_1$. Since $|A_0| \geq 2$, there exists at least a vertex $v_j \in A_0$ such that $v_j \in N_{C(\mathbb{T})}(v_i) \cap S$.

By Proposition 2.9 the upper bound is tight for $\mathbb{T} = S_{1,2,2}$ and $\mathbb{T} = S_{1,3,3}$. □

The next theorem gives some lower and upper bounds for the total domination number of the central graph of a disconnected graph, which none of its connected components is K_1.

Theorem 1.9. *Let G be a graph of order $n \geq 2$ with no isolated vertex. If $G = G_1 \cup \cdots \cup G_w$, that is $G_1, \ldots, G_w$ are all connected components of G with $w \geq 2$, then $\gamma_t(C(G))$ has the following tight bounds*:

$$\tau(G_1) + \cdots + \tau(G_w) \leq \gamma_t(C(G)) \leq n - w.$$

Proof. Let $|V(G_i)| = n_i \geq 2$ for $1 \leq i \leq w$. Obviously $C(G)$ is a graph which is obtained by replacing every maximal independent set of cardinality n_i in $K_{n_1, n_2, \ldots, n_m}$ by $C(G_i)$. If $V(G_i) = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$ and $\mathcal{C}_i = \{c_{i'j'}^i : v_{i'}^i v_{j'}^i \in E(G_i)\}$ for $1 \leq i \leq w$, then

$$V(C(G)) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_w) \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_w.$$

Since $S = \bigcup_{i=1}^{w} (V(G_i) \setminus \{v_{n_i}^i\})$ is a TDS of $C(G)$, so

$$\gamma_t(C(G)) \leq |S| = \sum_{i=1}^{w} (n_i - 1) = n - w.$$

Now let S be a γ_t-set of $C(G)$. Then for any $1 \leq i \leq w$, the set $S_i = S \cap V(G_i)$ is a vertex cover of G_i, since $\emptyset \neq N_{C(G)}(c_{i'j'}^i) \cap S \subseteq \{v_{i'}^i, v_{j'}^i\}$ for every $v_{i'}^i v_{j'}^i \in E(G_i)$. Thus either $v_{i'}^i \in S_i$ or $v_{j'}^i \in S_i$. So $\gamma_t(C(G)) \geq \tau(G_1) + \cdots + \tau(G_w)$.

The upper bound is sharp for $G = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_w}$. Because it can be easily seen $\tau(K_{n_i}) = n_i - 1$ for every $1 \leq i \leq w$. So $\gamma_t(C(G)) \geq \sum_{i=1}^{w} (n_i - 1) = n - w$. Also, the lower bound is sharp for $G = K_{1,n_1-1} \cup K_{1,n_2-1} \cup \cdots \cup K_{1,n_w-1}$. Because $\tau(K_{1,n_i-1}) = 1$ for every $1 \leq i \leq w$ and then $\gamma_t(C(G)) \geq \tau(K_{1,n_1-1}) + \cdots + \tau(K_{1,n_w-1})) = w$. Now since $S = \{v_1^1, v_1^2, \ldots, v_1^w\}$ is a TDS of $C(G)$ with cardinality w where $v_1^i \in V(K_{1,n_i-1})$ and $\deg_{K_{1,n_i-1}}(v_1^i) = n_i - 1$, we have $\gamma_t(C(G)) = w$. □

The next theorem gives some bounds for the total domination number of the central graph of join of a graph with an empty graph $\overline{K_p}$. We recall that the *join* $G \circ H$ of two graphs G and H is the graph obtained by the disjoint union of G and H and joining each vertex of G to all vertices of H.

Theorem 1.10. *For any connected graph G of order $n \geq 2$ and any integer $p \geq 1$,*

$$\gamma_t(C(G)) + 1 \leq \gamma_t(C(G \circ \overline{K_p})) \leq \gamma_t(C(G)) + \max\{2, p\}.$$

Also the bounds are tight.

Proof. Let G be a connected graph with the vertex set $V_1 = \{v_1, \ldots, v_n\}$ and $V(\overline{K_p}) = V_2 = \{v_{n+1}, \ldots, v_{n+p}\}$. Then $V(C(G \circ \overline{K_p})) = V(G \circ \overline{K_p}) \cup \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1 = \{c_{ij} : v_iv_j \in E(G)\}$ and $\mathcal{C}_2 = \{c_{(n+i)j} : 1 \leq i \leq p,\ 1 \leq j \leq n\}$. Let $p = 1$. Then for any γ_t-set S of $C(G)$, $S^{'} = S \cup \{v_{n+1}, c_{1(n+1)}\}$ is a TDS of $C(G \circ \overline{K_p})$. Thus $\gamma_t(C(G \circ \overline{K_p})) \leq \gamma_t(C(G)) + 2$. Now, let $p \geq 2$. Similarly for any γ_t-set S of $C(G)$, $S^{'} = S \cup V_2$ is a TDS of $C(G \circ \overline{K_p})$ and $\gamma_t(C(G \circ \overline{K_p})) \leq \gamma_t(C(G)) + p$, as desired. Now we prove the lower bound. Let S be a γ_t-set of $C(G \circ \overline{K_p})$. Two cases may happen.

