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GRADED INTEGRAL DOMAINS IN WHICH EACH NONZERO HOMOGENEOUS IDEAL IS DIVISORIAL

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Abstract. Let Γ be a nonzero commutative cancellative monoid (written additively), $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded integral domain with $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, and $S(H) = \{f \in R \,|\, C(f) = R\}$. In this paper, we study homogeneously divisorial domains which are graded integral domains whose nonzero homogeneous ideals are divisorial. Among other things, we show that if R is integrally closed, then R is a homogeneously divisorial domain if and only if $R_{S(H)}$ is an h-local Prüfer domain whose maximal ideals are invertible, if and only if R satisfies the following four conditions: (i) R is a graded-Prüfer domain, (ii) every homogeneous maximal ideal of R is invertible, (iii) each nonzero homogeneous prime ideal of R is contained in a unique homogeneous maximal ideal, and (iv) each homogeneous ideal of R has only finitely many minimal prime ideals. We also show that if R is a graded-Noetherian domain, then R is a homogeneously divisorial domain if and only if $R_{S(H)}$ is a divisorial domain of (Krull) dimension one.

0. Introduction

Let R be a commutative ring with identity and M be a unitary R-module. Then $\mathrm{Hom}_R(M, R)$, called the dual of M, is an R-module, and there is a natural R-module homomorphism φ from M into $\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ given by $\varphi(m)(f) = f(m)$ for all $m \in M$ and $f \in \mathrm{Hom}_R(M, R)$. An R-module M is said to be *reflexive* if φ is bijective. We say that R is a *reflexive ring* if every submodule of a finitely generated R-module is reflexive [11, page 6]. It is known that if R is a Noetherian domain, then R is reflexive if and only if every ideal of R is reflexive [11, Theorem 3.8]. Also, let D be an integral domain with quotient field K, and note that if I is a nonzero ideal of D, then $\mathrm{Hom}_D(I, D)$ is naturally isomorphic to $I^{-1} = \{x \in K \,|\, xI \subseteq D\}$; hence I is reflexive if and only if $(I^{-1})^{-1} = I$.

Let $F(D)$ be the set of all nonzero fractional ideals of D. An ideal of D means a fractional ideal I of D with $I \subseteq D$. For $I, J \in F(D)$, let $(I :_K$

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$J) = \{x \in K \mid xJ \subseteq I\}$; then $(I :_K J) \in F(D)$. Also, $I^{-1} = (D :_K I)$ and $I_v = (I^{-1})^{-1}$. It is known that if $\{A_\alpha\} \subseteq F(D)$ with $\bigcap_\alpha A_\alpha \neq (0)$, then $\bigcap_\alpha (A_\alpha)_v = (\bigcap_\alpha (A_\alpha)_v)_v$. Also, $(IJ)_v = (I_v J)_v = (I_v J_v)_v$ and $(aD)_v = aD$ for all $I, J \in F(D)$ and $0 \neq a \in K$. We say that I is a *divisorial ideal* or a v-ideal if $I_v = I$. Note that if A is a nonzero fractional ideal of D, then there exists a nonzero element $a \in D$ such that $aA \subseteq D$, and since $(aA)_v = a(A_v)$, A is divisorial if and only if aA is divisorial. Hence, every nonzero fractional ideal of D is divisorial if and only if every nonzero ideal of D is divisorial. We say that D is a *divisorial domain* if every nonzero ideal of D is divisorial. Hence, if D is Noetherian, then D is divisorial if and only if D is reflexive.

Divisorial domains have been studied by many researchers (see, for example, [5, 7, 11]). In [7], Heinzer showed that (i) if D is a divisorial domain, then D is an h-local domain (i.e., every nonzero ideal of D is contained in only finitely many maximal ideals and every nonzero prime ideal of D is contained in a unique maximal ideal) and (ii) if D is integrally closed, then D is a divisorial domain if and only if D is a Prüfer domain whose nonzero maximal ideals are invertible, each nonzero prime ideal of D is contained in a unique maximal ideal and each ideal of D has only finitely many minimal prime ideals. Matlis [11] studied a larger class of domains, the class of reflexive domains, where a nonzero ideal of D that is D-reflexive is just the divisorial ideal. Among other things, he showed that D is a Noetherian divisorial domain if and only if D has (Krull) dimension one and M^{-1} is two generated for all maximal ideals M of D [11, Theorem 3.8]. In [5, Proposition 5.4], Bazzoni and Salce proved that D is a divisorial domain if and only if D is an h-local domain and D_M is a divisorial domain for every maximal ideal M of D. In this paper, we study *homogeneously divisorial domains* which are graded integral domains whose nonzero homogeneous ideals are divisorial. (Definitions related with graded integral domains will be reviewed in the sequel.)

Let Γ be a nonzero torsionless grading monoid, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded integral domain with $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, H be the saturated multiplicative set of nonzero homogeneous elements of R, and $S(H) = \{f \in R \mid C(f) = R\}$. In Section 1, we show that if R is a homogeneously divisorial domain, then every nonzero homogeneous ideal (resp., homogeneous prime ideal) of R is contained in only finitely many homogeneous maximal ideals (resp., a unique homogeneous maximal ideal) and $R_{H \setminus P}$ is a homogeneously divisorial domain for all $P \in$ h-Max(R). In Section 2, among other things, we prove that if R is integrally closed, then R is a homogeneously divisorial domain if and only if $R_{S(H)}$ is a divisorial domain, if and only if R satisfies the following four conditions: (i) R is a graded-Prüfer domain, (ii) every homogeneous maximal ideal of R is invertible, (iii) each nonzero homogeneous prime ideal of R is contained in a unique homogeneous maximal ideal, and (iv) each homogeneous ideal of R has only finitely many minimal prime ideals. Suppose that R is a gr-Noetherian domain. In Section 3, we show that R is a homogeneously divisorial domain if and only if $R_{H \setminus P}$ is a homogeneously divisorial domain for all $P \in$ h-Max(R), if and only if $R_{S(H)}$ is a divisorial domain of (Krull) dimension one, if and only if each nonzero homogeneous prime ideal of R has height-one and M^{-1} is generated by two elements for every $M \in$ h-Max(R).

Definitions related to graded integral domains. Let Γ be a nonzero torsionless grading monoid, that is, Γ is a commutative cancellative monoid (written additively) and $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle\Gamma\rangle$ is a torsionfree abelian group. It is well known that a cancellative monoid is torsionless if and only if it can be given a total order compatible with the monoid operation [12, page 123]. By a (Γ-)graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, we mean an integral domain graded by Γ. That is, each nonzero $x \in R_\alpha$ has degree α, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Thus, each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. Clearly, $\mathrm{Supp}(R) = \{\alpha \in \Gamma \mid R_\alpha \neq \{0\}\}$ is a submonoid of Γ because R is an integral domain. Hence, by replacing Γ with $\mathrm{Supp}(R)$, we may assume that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$. A nonzero $x \in R_\alpha$ for every $\alpha \in \Gamma$ is said to be *homogeneous*.