Case 1. Assume that for every $v_i \in V_1$, $N_{C(G)}(v_i) \cap S \neq \emptyset$. Then this implies that $S \setminus (V_2 \cup \mathcal{C}_2)$ is a TDS of $C(G)$, since for any $c_{ij} \in \mathcal{C}_1$, $\emptyset \neq N_{C(G \circ \overline{K_p})}(c_{ij}) \cap S = N_{C(G)}(c_{ij}) \cap S \subseteq \{v_i, v_j\}$. Note that for every $1 \leq i \leq p$,

$$(1) \qquad \emptyset \neq N_{C(G \circ \overline{K_p})}(v_{n+i}) \cap S \subseteq (V_2 \setminus \{v_{n+i}\}) \cup \{c_{(n+i)j} : 1 \leq j \leq n\}.$$

Let $w \in N_{C(G \circ \overline{K_p})}(v_{n+i}) \cap S$ for some i. Then by (1), $w \in (V_2 \cup \mathcal{C}_2) \cap S$ which implies that $|S \setminus (V_2 \cup \mathcal{C}_2)| < |S|$. Hence $\gamma_t(C(G)) \leq |S \setminus (V_2 \cup \mathcal{C}_2)| \leq |S| - 1 = \gamma_t(C(G \circ \overline{K_p})) - 1$.

Case 2. Assume that there exists a vertex $v_k \in V_1$ such that $N_{C(G)}(v_k) \cap S = \emptyset$. Without loss of generality assume that $\{v_k \in V_1 : N_{C(G)}(v_k) \cap S = \emptyset\} = \{v_1, \ldots, v_m\}$ for some $1 \leq m \leq n$. Then for any $1 \leq k \leq m$, we have $\emptyset \neq N_{C(G \circ \overline{K_p})}(v_k) \cap S \subseteq \{c_{k(n+j)} : 1 \leq j \leq p\}$. Thus there exists $c_{k(n+j_k)} \in N_{C(G \circ \overline{K_p})}(v_k) \cap S$ for some $1 \leq j_k \leq p$. Also for any $1 \leq k \leq m$, fix an element $c_{km_k} \in \mathcal{C}_1$ (note that since G is connected such element exists). Now, set

$$S' = [(S \setminus \{c_{k(n+j_k)} : 1 \leq k \leq m\}) \cup \{c_{km_k} : 1 \leq k \leq m\}] \cap (V_1 \cup \mathcal{C}_1).$$

One can see that S' is a TDS of $C(G)$ with $|S'| \leq |S|$. If there exists an element $v_{n+i} \in V_2 \cap S$, then we have $|S'| \leq |S| - 1$ and then $\gamma_t(C(G)) + 1 \leq |S'| + 1 \leq |S| = \gamma_t(C(G \circ \overline{K_p}))$. Now let $V_2 \cap S = \emptyset$. Since $\emptyset \neq N_{C(G \circ \overline{K_p})}(c_{i(n+1)}) \cap S \subseteq \{v_i, v_{n+1}\}$ for every $1 \leq i \leq n$ and $v_{n+1} \notin S$, this forces $V_1 \subseteq S$. Thus for any $1 \leq k \leq m$, $N_{C(G)}(v_k) \cap V_1 \subseteq N_{C(G)}(v_k) \cap S = \emptyset$ which implies that v_k is nonadjacent to v_i in $C(G)$ for every $1 \leq i \leq n$. Therefore noting the fact that v_1 is adjacent to no vertex of V_1 and that $V_1 \subseteq S$, we have $S'' = S' \setminus \{v_1\}$ is a TDS of $C(G)$ with $|S''| \leq |S| - 1$ and we are done.

The lower bound is tight. Because if $G = K_n$, n is odd and $p = 1$, then $G \circ \overline{K_1} \cong K_{n+1}$ and $n + \lceil n/2 \rceil - 1 + 1 = \gamma_t(C(G)) + 1 = \gamma_t(C(G \circ \overline{K_1})) = n + 1 + \lceil (n+1)/2 \rceil - 1$ by Theorem 1.3. Also the upper bound is tight for $p = 1$. Because if $G = K_n$, where n is even, then $G \circ \overline{K_1} \cong K_{n+1}$ and $(n + \lceil n/2 \rceil - 1) + 2 = \gamma_t(C(G)) + 2 = \gamma_t(C(G \circ \overline{K_1})) = n + 1 + \lceil (n+1)/2 \rceil - 1$ by Theorem 1.3. □

The following lemma may be useful in turn.

Lemma 1.11. *For any connected graph G of order $n \geq 3$ and size m, $\alpha(C(G)) = m$.*

Proof. Let G be a connected graph of order $n \geq 3$ and size m with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and so $V(C(G)) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$. For $n = 3$ the result is clear. So we may assume that $n \geq 4$. Let S be an arbitrary independent set of $C(G)$ and $S = S_{\mathcal{C}} \cup S_V$ be a partition, where $S_{\mathcal{C}} = S \cap \mathcal{C}$ and $S_V = S \cap V$. Without loss of generality let $S_V = \{v_1, v_2, \ldots, v_k\}$. Then $S_{\mathcal{C}} = \mathcal{C} \setminus (\bigcup_{1 \leq i \leq k} N_{C(G)}(v_i))$, and so

$$|S| = |S_V| + |S_{\mathcal{C}}| = k + m - |\mathcal{C} \cap (\bigcup_{1 \leq i \leq k} N_{C(G)}(v_i))|.$$

We show that

$$|\mathcal{C} \cap (\bigcup_{1 \leq i \leq k} N_{C(G)}(v_i))| \geq k.$$

For $k = 1, 2$, the inequality is clear. So, let $k \geq 3$. Since S_V is independent in $C(G)$, the induced subgraph $G[S_V]$ is isomorphic to the complete graph K_k, and so

$$|\mathcal{C} \cap (\bigcup_{1 \leq i \leq k} N_{C(G)}(v_i))| \geq \binom{k}{2} \geq k,$$

So $|S| \leq m$. One can see that $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$ is an independent set of $C(G)$ of cardinality m. Thus $\alpha(C(G)) = m$. □

2. Central graph of known graphs and their total domination number

In this section, we obtain the total domination number of the central graph of some special families of graphs. The total domination number of the central graph of cycles and paths are given in the first two propositions.

Proposition 2.1. *For any path P_n of order $n \geq 2$,*

$$\gamma_t(C(P_n)) = \begin{cases} 2 & \text{if } n = 2, \\ 3 & \text{if } n = 3, 4, 5, \\ \lfloor n/2 \rfloor & \text{otherwise.} \end{cases}$$

Proof. Let $P_n : v_1v_2 \cdots v_n$ be a path of order $n \geq 2$ in which $v_iv_j \in E(P_n)$ if and only if $2 \leq j = i+1 \leq n$. Then $V(C(P_n)) = V \cup \mathcal{C}$ where $V = V(P_n)$ and $\mathcal{C} = \{c_{i(i+1)} : 1 \leq i \leq n-1\}$. Let S be a TDS of $C(P_n)$. Since $C(P_2) \cong P_3$, we have $\gamma_t(C(P_2)) = 2$. Let $n \in \{3, 4, 5, 6\}$. Then $\gamma_t(C(P_n)) = 3$ by Corollary 1.7. Now let $n \geq 7$. By Theorem 1.1 $\gamma_t(C(P_n)) \geq \tau(P_n) = \lfloor n/2 \rfloor$. Now since $S = \{v_{2i} : 1 \leq i \leq \lfloor n/2 \rfloor\}$ is a TDS of $C(P_n)$, we have $\gamma_t(C(P_n)) = \lfloor n/2 \rfloor$. □

The set $\{v_2, v_4, v_6\}$ is a min-TDS of $C(P_7)$ as illustrated in Figure 1.

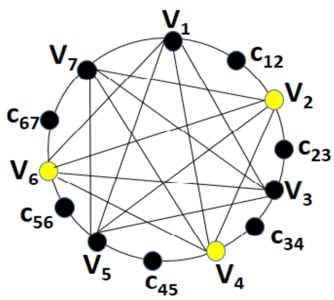

FIGURE 1. A min-TDS of $C(P_7)$

Proposition 2.2. *For any cycle C_n of order $n \geq 3$,*

$$\gamma_t(C(C_n)) = \begin{cases} 4 & \text{if } n = 3, 4, \\ \lceil n/2 \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $C_n : v_1v_2 \cdots v_n$ be a cycle of order $n \geq 3$ in which $v_iv_j \in E(C_n)$ if and only if $j \equiv i+1 \pmod{n}$. Then $V(C(C_n)) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{i(i+1)} : 1 \leq i \leq n-1\} \cup \{c_{1n}\}$. Let S be a TDS of $C(C_n)$. Let $n = 3$. Since $N_{C(C_n)}(c_{ij}) \cap S \neq \emptyset$ for every $c_{ij} \in \mathcal{C}$, so $|S \cap V| \geq 2$. Also since $N_{C(C_n)}(v_i) \cap S \neq \emptyset$ for every $v_i \in V$, so $|S \cap \mathcal{C}| \geq 2$. Hence $|S| = |S \cap V| + |S \cap \mathcal{C}| \geq 4$. Now since $S = \{v_1, v_2, c_{12}, c_{23}\}$ is a TDS of $C(C_3)$, we have $\gamma_t(C(C_3)) = 4$. Let $n = 4$. Since $N_{C(C_n)}(c_{i'j'}) \cap S \neq \emptyset$ for every $c_{i'j'} \in \mathcal{C}$, so $|S \cap V| \geq 2$, and there exist two indices i and j such that $|i - j| = 2$ and $v_i, v_j \in S$. Without loss of generality, we can assume that $v_1, v_3 \in S$. Since $N_{C(C_n)}(v_k) \cap S \neq \emptyset$ for $k = 2, 4$ and also $N_{C(C_n)}(v_2) \cap N_{C(C_n)}(v_4) = \emptyset$, so $|S| \geq 4$. Now since $S = V$ is a TDS of $C(C_4)$, we have $\gamma_t(C(C_n)) = 4$. Let $n \geq 5$. By Theorem 1.1 $\gamma_t(C(C_n)) \geq \tau(C_n) = \lceil n/2 \rceil$. Now since $S = \{v_{2i-1} : 1 \leq i \leq \lceil n/2 \rceil\}$ is a TDS of $C(C_n)$, we have $\gamma_t(C(C_n)) = \lceil n/2 \rceil$. □

Figure 2 illustrates the central graph of the cycle C_7 with a min-TDS $\{v_2, v_4, v_6, v_7\}$.

We use the following theorem which was proved in [1] to compare the total domination number of a path and a cycle with the total domination number its central graphs.