Let $H = \bigcup_{\alpha\in\Gamma}(R_\alpha \setminus \{0\})$; so H is the saturated multiplicative set of nonzero homogeneous elements of R. Then R_H, called the *homogeneous quotient field* of R, is a $\langle\Gamma\rangle$-graded integral domain whose nonzero homogeneous elements are units. We say that an overring E of R is a *homogeneous overring* of R if $E = \bigoplus_{\alpha\in\langle\Gamma\rangle}(E \cap (R_H)_\alpha)$; so E is a $\langle\Gamma\rangle$-graded integral domain such that $R \subseteq E \subseteq R_H$. Clearly, if $\Lambda = \{\alpha \in \langle\Gamma\rangle \mid E \cap (R_H)_\alpha \neq \{0\}\}$, then Λ is a torsionless grading monoid such that $\Gamma \subseteq \Lambda \subseteq \langle\Gamma\rangle$ and $E = \bigoplus_{\alpha\in\Lambda}(E \cap (R_H)_\alpha)$. It is obvious that R_S is a homogeneous overring of R for a multiplicative set S of nonzero homogeneous elements of R (with $\deg(\frac{a}{b}) = \deg(a) - \deg(b)$ for $a \in H$ and $b \in S$). For a fractional ideal A of R with $A \subseteq R_H$, let A^* be the fractional ideal of R generated by homogeneous elements in A. The A is said to be *homogeneous* if $A^* = A$. Clearly, $A^* \subseteq A$; and if A is a prime ideal, then A^* is a prime ideal; in particular, the minimal prime ideals of a nonzero homogeneous ideal are homogeneous. The letter R will denote a graded integral domain henceforth. A homogeneous ideal of R is called a *homogeneous maximal ideal* if it is maximal among proper homogeneous ideals of R. It is easy to see that each proper homogeneous ideal of R is contained in a homogeneous maximal ideal of R. Let h-Max(R) be the set of homogeneous maximal ideals of R. It is known that if A is a homogeneous ideal of R, then $A = \bigcap\{AR_{H\setminus P} \mid P \in \text{h-Max}(R)\}$ [16, Proposition 2.6]. For $f \in R_H$, let $C(f)$ denote the fractional ideal of R generated by the homogeneous components of f. It is well known that if $f, g \in R_H$, then $C(f)^{n+1}C(g) = C(f)^nC(fg)$ for some integer $n \geq 1$ ([12] or [2, Lemma 1.1]); so if $S(H) = \{f \in R \mid C(f) = R\}$, then $S(H)$ is a saturated multiplicative subset of R.

We say that R is a *graded-Prüfer domain* if each nonzero finitely generated homogeneous ideal of R is invertible. R is a *graded-valuation domain* (gr-valuation domain) if either $aR \subseteq bR$ or $bR \subseteq aR$ for all $a, b \in H$; equivalently, if R is a graded-Prüfer domain with unique homogeneous maximal ideal. We call R a *graded-Noetherian domain* (gr-Noetherian domain) if R satisfies the ascending chain condition on homogeneous ideals. Clearly, R is a gr-Noetherian domain if and only if each homogeneous ideal of R is finitely generated, if and only if each homogeneous prime ideal of R is finitely generated [15, Lemma 2.3].

Notation. Throughout this paper, Γ denotes a nonzero torsionless grading monoid (written additively), $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a Γ-graded integral domain with $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, h-Max(R) is the set of homogeneous maximal ideals of R, H is the saturated multiplicative set of nonzero homogeneous elements of R, and $S(H) = \{f \in R \,|\, C(f) = R\}$. In order to avoid a trivial case, we always assume that $R \subsetneq R_H$.

1. Some results in the general case

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. We will say that R is a *homogeneously divisorial domain* if each nonzero homogeneous ideal of R is divisorial. We begin this section with an example of graded integral domains R such that R is a homogeneously divisorial domain but not a divisorial domain.

Example 1.1. Let D be an integral domain with quotient field K and $R = D[X, X^{-1}]$ be the Laurent polynomial ring over D. Then R is a $\mathbb{Z}$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in D$ and an integer $n \in \mathbb{Z}$, and $H = \{aX^n \,|\, 0 \neq a \in D \text{ and } n \in \mathbb{Z}\}$; so $R_H = K[X, X^{-1}]$.

(1) If $D \neq K$, then $(a, 1 + X) \subsetneq (a, 1 + X)_v = R$ for any nonzero nonunit $a \in D$. Hence, R is a divisorial domain if and only if $D = K$.

(2) Note that each nonzero homogeneous element of R is of the form aX^n for some $0 \neq a \in D$ and $n \in \mathbb{Z}$. Note also that $aX^nR = aR$. Hence, a nonzero ideal A of R is homogeneous if and only if $A = IR$ for some nonzero ideal I of D. In this case, $A_v = I_vR$ [8, Proposition 2.2], $IR \cap D = I$ and $I_vR \cap D = I_v$. Thus, R is a homogeneously divisorial domain if and only if D is a divisorial domain.

(3) Let $D = \mathbb{Z}$ be the ring of integers. Then, R is a homogeneously divisorial domain by (2), while there is a nonzero ideal of R that is not divisorial (for example, $(2, 1 + X) \subsetneq (2, 1 + X)_v = R$).

The next result appears in [1, Proposition 2.5] which is essential in the subsequent arguments, and we use it without citation.

Lemma 1.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with quotient field K. If I and J are homogeneous fractional ideals of R, then $(I :_K J)$ is also a homogeneous fractional ideal and $(I :_K J) = (I :_{R_H} J)$. Thus if I is a homogeneous fractional ideal of R, then both I^{-1} and I_v are homogeneous.*

Let I be a nonzero fractional ideal of an integral domain D. Then I_v is the intersection of all principal fractional ideals of D containing I [6, Theorem 34.1]. The next result is a graded integral domain analog.