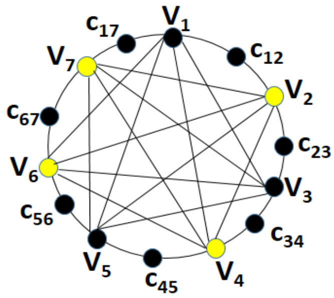

FIGURE 2. A min-TDS of $C(C_7)$

Theorem 2.3. *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. In other word,*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 4, \\ \lfloor n/2 \rfloor + 1 & \text{otherwise.} \end{cases}$$

As an immediate consequence of Propositions 2.1, 2.2 and Theorem 2.3, we have the following corollary.

Corollary 2.4. *For any integer $n \geq 6$,*

$$\gamma_t(C(P_n)) = \begin{cases} \gamma_t(P_n) & \text{if } n \equiv 0 \pmod 4, \\ \gamma_t(P_n) - 1 & \text{otherwise} \end{cases}$$

and for any integer $n \geq 5$,

$$\gamma_t(C(C_n)) = \begin{cases} \gamma_t(C_n) - 1 & \text{if } n \equiv 2 \pmod 4, \\ \gamma_t(C_n) & \text{otherwise.} \end{cases}$$

As a result of Theorem 1.8, Lemma 1.11 and Propositions 2.1, 2.2, we have $\gamma_t(C(P_3)) > \alpha(C(P_3))$, $\gamma_t(C(C_3)) > \alpha(C(C_3))$, $\gamma_t(C(P_4)) = \alpha(C(P_4))$ and for any tree $\mathbb{T}$ of order $n \geq 5$, $\gamma_t(C(\mathbb{T})) \leq \lfloor 2n/3 \rfloor < n - 1 = \alpha(C(\mathbb{T}))$.

As a research problem, it is natural to state the next problem.

Problem 2.5. *Find some families graphs G of order n and size m where $m \geq n \geq 5$ with $\gamma_t(C(G)) = \alpha(C(G))$.*

Now, we consider the central graph of complete multipartite graphs. In the first step, we calculate the total domination number of the central graph of a complete bipartite graph.

Proposition 2.6. *Let $n \geq m \geq 1$ be integers such that $mn \neq 1$. Then $\gamma_t(C(K_{m,n})) = m + 2$.*

Proof. Set $G = K_{m,n}$ such that $n \geq m \geq 1$ and $mn \neq 1$. Let $V \cup U$ be the partition of the vertex set of G to the independent sets $V = \{v_i : 1 \leq i \leq m\}$ and $U = \{u_j : 1 \leq j \leq n\}$. Then $V \cup U \cup \mathcal{C}$ is a partition of the vertex set of

$C(G)$ in which $\mathcal{C} = \{c_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ and $E(C(G)) = \{v_iv_j : 1 \le i < j \le m\} \cup \{u_iu_j : 1 \le i < j \le n\} \cup \{v_ic_{ij}, u_jc_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$. Let S be an arbitrary TDS of $C(G)$. We have either $V \subseteq S$ or $U \subseteq S$. Otherwise if $v_i \notin S$ and $u_j \notin S$ for some i and j, then $N_{C(G)}(c_{ij}) \cap S = \emptyset$, which is a contradiction. Without loss of generality we may assume that $V \subseteq S$. Since $\emptyset \ne N_{C(G)}(u_1) \cap S \subseteq U \cup \{c_{i1} : 1 \le i \le m\}$, we have $c_{i1} \in S$ for some $1 \le i \le m$ or $u_k \in S$ for some $2 \le k \le n$. So $|S| \ge m + 1$. By contradiction assume that $|S| = m + 1$. Then we have $S = V \cup \{c_{i1}\}$ for some i or $S = V \cup \{u_k\}$ for some k. If $S = V \cup \{c_{i1}\}$, then $N_{C(G)}(u_j) \cap S = \emptyset$ for every $2 \le j \le n$, which is a contradiction. If $S = V \cup \{u_k\}$, then $N_{C(G)}(u_k) \cap S = \emptyset$, a contradiction. Thus $|S| \ge m + 2$. Since S was arbitrary, we have $\gamma_t(C(G)) \ge m + 2$. Now since $S = V \cup \{u_1, c_{11}\}$ is a TDS of $C(G)$ of cardinality $m + 2$, we have $\gamma_t(C(G)) = m + 2$. □

The set $\{v_1, v_2, v_3, u_1, c_{11}\}$ is a min-TDS of $C(K_{3,4})$ as depicted in Figure 3.

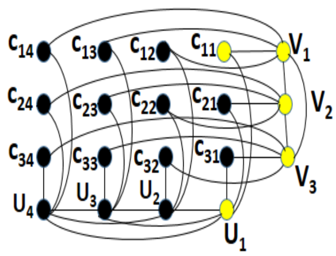

FIGURE 3. A min-TDS of $C(K_{3,4})$

Proposition 2.7. *Let $K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph of order $n \ge 4$ such that $p \ge 3$ and $n_1 \le n_2 \le \cdots \le n_p$. Then*

$$\gamma_t(C(K_{n_1,n_2,\ldots,n_p})) = \begin{cases} \sum_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1 & \text{if } n_p = 2 \text{ or } q \text{ is odd,} \\ \sum_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 2 & \text{otherwise,} \end{cases}$$

where $q = |\{\ i : n_i = 1\}|$.