Lemma 1.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. If I is a homogeneous fractional ideal of R, then I_v is the intersection of the set of homogeneous principal fractional ideals of R containing I.*

Proof. Assume that $\{(\xi_\lambda)\}_{\lambda\in\Lambda}$ is the set of homogeneous principal fractional ideals of R which contain I. If $\lambda \in \Lambda$, then $I \subseteq (\xi_\lambda)$, and hence $I_v \subseteq (\xi_\lambda)_v = (\xi_\lambda)$. Thus, $I_v \subseteq \bigcap_{\lambda\in\Lambda}(\xi_\lambda)$. For the reverse containment, let $x \in R_H \setminus I_v$. Then $xI^{-1} \nsubseteq R$, and hence $xa \notin R$ for some homogeneous element $a \in I^{-1}$. Thus, $x \notin (a^{-1})$, but $I \subseteq (a^{-1})$ because $a \in I^{-1}$. Consequently, $\bigcap_{\lambda\in\Lambda}(\xi_\lambda) \subseteq I_v$. $\square$

Lemma 1.4 (cf. [7, Lemma 2.1])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain and $\{A_\alpha\}$ be a set of homogeneous fractional ideals of R such that $\bigcap_\alpha A_\alpha \neq (0)$. Then $(\bigcap_\alpha A_\alpha)^{-1} = \sum_\alpha A_\alpha^{-1}$.*

Proof. Note that $\bigcap_\alpha A_\alpha \subseteq A_\alpha$ for each α; so $A_\alpha^{-1} \subseteq (\bigcap_\alpha A_\alpha)^{-1}$, and hence $\sum_\alpha A_\alpha^{-1} \subseteq (\bigcap_\alpha A_\alpha)^{-1}$ and $\sum_\alpha A_\alpha^{-1}$ is a homogeneous fractional divisorial ideal. For the reverse containment, let $A = \sum_\alpha A_\alpha^{-1}$. Then $A_\alpha^{-1} \subseteq A$ implies that $A^{-1} \subseteq (A_\alpha^{-1})^{-1} = A_\alpha$ for each α. Thus, $A^{-1} \subseteq \bigcap_\alpha A_\alpha$, and hence $(\bigcap_\alpha A_\alpha)^{-1} \subseteq (A^{-1})^{-1} = \sum_\alpha A_\alpha^{-1}$. $\square$

Lemma 1.5 (cf. [7, Lemma 2.2])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain and P be a homogeneous maximal ideal of R. Then $R \subsetneq P^{-1}$ and $P^{-1} = R + xR$ for all homogeneous elements $x \in P^{-1}\setminus R$.*

Proof. Note that $P = (P^{-1})^{-1}$; so $R \subsetneq P^{-1}$, and since P^{-1} is homogeneous, the homogeneous elements of $P^{-1}\setminus R$ is non empty. Set $F = xR + R$ for a homogeneous element $x \in P^{-1} \setminus R$. Then $R \subsetneq F \subseteq P^{-1}$, and hence $P = (P^{-1})^{-1} \subseteq F^{-1} \subsetneq R$; so $P = F^{-1}$. Thus, $P^{-1} = (F^{-1})^{-1} = F$. $\square$

Lemma 1.6 (cf. [7, Lemma 2.3])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain, A be a nonzero proper homogeneous ideal of R, and P be a homogeneous maximal ideal of R containing A. If $\{B_\alpha\}$ is the set of homogeneous ideals of R which contain A and are not contained in P, then $B = \bigcap_\alpha B_\alpha$ is not contained in P. Hence, $A \subsetneq B$.*

Proof. If $B_1 \in \{B_\alpha\}$, then $B_1 \nsubseteq P$, and hence $B_1 + P = R$ because P is a homogeneous maximal ideal and $B_1 + P$ is homogeneous. Thus, $R = (B_1 + P)^{-1} = B_1^{-1} \cap P^{-1}$. Choose a homogeneous element $x \in P^{-1}\setminus R$. If $\{B_i\}_{i=1}^n$ is a finite subset of $\{B_\alpha\}$, then $\bigcap_{i=1}^n B_i$ is a homogeneous ideal of R which is not contained in P; so $\bigcap_{i=1}^n B_i \in \{B_\alpha\}$. Thus, $x \notin (\bigcap_{i=1}^n B_i)^{-1} = \sum_{i=1}^n B_i^{-1}$ by Lemma 1.4 because $(\bigcap_{i=1}^n B_i)^{-1} \cap P^{-1} = R$ by the second sentence above. It follows that $x \notin \sum_\alpha B_\alpha^{-1} = (\bigcap_\alpha B_\alpha)^{-1}$. Hence, $P^{-1} \nsubseteq (\bigcap_\alpha B_\alpha)^{-1}$, and thus $\bigcap_\alpha B_\alpha \nsubseteq P$. $\square$

Let D be an integral domain, $\mathrm{Max}(D)$ be the set of maximal ideals of D, and $\{D_\lambda\}$ be a set of overrings of D such that $D = \bigcap_\lambda D_\lambda$. We say that the intersection $\bigcap_\lambda D_\lambda$ is *locally finite* if each nonzero nonunit of D is a unit in D_λ

for all but a finitely many D_λ of $\{D_\lambda\}$. Following [10], we call D an *h-local domain* if (i) each nonzero prime ideal of D is contained in a unique maximal ideal and (ii) D has finite character, i.e., the intersection $\bigcap_{M\in \mathrm{Max}(D)} D_M$ is locally finite. As a graded integral domain analog, we will say that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is *homogeneously h-local* if (i) each nonzero homogeneous prime ideal of R is contained in a unique homogeneous maximal ideal and (ii) the intersection $\bigcap_{M\in \text{h-Max}(R)} R_{H\setminus M}$ is locally finite.

We next give a graded integral domain analog of [7, Theorems 2.4 and 2.5] (or [11, Theorem 2.7]) that D is an h-local domain when every nonzero ideal of D is divisorial.

Theorem 1.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain. Then R is a homogeneously h-local domain.*

Proof. (1) Let Q be a nonzero homogeneous prime ideal of R, and assume that there exist two distinct homogeneous maximal ideals P_1 and P_2 of R which contain Q. Let $\{B_\alpha\}$ be the set of all homogeneous ideals of R such that $Q \subseteq B_\alpha \nsubseteq P_1$, and set $B = \bigcap_\alpha B_\alpha$. Then B is a homogeneous ideal of R with $B \nsubseteq P_1$ by Lemma 1.6. Choose a homogeneous element $y \in B\setminus P_1$. Then $y^2 \notin P_1$, and hence $Q + (y^2) \in \{B_\alpha\}$. Hence, $y \in Q + (y^2)$; so $y = q + ry^2$ for some $q \in Q$ and $r \in R$. Thus, $y(1 - ry) = q \in Q$. Note that $y \notin Q$; so $1 - ry \in Q$. However, note also that $P_2 \in \{B_\alpha\}$ because $Q \subseteq P_2$ and $P_2 \nsubseteq P_1$; so $y \in B \subseteq P_2$. Thus, $1 = (1 - ry) + ry \in P_2$, a contradiction.

(2) Let A be a nonzero homogeneous ideal of R, and let $\{P_\alpha \mid \alpha \in \Lambda\}$ be the set of homogeneous maximal ideals of R which contain A. For each $\alpha \in \Lambda$, let F_α be the intersection of homogeneous integral ideals of R which contain A but are not contained in P_α. By Lemma 1.6, F_α is not contained in P_α. Hence, $A \subseteq \sum F_\alpha$ and $\sum F_\alpha \nsubseteq P_\beta$ for all $\beta \in \Lambda$. Thus, $\sum F_\alpha = R$ and $1 = \sum_{i=1}^n t_i$ for some $t_i \in F_i \in \{F_\alpha\}$. Hence, $\sum_{i=1}^n F_i = R$, and thus $\{P_\alpha\} = \{P_i\}_{i=1}^n$. Indeed, if $P_\alpha \notin \{P_1, \ldots, P_n\}$, then $A \subseteq P_\alpha$ and $P_\alpha \nsubseteq P_i$ for $i = 1, \ldots, n$. Hence, $F_i \subseteq P_\alpha$ for $i = 1, \ldots, n$, and therefore $R = \sum_{i=1}^n F_i \subseteq P_\alpha \subsetneq R$, a contradiction. □

Corollary 1.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain and $S(H) = \{f \in R \mid C(f) = R\}$.*

(1) $\mathrm{Max}(R_{S(H)}) = \{Q_{S(H)} \mid Q \in h\text{-}\mathrm{Max}(R)\}$.
(2) *$R_{S(H)}$ has finite character.*
(3) *Every maximal ideal of $R_{S(H)}$ is divisorial.*

Proof. Clearly, $S(H) = \{f \in R \mid C(f)_v = R\}$. Also, by Theorem 1.7, the intersection $\bigcap_{Q\in \text{h-Max}(R)} R_Q$ is locally finite. Thus, $\mathrm{Max}(R_{S(H)}) = \{Q_{S(H)} \mid Q \in \text{h-Max}(R)\}$ [2, Proposition 1.4 and Lemma 2.2]. Hence, $R_{S(H)} = \bigcap_{Q\in \text{h-Max}(R)} R_Q$, and thus $R_{S(H)}$ has finite character. Also, if $Q \in \text{h-Max}(R)$, then $(QR_{S(H)})_v = Q_v R_{S(H)} = QR_{S(H)}$ [2, Proposition 1.3]. □

We next give in Theorem 1.13 a graded integral domain analog of [5, Proposition 5.4] that D is a divisorial domain if and only if D is h-local and D_M is a divisorial domain for all $M \in \mathrm{Max}(D)$. We prove this result by a series of lemmas.

Lemma 1.9 (cf. [7, Lemma 3.4]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain, P be a homogeneous maximal ideal of R, and A be a nonzero proper homogeneous ideal of R such that $AR_{H\setminus P} \cap R = A$. Then P is a unique homogeneous maximal ideal of R containing A.*

Proof. Note that $AR_{H\setminus P} \cap R = A$ implies $\sqrt{A} = \sqrt{AR_{H\setminus P}} \cap R$. Clearly, $\sqrt{AR_{H\setminus P}}$ is the intersection of all homogeneous prime ideals of $R_{H\setminus P}$ containing $AR_{H\setminus P}$. Hence, $\sqrt{A} = \bigcap_\alpha Q_\alpha$ where $\{Q_\alpha\}$ is the set of minimal primes of A that are contained in P. Note that each Q_α is homogeneous; so if P_1 is a homogeneous maximal ideal of R distinct from P, then $Q_\alpha \nsubseteq P_1$ by Theorem 1.7. Hence, $\sqrt{A} = \bigcap_\alpha Q_\alpha \nsubseteq P_1$ by Lemma 1.6. Thus, P is a unique homogeneous maximal ideal of R containing A. □

Lemma 1.10 (cf. [7, Lemma 3.5]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, P be a homogeneous maximal ideal of R, and A be a homogeneous ideal of R such that P is a unique homogeneous maximal ideal of R containing A. Then $AR_{H\setminus P} \cap R = A$.*

Proof. Since A is homogeneous,

$$\begin{aligned} A &= \bigcap\{AR_{H\setminus Q} \mid Q \in \text{h-Max}(R)\} \\ &= AR_{H\setminus P} \cap \left(\bigcap\{AR_{H\setminus Q} \mid Q \in \text{h-Max}(R) \text{ with } Q \neq P\}\right) \\ &= AR_{H\setminus P} \cap R \end{aligned}$$

(see [16, Proposition 2.6] for the first equality). □

Lemma 1.11 (cf. [7, Theorem 3.6]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain and P be a homogeneous maximal ideal of R. Then $R_{H\setminus P}$ is a homogeneously divisorial domain.*

Proof. Let A' be a nonzero homogeneous ideal of $R_{H\setminus P}$ and $A = A' \cap R$. Then A is a homogeneous ideal of R such that $A' = AR_{H\setminus P}$ and $AR_{H\setminus P} \cap R = A$. Next, let $B = (AR_{H\setminus P})_v \cap R$. Then B is homogeneous, $BR_{H\setminus P} = (AR_{H\setminus P})_v$, and $A \subseteq B$. Choose $y \in H \setminus A$. Since A is homogeneous, $A_v = A$, and hence there exists a homogeneous element $x \in R_H$ such that $A \subseteq xR$ and $y \notin xR$ by Lemma 1.3. If $E = xR \cap R$, then E is homogeneous, $y \notin E$, and $A \subseteq E$, and since $AR_{H\setminus P} \cap R = A$, by Lemma 1.9, P is a unique homogeneous maximal ideal of R containing E. Hence, by Lemma 1.10, $ER_{H\setminus P} \cap R = E$. Moreover, $E = xR \cap R$ implies $ER_{H\setminus P} = xR_{H\setminus P} \cap R_{H\setminus P}$ which is a homogeneous divisorial ideal of $R_{H\setminus P}$ and contains $AR_{H\setminus P}$. Thus, $(AR_{H\setminus P})_v = BR_{H\setminus P} \subseteq ER_{H\setminus P}$, and hence $B \subseteq BR_{H\setminus P} \cap R \subseteq E$. We conclude that $y \notin B$; so $A = B$. Therefore, $A' = AR_{H\setminus P} = BR_{H\setminus P} = (AR_{H\setminus P})_v = (A')_v$. □

Lemma 1.12 (cf. [5, Lemma 2.3]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously h-local domain, M be a homogeneous maximal ideal of R, and A be a nonzero homogeneous fractional ideal of R. Then*

$$(AR_{H\setminus M})^{-1} = A^{-1}R_{H\setminus M} \text{ and } (AR_{H\setminus M})_v = A_v R_{H\setminus M}.$$

Proof. It suffices to show that $(AR_{H\setminus M})^{-1} = A^{-1}R_{H\setminus M}$ because A^{-1} is homogeneous. Note that

$$\begin{aligned}(R:A) &= \Big(\bigcap_{N\in \text{h-Max}(R)} R_{H\setminus N}\Big) : A\\ &= \bigcap_{N\in \text{h-Max}(R)} (R_{H\setminus N} : A)\\ &= \bigcap_{N\in \text{h-Max}(R)} (R_{H\setminus N} : A_{H\setminus N})\\ &= (R_{H\setminus M} : A_{H\setminus M}) \cap \Big(\bigcap_{N\neq M} (R_{H\setminus N} : A_{H\setminus N})\Big).\end{aligned}$$