Proof. Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph of order $n \ge 4$ such that $n_1 \le n_2 \le \cdots \le n_p$, $p \ge 3$ and $V_1 \cup \cdots \cup V_p$ is the partition of $V = V(G) = \{v_i : 1 \le i \le n\}$ to the maximal independent sets $V_1, \ldots, V_p$ with the cardinalities $n_1, \ldots, n_p$, respectively. Set $V' = \{v_i : \{v_i\}$ is a partite of $V\}$ and without loss of generality assume that $V' = \{v_1, \ldots, v_q\}$. Then the induced subgraph of G on the set V' is a complete graph of order q. Let S be an arbitrary TDS of $C(G)$. We claim that $V_i \subseteq S$ for at least $p - 1$ values of $1 \le i \le p$.

Otherwise there exist two sets V_k and V_m such that $V_k \nsubseteq S$ and $V_m \nsubseteq S$. Then for a vertex $v_i \in V_k \setminus S$ and a vertex $v_j \in V_m \setminus S$ we have $N_{C(G)}(c_{ij}) \cap S = \emptyset$, a contradiction. Since $n_1 \leq n_2 \leq \cdots \leq n_p$, without loss of generality we may assume that $V_1 \cup \cdots \cup V_{p-1} \subseteq S$. Then we have $|S \cap V| \geq n_1 + n_2 + \cdots + n_{p-1}$. We consider the following cases.

Case 1. Let $|V_p| = 2$. Without loss of generality assume that $V_p = \{v_{n-1}, v_n\}$ and set $A = \{v_1, \ldots, v_q, v_{n-1}, v_n\}$. For any $v_i \in A$, we have $\emptyset \neq N_{C(G)}(v_i) \cap S \subseteq \mathcal{C} \cup V_p$. Moreover, any vertex of $C(G)$ belongs to $N_{C(G)}(v_i)$ for at most two values of $i \in \{1, \ldots, q, n-1, n\}$. Thus there exists a set $S^{'} \subseteq \mathcal{C} \cup V_n$ such that $S^{'} \subseteq S$ and $|S^{'}| \geq \lceil (q+2)/2 \rceil$. Hence $V_1 \cup \cdots \cup V_{p-1} \cup S^{'} \subseteq S$ and $|S| \geq \sum_{i=1}^{p-1} n_i + |S^{'}| \geq \sum_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1$. Now since for $q > 1$, the set $S = V_1 \cup \cdots \cup V_{p-1} \cup \{c_{(q-1)(n-1)}, c_{qn}\} \cup \{c_{(2i-1)(2i)} : 1 \leq i \leq \lceil (q-2)/2 \rceil\}$ is a TDS of $C(G)$ and for $q = 1$, the set $S = V_1 \cup \cdots \cup V_{p-1} \cup \{c_{1(n-1)}, c_{1n}\}$ is a TDS of $C(G)$, we have $\gamma_t(C(G)) = \sum_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1$.

Case 2. Let $|V_p| \geq 3$. First we assume that there exists a vertex $v_j \in S \cap V_p$. Set $B = \{v_1, v_2, \ldots, v_q, v_j\}$. Then $\emptyset \neq N_{C(G)}(v_i) \cap S \subseteq \mathcal{C} \cup V_p$ for every $v_i \in B$ and any vertex of $C(G)$ belongs to the neighbourhood at most two vertices in B. Thus there exists a set $S^{'} \subseteq \mathcal{C} \cup V_p$ such that $S^{'} \subseteq S$ and $|S^{'}| \geq \lceil (q+1)/2 \rceil$. Hence $V_1 \cup \cdots \cup V_{p-1} \cup S^{'} \subseteq S$ and

$$
\begin{aligned}
|S| &\geq \sum_{i=1}^{p-1} n_i + |S^{'}| + 1 \\
&\geq \sum_{i=1}^{p-1} n_i + \lceil (q+1)/2 \rceil + 1 = \begin{cases} \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1 & \text{if } q \text{ is odd,} \\ \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 2 & \text{if } q \text{ is even.} \end{cases}
\end{aligned}
$$

Now, let $S \cap V_p = \emptyset$. We set $B = \{v_1, v_2, \ldots, v_q\} \cup V_p$. For any $v_i \in B$, we have $\emptyset \neq N_{C(G)}(v_i) \cap S \subseteq \mathcal{C}$ and any vertex of $\mathcal{C}$ belongs to $N_{C(G)}(v_i)$ for at most two vertices $v_i \in B$. Therefore there exists a set $S^{'} \subseteq \mathcal{C}$ such that $S^{'} \subseteq S$ and $|S^{'}| \geq \lceil (q + |V_p|)/2 \rceil \geq \lceil (q+3)/2 \rceil$. Hence $V_1 \cup \cdots \cup V_{p-1} \cup S^{'} \subseteq S$ and

$$
|S| \geq \sum_{i=1}^{p-1} n_i + |S^{'}| \geq \sum_{i=1}^{p-1} n_i + \lceil (q+3)/2 \rceil = \begin{cases} \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1 & \text{if } q \text{ is odd,} \\ \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 2 & \text{if } q \text{ is even.} \end{cases}
$$

Now since $S = V_1 \cup \cdots \cup V_{p-1} \cup \{c_{qn}, v_n\} \cup \{c_{(2i-1)(2i)} : 1 \leq i \leq \lceil (q-1)/2 \rceil\}$ is a TDS of $C(G)$, we have

$$
\gamma_t(C(G)) = \begin{cases} \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 1 & \text{if } q \text{ is odd,} \\ \sum\limits_{i=1}^{p-1} n_i + \lceil q/2 \rceil + 2 & \text{if } q \text{ is even.} \end{cases}
$$

□

In Figure 4, $\{v_i : 1 \leq i \leq 5\} \cup \{c_{45}\}$ is a min-TDS of $C(K_{2,2,3})$.