Also, if N is a homogeneous maximal ideal of R with $N \neq M$, then $(R_{H\setminus N})_{H\setminus M} = R_H$ because each nonzero homogeneous prime ideal of R is contained in a unique homogeneous maximal ideal. Note that if $A \not\subseteq R$, then there is an $x \in H$ such that $xA \subseteq R$ and xA is homogeneous; so we may assume that $A \subseteq R$. Hence, $R_H = \bigcap_{N\neq M}(R_{H\setminus N})_{H\setminus M} \subseteq \bigcap_{N\neq M}(R_{H\setminus N} : A_{H\setminus N})_{H\setminus M} \subseteq R_H$. Thus,

$$\begin{aligned}A^{-1}R_{H\setminus M} &= (R:A)_{H\setminus M}\\ &= (R_{H\setminus M} : A_{H\setminus M}) \cap \Big(\bigcap_{N\neq M} (R_{H\setminus N} : A_{H\setminus N})_{H\setminus M}\Big)\\ &= (R_{H\setminus M} : A_{H\setminus M}) \cap R_H\\ &= (R_{H\setminus M} : A_{H\setminus M})\\ &= (AR_{H\setminus M})^{-1},\end{aligned}$$

where the second equality follows because each nonzero nonunit of R is contained in only finitely many homogeneous maximal ideals of R. □

Theorem 1.13. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a homogeneously divisorial domain if and only if R is homogeneously h-local and $R_{H\setminus M}$ is a homogeneously divisorial domain for every $M \in h$-Max(R).*

Proof. If R is a homogeneously divisorial domain, then R is homogeneously h-local by Theorem 1.7 and $R_{H\setminus M}$ is a homogeneously divisorial domain for every $M \in$ h-Max(R) by Lemma 1.11. For the reverse implication, let A be a nonzero homogeneous ideal of R. Then $AR_{H\setminus M} = (AR_{H\setminus M})_v = A_vR_{H\setminus M}$ for all $M \in$ h-Max(R) by assumption and Lemma 1.12. Thus, $A = \bigcap\{AR_{H\setminus M} \mid M \in \text{h-Max}(R)\} = \bigcap\{A_vR_{H\setminus Q} \mid Q \in \text{h-Max}(R)\} = A_v$ [16, Proposition 2.6]. □

2. Integrally closed graded integral domains

In this section, we completely characterize integrally closed homogeneously divisorial domains. We first need a gr-valuation domain analog of [7, Lemma 5.2] that if V is a valuation domain with maximal ideal M, then M is principal if and only if every nonzero ideal of V is divisorial.

Lemma 2.1. *Let $V = \bigoplus_{\alpha\in\Gamma} V_\alpha$ be a gr-valuation domain with homogeneous maximal ideal M. Then the following statements are equivalent.*

(1) *M is principal.*
(2) *V be a homogeneously divisorial domain.*
(3) *M is divisorial, i.e., $M_v = M$.*

Proof. (1) ⇒ (2) Let $x \in V$ be a homogeneous element such that $M = xV$, and A be a homogeneous ideal of V. If $y \in V \setminus A$ is a homogeneous element, then $A \subsetneq yV$, and hence $\frac{1}{y}A \subsetneq V$. Hence, $\frac{1}{y}A \subseteq xV$, and thus $A \subseteq xyV \subsetneq yV$. Thus, $A_v \subseteq xyV$ and $y \notin A_v$. Therefore, $A = A_v$.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Note that $V \subsetneq M^{-1}$ and M^{-1} is homogeneous; so we can choose a homogeneous element $a \in M^{-1} \setminus V$. Then $\frac{1}{a} \in V$, and thus $M = M_v = (1, a)^{-1} = V \cap \frac{1}{a}V = \frac{1}{a}V$. □

We next give a complete characterization of integrally closed graded integral domains in which each nonzero homogeneous ideal is divisorial.

Theorem 2.2 (cf. [7, Theorem 5.1])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded integral domain. Then the following statements are equivalent.*

(1) *R is a homogeneously divisorial domain.*
(2) *R satisfies the following four conditions.*
  (a) *R is a graded-Prüfer domain.*
  (b) *Each homogeneous maximal ideal of R is invertible.*
  (c) *Each nonzero homogeneous prime ideal of R is contained in a unique homogeneous maximal ideal.*
  (d) *Each homogeneous ideal of R has only finitely many minimal prime ideals.*
(3) *R is a homogeneously h-local graded-Prüfer domain in which each homogeneous maximal ideal is invertible.*

Proof. (1) ⇒ (2): (a) This follows from [17, Corollary 3.4].

(b) Let P be a homogeneous maximal ideal of R. Then $P_v = P$, and hence there are homogeneous elements $a, b \in R$ such that $P = (1, \frac{b}{a})^{-1}$ by Lemma 1.5. Note that $(1, \frac{b}{a})$ is homogeneous; so $P^{-1} = (1, \frac{b}{a})_v = (1, \frac{b}{a})$ by (1). Thus, P is invertible by (a).

(c) This follows from Theorem 1.7.

(d) Let A be a nonzero homogeneous ideal of R and $\{P_\alpha\}$ be the set of minimal prime ideals of A. (Note that each P_α is homogeneous because A is homogeneous.) Then by Theorem 1.7, A is contained in only finitely many homogeneous maximal ideals of R, say $\{M_1, \ldots, M_n\}$. Since R is a graded-Prüfer domain, the homogeneous prime ideals of R contained in a fixed homogeneous maximal ideal are linearly ordered with respect to inclusion. Thus, $|\{P_\alpha\}| \le n$.

(2) $\Rightarrow$ (1) Let A be a nonzero homogeneous ideal of R and P be a homogeneous maximal ideal of R. Then $R_{H\setminus P}$ is a gr-valuation domain with homogeneous maximal ideal $PR_{H\setminus P}$ by (a) and [16, Lemma 4.3 and Theorem 4.4]. Also, $PR_{H\setminus P}$ is principal because P is finitely generated by (b). Hence, each nonzero homogeneous ideal of $R_{H\setminus P}$ is divisorial by Lemma 2.1. Thus, $AR_{H\setminus P}$ is divisorial, and hence $AR_{H\setminus P} = \bigcap_\alpha x_\alpha R_{H\setminus P}$ for some homogeneous elements $x_\alpha \in R$ (because each homogeneous element of $R_{H\setminus P}$ is of the form $\frac{a}{b}$ with $a \in H$ and $b \in H \setminus P$). Hence, if each $x_\alpha R_{H\setminus P} \cap R$ is divisorial, then $AR_{H\setminus P} \cap R = \bigcap_\alpha (x_\alpha R_{H\setminus P} \cap R)$ is divisorial [6, Theorem 32.2]. Also, since $A = \bigcap\{AR_{H\setminus Q} \cap R \,|\, Q \in \text{h-Max}(R)\}$ [16, Proposition 2.6], A is divisorial. Thus, it suffices to show that $aR_{H\setminus P} \cap R$ is divisorial for each nonzero homogeneous element $a \in R$.

Let $J = aR_{H\setminus P} \cap R$. If $a \notin P$, then $aR_{H\setminus P} = R_{H\setminus P}$, and hence $J = R$ is divisorial. Next, assume that $a \in P$. Then a is contained in only finitely many homogeneous maximal ideals of R by (c) and (d), say $\{P, M_1, \ldots, M_n\}$. Since $R_{H\setminus P}$ is a gr-valuation domain, $\sqrt{aR_{H\setminus P}}$ is a homogeneous prime ideal, and hence $\sqrt{J} \subseteq P$ is a nonzero homogeneous prime ideal of R because $\sqrt{aR_{H\setminus P}} \cap R = \sqrt{J}$. Therefore, $\sqrt{J}$, and hence J is contained in no M_i by (c). Choose a homogeneous element $y_i \in J \setminus M_i$ for each i, and let $I = (a, y_1, \ldots, y_n)R$. Then P is a unique homogeneous maximal ideal of R containing I. Note that $a \in I \subseteq J$; so

$$I = IR_{H\setminus P} \cap R = aR_{H\setminus P} \cap R = J.$$

Thus, J is a finitely generated homogeneous ideal of R, and since R is a graded-Prüfer domain, J is invertible; so J is divisorial [6, Lemma 32.17].

(2) $\Leftrightarrow$ (3) This follows because (d) is equivalent to that each nonzero homogeneous ideal of R is contained in only finitely many homogeneous maximal ideals of R by (a) and (c). □

Corollary 2.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded integral domain and $S(H) = \{f \in R \,|\, C(f) = R\}$. Then the following statements are equivalent.*

(1) *R is a homogeneously divisorial domain.*
(2) *$R_{S(H)}$ is a divisorial domain.*
(3) *$R_{S(H)}$ is an h-local Prüfer domain whose maximal ideals are invertible.*

Proof. (1) $\Rightarrow$ (2) Clearly, $S(H) = \{f \in R \,|\, C(f)_v = R\}$, and since $\text{Max}(R_{S(H)}) = \{Q_{S(H)} \,|\, Q \in \text{h-Max}(R)\}$ by Corollary 1.8, every ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R by Theorem 2.2 and [2, Corollary 1.10]. Hence, if A' is a nonzero ideal of $R_{S(H)}$, then $A' = AR_{S(H)}$ for some nonzero homogeneous ideal A of R. Thus, $(A')_v = (AR_{S(H)})_v = A_v R_{S(H)} = AR_{S(H)} = A'$ (cf. [2, Proposition 1.3] for the second equality).

(2) $\Rightarrow$ (1) Note that $R_{S(H)}$ is integrally closed; so $R_{S(H)}$ is a Prüfer domain [7, Theorem 5.1]. Hence, if $Q \in$ h-Max(R), then $Q_{S(H)} \subsetneq R_{S(H)}$, and thus $R_Q = (R_{S(H)})_{Q_{S(H)}}$ is a valuation domain. Thus, R is a graded-Prüfer domain [4, Theorem 2.1] and $S(H) = \{f \in R \,|\, C(f)_v = R\}$. Let A be a nonzero homogeneous ideal of R. Then $AR_{S(H)} = (AR_{S(H)})_v = A_v R_{S(H)}$ [2, Proposition 1.3], and since both A and A_v are homogeneous, $A = AR_{S(H)} \cap R = A_v R_{S(H)} \cap R = A_v$ [3, Lemma 2].

(2) $\Leftrightarrow$ (3) [7, Theorem 5.1]. □

We recall that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a *graded-Dedekind domain* (gr-Dedekind domain) if R is a gr-Noetherian and graded-Prüfer domain. Hence, R is a gr-Dedekind domain if and only if each nonzero homogeneous ideal of R is invertible, if and only if each nonzero homogeneous prime ideal of R is invertible. In [3, Corollary 7], Anderson and Chang showed that if $\Gamma \cap (-\Gamma) = \{0\}$, then R is a gr-Dedekind domain if and only if R is a Dedekind domain, if and only if R is a PID.

Theorem 2.4 (cf. [7, Proposition 5.5]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a completely integrally closed graded integral domain and $S(H) = \{f \in R \,|\, C(f) = R\}$. Then the following statements are equivalent.*

(1) *R is a homogeneously divisorial domain.*
(2) *R is a gr-Dedekind domain.*
(3) *$R_{S(H)}$ is a Dedekind domain.*
(4) *$R_{S(H)}$ is a PID.*
(5) *R is a gr-Noetherian domain in which each homogeneous maximal ideal is invertible.*

Proof. (1) $\Rightarrow$ (2) Let A be a nonzero homogeneous ideal of R. Since R is completely integrally closed, $(AA^{-1})_v = R$ [6, Theorem 34.3], and since AA^{-1} is homogeneous, $(AA^{-1})_v = AA^{-1}$. Thus, $AA^{-1} = R$.

(2) $\Rightarrow$ (1) If R is a gr-Dedekind domain, then each nonzero homogeneous ideal of R is invertible. Thus, R is a homogeneously divisorial domain.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) [3, Theorem 4].

(5) $\Rightarrow$ (2) It suffices to show that $R_{H\setminus P}$ is a gr-valuation domain for all $P \in$ h-Max(R). Let M be a homogeneous maximal ideal of R. Then $MR_{H\setminus M}$ is invertible by (5), and hence $MR_{H\setminus M}$ is principal. Also, since $R_{H\setminus M}$ is gr-Noetherian, $MR_{H\setminus M}$ is a unique nonzero homogeneous prime ideal of $R_{H\setminus M}$. Hence, $R_{H\setminus M}$ is a gr-Dedekind domain with a unique homogeneous maximal ideal. Thus, $R_{H\setminus M}$ is a gr-valuation domain. □

Corollary 2.5. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a completely integrally closed graded integral domain such that $\Gamma \cap (-\Gamma) = \{0\}$. Then R is a divisorial domain if and only if R is a homogeneously divisorial domain.*

Proof. If R is a homogeneously divisorial domain, then R is a gr-Dedekind domain by Theorem 2.4, and thus R is a Dedekind domain [3, Corollary 7]. Hence, R is a divisorial domain [7, Proposition 5.5]. The converse is trivial. $\square$

3. Graded-Noetherian domains

In this section, we study gr-Noetherian domains which are also homogeneously divisorial domains.