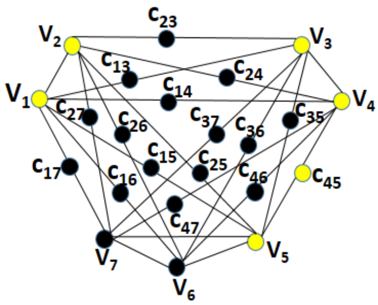

FIGURE 4. A min-TDS of $C(K_{2,2,3})$

In the sequel, we calculate the total domination number of the central graph of a *corona* $G \circ P_1$. We recall that the m-*corona* $G \circ P_m$ of a graph G is the graph obtained from G by adding a path of order m to each vertex of G.

Proposition 2.8. *For any connected graph G of order $n \geq 3$,*

$$\gamma_t(C(G \circ P_1)) = \begin{cases} n+1 & \text{if } G \text{ is a complete graph,} \\ n & \text{otherwise.} \end{cases}$$

Proof. Let G be a connected graph with the vertex set $V(G) = \{v_i : 1 \leq i \leq n\}$. Then $V(G \circ P_1) = \{v_i : 1 \leq i \leq 2n\}$, $E(G \circ P_1) = E(G) \cup \{v_i v_{n+i} : 1 \leq i \leq n\}$ and $V(C(G \circ P_1)) = V(G \circ P_1) \cup \mathcal{C}$ where $\mathcal{C} = \{c_{ij} : v_i v_j \in E(G \circ P_1)\}$. By Theorem 1.1, $\gamma_t(C(G \circ P_1)) \geq \tau(G \circ P_1) = n$. Assume that G is not a complete graph and without loss of generality let $\deg_G(v_n) < n-1$. Then $S^{'} = \{v_i : 1 \leq i \leq n-1\} \cup \{v_{2n}\}$ is a TDS of $C(G \circ P_1)$ of cardinality n. Thus $\gamma_t(C(G \circ P_1)) = n$. Now let G be a complete graph and S be an arbitrary TDS of $C(G \circ P_1)$. We have $|S \cap V(G)| \geq n-1$, since otherwise there would exist a vertex $c_{ij} \in \mathcal{C}$ such that $N_{C(G \circ P_1)}(c_{ij}) \cap S = \emptyset$, a contradiction. So without loss of generality we assume that $\{v_1, \ldots, v_{n-1}\} \subseteq S$. Since $\emptyset \neq N_{C(G \circ P_1)}(c_{n(2n)}) \cap S \subseteq \{v_n, v_{2n}\}$, we have either $v_n \in S$ or $v_{2n} \in S$. Therefore $\{v_1, \ldots, v_{n-1}, v_n\} \subseteq S$ or $\{v_1, \ldots, v_{n-1}, v_{2n}\} \subseteq S$. One can see that none of the sets $\{v_1, \ldots, v_{n-1}, v_n\}$ and $\{v_1, \ldots, v_{n-1}, v_{2n}\}$ is a TDS of $C(G \circ P_1)$. Thus $|S| \geq n+1$. Now since $S^{'} = \{v_i : 1 \leq i \leq n-1\} \cup \{v_{2n}, c_{n(2n)}\}$ is a TDS of $C(G \circ P_1)$ of cardinality $n+1$, we have $\gamma_t(C(G \circ P_1)) = n+1$. □

A min-TDS of $C(P_4 \circ P_1)$ is shown in Figure 5 which is the set $\{v_1, v_2, v_3, v_8\}$.

In the next step, we calculate the total domination number of a *double star graph* $S_{1,n,n}$. We recall that a *double star graph* $S_{1,n,n}$ is obtained from the complete bipartite graph $K_{1,n}$ by replacing every edge by a path of length 2.