Lemma 3.1 (cf. [7, Corollary 3.2]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, S be a multiplicative set of nonzero homogeneous elements of R, and A be a finite intersection of principal homogeneous fractional ideals of R. Then AR_S is a homogeneous divisorial ideal.*

Proof. Let $A = \bigcap_{i=1}^n x_i R$ be a finite intersection of nonzero principal homogeneous ideals of R. Then $AR_S = \bigcap_{i=1}^n x_i R_S$ [7, Lemma 3.1], and thus AR_S is a homogeneous divisorial ideal of R_S. $\square$

Let V be a rank 2 valuation domain with prime ideals $(0) \subsetneq P \subsetneq M$ such that M is principal but V_P is not discrete. Then each nonzero homogeneous ideal of $R = V[X, X^{-1}]$ is divisorial, while $R_{H\setminus PR} = V_P[X, X^{-1}]$ has a non-divisorial homogeneous ideal $PR_{H\setminus PR}$ by Example 1.1. Hence, Lemma 1.11 is not true for homogeneous non-maximal ideals. However, our next result shows that every nonzero homogeneous ideal of R_S is divisorial for any multiplicative set $S \subseteq H$ when $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a gr-Noetherian homogeneously divisorial domain.

Proposition 3.2 (cf. [7, Remark 3.3]). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain and S be a multiplicative set of nonzero homogeneous elements of R. If R is gr-Noetherian, then R_S is a homogeneously divisorial domain.*

Proof. Let A be a nonzero homogeneous ideal of R and $B = A^{-1}$. Then B is a finitely generated homogeneous fractional ideal of R, say $B = (x_1, \ldots, x_n)$ for some homogeneous elements $x_i \in R_H$, and hence

$$A = B^{-1} = \bigcap_{i=1}^n (R : x_i) = \bigcap_{i=1}^n (1/x_i)R.$$

Thus, AR_S is divisorial by Lemma 3.1. $\square$

Let D be a Noetherian domain. It is known that if D is a divisorial domain, then D has (Krull) dimension one [7, Corollary 4.3]. Also, if D has (Krull) dimension one, then D is an h-local domain. Thus, D is a divisorial domain if and only if D_M is a divisorial domain for all $M \in \mathrm{Max}(D)$ [5, Proposition 5.4].

Corollary 3.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a gr-Noetherian domain. Then R is a homogeneously divisorial domain if and only if $R_{H\setminus P}$ is a homogeneously divisorial domain for all $P \in h$-$\mathrm{Max}(R)$.*

Proof. Suppose that $R_{H\setminus P}$ is a homogeneously divisorial domain for all $P \in$ h-Max(R). Let A be a nonzero homogeneous ideal of R. Then A is finitely generated, and hence $AR_{H\setminus P} = (AR_{H\setminus P})_v = (A_vR_{H\setminus P})_v$ [18, Lemma 4]; so $AR_{H\setminus P} = A_vR_{H\setminus P}$ for all $P \in$ h-Max(R). Thus,

$$\begin{aligned} A &= \bigcap\{AR_{H\setminus Q} \mid Q \in \text{h-Max}(R)\} \\ &= \bigcap\{A_vR_{H\setminus Q} \mid Q \in \text{h-Max}(R) \\ &= A_v \end{aligned}$$

[16, Proposition 2.6]. The converse is from Proposition 3.2 or Lemma 1.11. □

Let D be an almost Dedekind domain that is not Dedekind (see, for example, [6, Example 42.6] for such an integral domain), and let $R = D[X, X^{-1}]$. Then D is not an h-local domain, h-Max$(R) = \{P[X, X^{-1}] \mid P \in \text{Max}(D)\}$, and $R_{H\setminus P[X,X^{-1}]} = D_P[X, X^{-1}]$ is a gr-valuation domain such that $PD_P[X, X^{-1}]$ is the homogeneous maximal ideal and $PD_P[X, X^{-1}]$ is principal. Hence, every nonzero homogeneous ideal of $R_{H\setminus Q}$ is divisorial for all $Q \in$ h-Max(R) by Lemma 2.1, but R has a nonzero homogeneous ideal that is not divisorial by Example 1.1. Thus, Corollary 3.3 does not hold if R is not a gr-Noetherian domain.

We next prove in Theorem 3.9 that if $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a homogeneously divisorial domain which is also gr-Noetherian, then every nonzero homogeneous prime ideal of R has height-one. For which we first need several lemmas.

Lemma 3.4 (cf. [7, Lemma 4.1])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain with a unique homogeneous maximal ideal P and xR be a nonzero homogeneous principal ideal of R. Then xP^{-1} is contained in any homogeneous fractional ideal which properly contains xR.*

Proof. Let A be a homogeneous fractional ideal of R with $xR \subsetneq A$. Then $R = \frac{1}{x}(xR) \subsetneq \frac{1}{x}A$, and hence $(\frac{1}{x}A)^{-1} \subsetneq R$. Note that $\frac{1}{x}A$ is homogeneous; so $(\frac{1}{x}A)^{-1} \subseteq P$. Hence, $P^{-1} \subseteq (\frac{1}{x}A)_v = \frac{1}{x}A$, and thus $xP^{-1} \subseteq A$. □

We recall that a homogeneous ideal is *h-irreducible* if it is not a finite intersection of homogeneous ideals strictly containing it.

Corollary 3.5 (cf. [7, Corollary 4.2])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain with a unique homogeneous maximal ideal. Then every homogeneous principal ideal of R is h-irreducible.*

Proof. Let $x \in R$ be a nonzero homogeneous element such that $xR = A \cap B$ for some homogeneous ideals A, B of R with $xR \subsetneq A$ and $xR \subsetneq B$, and P be the homogeneous maximal ideal of R. Then $xP^{-1} \subseteq A \cap B = xR$ by Lemma 3.4, and thus $P^{-1} \subseteq R$, a contradiction. □

An ideal I of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is called *h-primary* if $ab \in I$ for homogeneous elements a, b of R implies that $a \in I$ or $b^n \in I$ for some integer $n \geq 1$. Clearly, a primary ideal is h-primary. Also, it is known that if Q is an h-primary homogeneous ideal of R, then Q is a primary ideal of R [9, Proposition 5.6.20].

Lemma 3.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a gr-Noetherian domain. Then each h-irreducible ideal of R is a primary ideal.*

Proof. Let I be a nonzero h-irreducible ideal of R. Assume that I is not primary. Then there exist homogeneous elements $a, b \in R$ such that $ab \in I$, $a \notin I$, and $b^n \notin I$ for all integers $n \geq 1$. Since R is a gr-Noetherian domain, the ascending chain of homogeneous ideals $(I :_R b) \subseteq (I :_R b^2) \subseteq \cdots$ must be stationary. Hence, there exists an integer $m \geq 0$ such that $(I :_R b^m) = (I :_R b^{m+1})$. Consider the homogeneous ideals $aR+I$ and b^mR+I. If $x \in (aR+I)\cap(b^mR+I)$, then $x = t_1a+i = t_2b^m+j$ for some $t_1, t_2 \in R$ and $i, j \in I$. Hence, $t_2b^{m+1} = (t_2b^m)b = (t_1a+(i-j))b \in I$, and thus $t_2 \in (I :_R b^{m+1}) = (I :_R b^m)$; so $x = t_2b^m + j \in I$. Thus, $(aR + I) \cap (b^mR + I) = I$, and hence I is not h-irreducible, a contradiction. □