Proposition 2.9. *For any integer $n \geq 2$, $\gamma_t(C(S_{1,n,n})) = n+1$.*

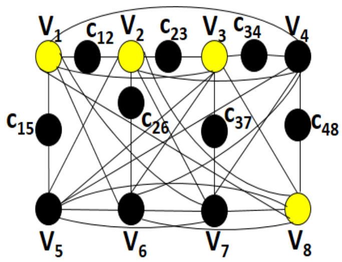

FIGURE 5. A min-TDS of $C(P_4 \circ P_1)$

Proof. Let $G = S_{1,n,n}$ be a double star graph with the vertex set $V(G) = \{v_i : 0 \leq i \leq 2n\}$ and the edge set $E(G) = \{v_0v_i, v_iv_{n+i} : 1 \leq i \leq n\}$. Then $V(C(G)) = V(G) \cup \mathcal{C}$ and $E(C(G)) = \{v_ic_{ij}, v_jc_{ij} : c_{ij} \in \mathcal{C}, v_iv_j \in E(G)\} \cup \{v_iv_j : v_iv_j \notin E(G)\}$ where $\mathcal{C} = \{c_{0i}, c_{i(n+i)} : 1 \leq i \leq n\}$. Let S be a TDS of $C(G)$. For any $1 \leq i \leq n$, $\emptyset \neq N_{C(G)}(c_{i(n+i)}) \cap S \subseteq \{v_i, v_{n+i}\}$. So either $v_i \in S$ or $v_{n+i} \in S$ for every $1 \leq i \leq n$. Hence $|S \cap \{v_i : 1 \leq i \leq 2n\}| \geq n$. If $\{v_1, \ldots, v_n\} \subseteq S$, then consider a vertex $w \in N_{C(G)}(v_0) \cap S$. Since $w \notin \{v_1, \ldots, v_n\}$, we have $|S| \geq n+1$. If $\{v_1, \ldots, v_n\} \nsubseteq S$ and $v_j \notin S$ for some $1 \leq j \leq n$, then $\emptyset \neq N_{C(G)}(c_{0j}) \cap S \subseteq \{v_0, v_j\}$. So $v_0 \in S$ and $|S| \geq n+1$. Therefore $\gamma_t(C(G)) \geq n+1$. Now since $S' = \{v_0\} \cup \{v_{n+i} : 1 \leq i \leq n\}$ is a TDS of $C(G)$, we have $\gamma_t(C(G)) = n+1$. □

In Figure 6, $\{v_0, v_4, v_5, v_6\}$ is a min-TDS of $C(S_{1,3,3})$.

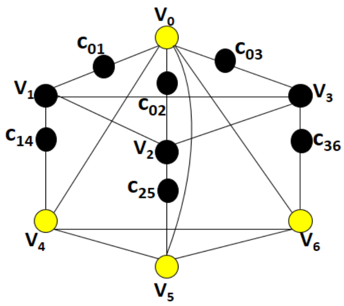

FIGURE 6. A min-TDS of $C(S_{1,3,3})$

In the next proposition the total domination number of the central graph of a wheel graph is obtained.

Proposition 2.10. *For any wheel W_n of order $n+1 \geq 4$,*

$$\gamma_t(C(W_n)) = \begin{cases} 5 & \text{if } n = 3, 4, \\ \lceil n/2 \rceil + 2 & \text{otherwise.} \end{cases}$$

Proof. Since W_3 is isomorphic to the complete graph K_4, and $\gamma_t(C(K_4)) = 5$ by Theorem 1.3, we may assume that $n \geq 4$. Consider W_n with the vertex set $V = \{v_i : 0 \leq i \leq n\}$, and the edge set $E = \{v_0v_i, v_iv_{i+1} : 1 \leq i \leq n\}$. Then $V(C(W_n)) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{0i}, c_{i(i+1)} : 1 \leq i \leq n\}$. Since $W_n = C_n \circ K_1$ where $V(K_1) = \{v_0\}$ and $V(C_n) = V \setminus \{v_0\}$, Theorem 1.10 implies that

$$\gamma_t(C(C_n)) + 1 \leq \gamma_t(C(W_n)) \leq \gamma_t(C(C_n)) + 2. \tag{2}$$

Let $n = 4$. Then $\gamma_t(C(W_4)) \geq \gamma_t(C(C_4)) + 1 = 5$ by Proposition 2.2. Now since $S = \{v_0, v_1, v_3, c_{12}, c_{04}\}$ is a TDS of $C(W_4)$, we have $\gamma_t(C(W_4)) = 5$. Now let $n \geq 5$. By Proposition 2.2 and (2), $\gamma_t(C(W_n)) \leq \gamma_t(C(C_n)) + 2 = \lceil n/2 \rceil + 2$. Therefore it is sufficient to show that $\gamma_t(C(W_n)) \geq \lceil n/2 \rceil + 2$. Let S be a TDS of $C(W_n)$. If $v_0 \notin S$, then $\emptyset \neq N_{C(W_n)}(c_{0j}) \cap S \subseteq \{v_0, v_j\}$ for every $1 \leq i \leq n$. Thus $v_j \in S$ for every $1 \leq j \leq n$ and $|S| \geq n \geq \lceil n/2 \rceil + 2$. Now, let $v_0 \in S$. Then $\emptyset \neq N_{C(W_n)}(v_0) \cap S \subseteq \{c_{0j} : 1 \leq j \leq n\}$. Thus $c_{0j} \in S$ for some $1 \leq j \leq n$. Also since $\emptyset \neq N_{C(W_n)}(c_{i(i+1)}) \cap S \subseteq \{v_i, v_{i+1}\}$ for every $1 \leq i \leq n$, we have $|S \cap \{v_1, \ldots, v_n\}| \geq \lceil n/2 \rceil$. Hence $|S| \geq n \geq \lceil n/2 \rceil + 2$. □

In Figure 7, $\{v_0, v_2, v_4, v_6, c_{05}\}$ is a min-TDS of $C(W_6)$.

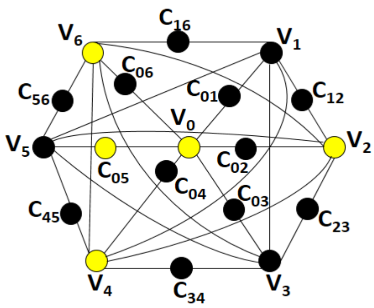

FIGURE 7. A min-TDS of $C(W_6)$

Comparing Theorem 2.3 and Proposition 2.1 we conclude that if $n \equiv 0 \pmod 4$, then $\gamma_t(P_n) = \gamma_t(C(P_n))$ and if $n \equiv 1 \pmod 4$, then $\gamma_t(P_n) > \gamma_t(C(P_n))$. Also, obviously $\gamma_t(C(K_{m,n})) = m + 2 > 2 = \gamma_t(K_{m,n})$ by Proposition 2.6. So Theorem 2.3 and Propositions 2.1 and 2.6 confirm the truth of the next remark.