Corollary 3.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain with a unique homogeneous maximal ideal. If R is gr-Noetherian, then every homogeneous principal ideal of R is primary.*

Proof. It follows from Corollary 3.5 and Lemma 3.6. □

Lemma 3.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain, P be a homogeneous maximal ideal of R, and $0 \neq x \in P$ be homogeneous. Then there is a homogeneous element $y \in R\backslash xR$ such that $P = (xR :_R y)$.*

Proof. Note that $R \subsetneq P^{-1}$ and P^{-1} is homogeneous; so there are homogeneous elements $a, b \in R$ such that $\frac{b}{a} \in P^{-1} \setminus R$. Clearly, $P = P_v = (\frac{b}{a}, 1)^{-1} = \frac{a}{b}R \cap R = (aR :_R b)$. Thus, if $y = x\frac{b}{a}$, then $y \in R \setminus xR$ and $P = (aR :_R b) = (R :_R \frac{b}{a}) = (xR :_R y)$. □

The *h-height* of a homogeneous prime ideal Q of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ (denoted by h-ht$Q$) is defined to be the supremum of the lengths of chains of homogeneous prime ideals $(0) \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n = Q$. Clearly, h-ht$Q \leq$ ht$Q$, and equality holds when every prime ideal P of R with $P \subseteq Q$ is homogeneous. The *h-dimension* of R (denoted by *h*-dim$\,R$) is defined to be $\sup\{\text{h-ht}Q \mid Q \in \text{h-Spec}(R)\}$.

Theorem 3.9 (cf. [7, Corollary 4.3])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a homogeneously divisorial domain which is also gr-Noetherian. Then each nonzero homogeneous prime ideal of R has height-one.*

Proof. Let P be a homogeneous maximal ideal of R. Then $R_{H\setminus P}$ is a gr-Noetherian domain whose nonzero homogeneous ideals are divisorial by Corollary 3.3. Hence, by replacing R and P with $R_{H\setminus P}$ and $PR_{H\setminus P}$, respectively, we may assume that R is a gr-Noetherian domain with a unique homogeneous maximal ideal P. Let $a \in P$ be a nonzero homogeneous element. By Lemma

3.8, there is a homogeneous element $b \in R\backslash aR$ such that $P = (aR : b)$. We show that $\sqrt{aR} = (aR : b)$. It is clear that $\sqrt{aR} \subseteq \sqrt{(aR : b)} = (aR : b)$. If $x \in (aR : b)$, then $xb \in aR$, and since aR is a primary ideal by Corollary 3.7 and $b \notin aR$, we have $x^n \in aR$ for some integer $n \geq 1$; so $x \in \sqrt{aR}$. Consequently, P is minimal over aR, and since R is a gr-Noetherian domain, h-h$P \leq 1$ [13, Theorem 3.5]. Hence, h-dim$(R) = 1$, and if $\bar{R}$ is the integral closure of R, then $\bar{R}$ is a homogeneous overring of R with h-dim$(\bar{R}) = 1$ [14, Lemmas 2.2, 2.3, and Corollary 1.6]. Let Q be a nonzero prime ideal of R such that $Q \subseteq P$. Then there are prime ideals $Q' \subseteq P'$ of $\bar{R}$ such that $Q' \cap R = Q$ and $P' \cap R = P$. Note that P' is a homogeneous ideal of $\bar{R}$; so ht$P' = 1$ (cf. [14, Theorem 2.10] and [1, Proposition 5.5]). Hence, $Q = Q' \cap R = P' \cap R = P$. Thus, ht$P = 1$. □

Corollary 3.10 (cf. [11, Theorem 3.8]). *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a gr-Noetherian domain and $S(H) = \{f \in R \,|\, C(f) = R\}$. Then the following statements are equivalent.*

(1) *R is a homogeneously divisorial domain.*
(2) *Each nonzero homogeneous prime ideal of R has height-one and M^{-1} is generated by two elements for all $M \in h$-Max(R).*
(3) *$R_{S(H)}$ is a divisorial domain of (Krull) dimension one.*
(4) *$R_{H \backslash P}$ is a homogeneously divisorial domain for all $P \in h$-Max(R).*

Proof. (1) ⇒ (2) Each nonzero homogeneous prime ideal of R has height-one by Theorem 3.9 and $M^{-1} = R + xR$ for any homogeneous element $x \in M^{-1} \setminus R$ by Lemma 1.5.

(2) ⇒ (3) Let M be a homogeneous maximal ideal of R. Then R_M is a one-dimensional Noetherian domain such that $(MR_M)^{-1} = M^{-1}R_M$ (because M is finitely generated); so $(MR_M)^{-1}$ is two generated. Thus, each nonzero ideal of R_M is divisorial [11, Theorem 3.8]. Next, note that $\bigcap_{Q \in \text{h-Max}(R)} R_Q$ is locally finite because R is gr-Noetherian and each homogeneous maximal ideal of R has height-one; so Max$(R_{S(H)}) = \{Q_{S(H)} \,|\, Q \in \text{h-Max}(R)\}$ by the proof of Corollary 1.8. Thus, $R_{S(H)}$ is a one-dimensional Noetherian domain, and hence $R_{S(H)}$ is an h-local domain. Note that $(R_{S(H)})_{M_{S(H)}} = R_M$ for all $M \in$ h-Max(R). Thus, $R_{S(H)}$ is a divisorial domain [5, Proposition 5.4].

(3) ⇒ (1) Let A be a nonzero homogeneous ideal of R. Then $A = AR_{S(H)} \cap R = (AR_{S(H)})_v \cap R = A_vR_{S(H)} \cap R = A_v$ by [3, Lemma 2] and the proof of Corollary 2.3. Thus, R is a homogeneously divisorial domain.

(1) ⇔ (4) Corollary 3.3. □

Corollary 3.11. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a homogeneously divisorial domain which is also gr-Noetherian. Then the integral closure of R is a gr-Dedekind domain.*

Proof. Let $\bar{R}$ be the integral closure of R and $S(H) = \{f \in R \,|\, C(f) = R\}$. Then $\bar{R}_{S(H)}$ is a one-dimensional integrally closed Noetherian domain by Corollary 3.10 because $\bar{R}_{S(H)}$ is the integral closure of $R_{S(H)}$. Thus, $\bar{R}_{S(H)}$ is a

Dedekind domain [6, Theorem 37.8]. Also, $\bar{R}$ is a homogeneous overring of R [14, Lemmas 2.2 and 2.3]. Hence, if $\bar{S}(H) = \{f \in \bar{R} \,|\, C(f) = \bar{R}\}$, then $S(H) \subseteq \bar{S}(H)$, and thus $\bar{R}_{\bar{S}(H)}$ is a Dedekind domain. Thus, $\bar{R}$ is a gr-Dedekind domain [3, Theorem 4]. □

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