Remark 2.11. If G is a connected graph of order n, then one may not conclude that

$$\gamma_t(G) \geq \gamma_t(C(G)) \text{ or } \gamma_t(G) \leq \gamma_t(C(G)).$$

We end this section with the following natural problem.

Problem 2.12. *Characterize the trees* $\mathbb{T}$ *satisfying* $\gamma_t(C(\mathbb{T})) = \lfloor 2n/3 \rfloor$.

3. Nordhaus-Gaddum-like relations

Finding a Nordhaus-Gaddum-like relation for any parameter in graph theory is one of the traditional works which is started after the following theorem by Nordhaus and Gaddum in 1956 [4].

Theorem 3.1 ([4])**.** *For any graph G of order n, $2\sqrt{n} \leq \chi(G)+\chi(\overline{G}) \leq n+1$.*

Here, we present some Nordhaus-Gaddum-like relations for the total domination number of central graphs.

Theorem 3.2. *Let $G \neq K_{1,n-1}$ be a connected graph of order $n \geq 4$. Then $\gamma_t(\overline{C(G)}) = 2$*

Proof. Let $G \neq K_{1,n-1}$ be a connected graph of order $n \geq 4$ with the vertex set $V = \{v_1, \ldots, v_n\}$. Then $V(C(G)) = V(\overline{C(G)}) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$ and $E(\overline{C(G)}) = E(G) \cup \{c_{ij}v_k : c_{ij} \in \mathcal{C},\ v_k \in V,\ \text{and}\ k \neq i,j\} \cup \{c_{ij}c_{i'j'} : c_{ij}, c_{i'j'} \in \mathcal{C}\}$. Since $G \neq K_{1,n-1}$, so there exist at least two edges $v_iv_j, v_{i'}v_{j'} \in E(G)$ such that $\{i,j\} \cap \{i^{'}, j^{'}\} = \emptyset$. Now since $S = \{c_{ij}, c_{i'j'}\}$ is a min-TDS of $\overline{C(G)}$, we have $\gamma_t(\overline{C(G)}) = 2$. □

Proposition 3.3. *Let $n \geq 3$ be an integer. Then $\gamma_t(\overline{C(K_{1,n-1})}) = 3$.*

Proof. Let $G = K_{1,n-1}$ be a star graph of order $n \geq 3$ with the vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ where $\deg(v_0) = n-1$. Then $V(C(G)) = V(\overline{C(G)}) = V \cup \mathcal{C}$ where $\mathcal{C} = \{c_{0i} : 1 \leq i \leq n-1\}$ and $E(\overline{C(G)}) = E(G) \cup \{c_{0i}v_k : c_{0i} \in \mathcal{C},\ v_k \in V,\ \text{and}\ k \neq 0, i\} \cup \{c_{0i}c_{0j} : i \neq j\}$. We show that no set of cardinality 2 is a TDS of $C(G)$. If $S = \{c_{0i}, c_{0j}\}$ for some i, j, then $N_{\overline{C(G)}}(v_0) \cap S = \emptyset$. If $S = \{v_0, v_i\}$ for some i, then $N_{\overline{C(G)}}(c_{0i}) \cap S = \emptyset$. If $S = \{v_i, c_{0j}\}$ for some $i \neq j$, where $1 \leq i \leq n$, then $N_{\overline{C(G)}}(v_j) \cap S = \emptyset$. Hence in each case S is not a TDS of $C(G)$. Thus $\gamma_t(\overline{C(G)}) \geq 3$. Now since $S = \{v_0, v_1, c_{02}\}$ is a TDS of $\overline{C(G)}$, we have $\gamma_t(\overline{C(G)}) = 3$. □

As an immediate consequence of Theorem 1.6 for $\Delta = n-1$ and Proposition 3.3, we have the following result.

Corollary 3.4. *There exists a connected graph G of order $n \geq 3$ with $\gamma_t(C(G)) = \gamma_t(\overline{C(G)})$.*

As a result of Theorems 1.2, 1.4, 1.6, 1.8, 3.2 and Proposition 2.1, we have the next corollaries as three Nordhaus-Gaddum-like relations.

Corollary 3.5. *For any connected graph $G \neq K_{1,n-1}$ of order $n \geq 4$,*

$$5 \leq \gamma_t(C(G)) + \gamma_t(\overline{C(G)}) \leq n + \lceil \frac{n}{2} \rceil + 1.$$

Corollary 3.6. *For any connected graph $G \neq K_{1,n-1}$ of order $n \geq 4$ with $\Delta(G) \leq n-2$,*

$$5 \leq \gamma_t(C(G)) + \gamma_t(\overline{C(G)}) \leq n + 2.$$

Corollary 3.7. *For any tree* $\mathbb{T} \neq K_{1,n-1}$ *of order* $n \geq 3$*,*

$$5 \leq \gamma_t(C(\mathbb{T})) + \gamma_t(\overline{C(\mathbb{T})}) \leq \lfloor 2n/3 \rfloor + 2.$$

In particular, if $\mathbb{T}$ *is a path, then*

$$5 \leq \gamma_t(C(\mathbb{T})) + \gamma_t(\overline{C(\mathbb{T})}) \leq \lfloor n/2 \rfloor + 2.$$

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