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COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE THEOREMS FOR WEIGHTED SUMS OF ARRAYS OF ROWWISE EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. In the present work, the complete convergence and complete moment convergence properties for arrays of rowwise extended negatively dependent (END) random variables are investigated. Some sharp theorems on these strong convergence for weighted sums of END cases are established. These main results not only generalize the known corresponding ones of Cai [2], Wang et al. [17] and Shen [14], but also improve them, respectively.

1. Introduction

By weakening the assumptions of validity of limit theory, we provide an extension for applications of probability theory to various fields, especially to statistics research. In many statistical theoretical frameworks, we usually assume that variables are independent. But in many practical studies, this assumption is not plausible. And then, many researchers have revised this assumption in order to consider dependent cases, such as negatively associated random variables, positively associated random variables, negatively orthant dependent random variables, extended negatively dependent random variables (END), and many others. In this work, we consider the END structure, which includes

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independent random variables, negatively associated random variables and negatively orthant dependent random variables as special cases, and present some sharp results on complete convergence and complete moment convergence for weighted sums of arrays of rowwise END random variables.

Firstly, let us recall some concepts of dependent structures.

Definition 1.1. A finite collection of random variables $X_1, X_2, \dots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \dots, n\}$ and any real non-decreasing functions f_1 on $\mathbb{R}^{A_1}$ and f_2 on $\mathbb{R}^{A_2}$,

$$\operatorname{Cov}\left(f_1\left(X_i, i \in A_1\right), f_2\left(X_j, j \in A_2\right)\right) \leq 0, \tag{1.1}$$

whenever the covariance exists. An infinite collection of random variables $\{X_n, n \geq 1\}$ is NA if every finite sub-collection is NA.

Definition 1.2. A finite collection of random variables $X_1, X_2, \dots, X_n$ is said to be negatively orthant dependent (NOD) if all $x_1, x_2, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq \prod_{j=1}^{n} P(X_j \leq x_j), \tag{1.2}$$

and

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq \prod_{j=1}^{n} P(X_j > x_j). \tag{1.3}$$

An infinite sequence of random variables $\{X_n, n \geq 1\}$ is said to be NOD if every finite sub-family is NOD. An array of random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ is called rowwise NOD if for every $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ is a sequence of NOD random variables.

The concept of extended negatively dependent random variables was introduced by Liu [9] as follows.

Definition 1.3. A finite collection of random variables $X_1, X_2, \dots, X_n$ is said to be extended negatively dependent (END) if there exists a constant $C > 0$ such that both inequalities

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq C \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq C \prod_{i=1}^{n} P(X_i \leq x_i)$$

hold for all real numbers $x_1, x_2, \dots, x_n$. An infinite sequence of random variables $\{X_n, n \geq 1\}$ is said to be END if every finite sub-collection is END.

An array of random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ is called rowwise END random variables if for every $n \geq 1, \{X_{ni}, 1 \leq i \leq n\}$ are END random variables.

Obviously, the NOD structure is a special case of the END structure with $C = 1$. In other words, the END structure is superordinate to the NOD structure which was introduced by Lehmann [8] and after developed by Ebrahimi and Ghosh [5] (cf. also Joag-Dev and Proschan [7]).

Note that Joag-Dev and Proschan [7] proved that NA random variables must be NOD (but NOD is not necessarily NA); thus NA random variables are also END. Many well known multivariate distributions possess the NA property. Hence, extending and improving the limit theory of NA and NOD structures to the wider END case are of interest in theoretical research and applications.

The END structure can reflect not only a negative dependence structure but also a positive one to some extent. Liu [9] pointed out that some sequences of END random variables obey both negatively and positively dependent properties, and provided some interesting specific examples to support this idea.

Since the paper of Liu [9] appeared, many applications of END random variables have been found in various aspects by many authors. For example, Liu [10] studied the sufficient and necessary conditions of moderate deviations for END random variables with heavy tails; Chen et al. [3] established the strong law of large numbers for END random variables and showed applications to risk theory and renewal theory; Shen [13] presented some probability inequalities for END random variables and gave some applications; Wang and Wang [19] investigated the extended precise large deviations of random sums in the presence of END structure and consistent variation; Wu and Guan [26] presented some convergence properties for the partial sums of END random variables; Wang and Wang [20] investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. [12], Wang et al. [16,18,21] and Hu et al. [6] provided some results on complete convergence for END random variables; Wu et al. [25,27] established the complete moment convergence for arrays of rowwise END random variables; Wang et al. [22] studied the complete consistency for the estimator of nonparametric regression models based on END errors, and many others.

Recently, Wang et al. [17] established the following theorems, which extended the result of Cai [2] for NA random variables to NOD cases without assumption of identical distribution.

Theorem A. *Let $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NOD random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}; i \geq 1, n \geq 1\}$ be an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n^{\delta}\right)$ for some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$. $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Assume that $EX_{ni} = 0$ if $1 < \alpha < 2$. Then for some $h > 0$*

and $\gamma > 0$, $E(\exp(h|X|^\gamma)) < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0, \tag{1.4}$$

where $\alpha p \ge 1$.

Theorem B. *Under the conditions of Theorem A, let* $\{a_{ni}; i \ge 1, n \ge 1\}$ *be an array of real numbers satisfying* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ *for some* α *with* $0 < \alpha < 2$. *Then for some* $h > 0$ *and* $\gamma > 0$, $E(\exp(h|X|^\gamma)) < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0. \tag{1.5}$$

Shen [14] also investigated the complete convergence for weighted sums of arrays of rowwise NOD random variables.

Theorem C. *Let* $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ *be an array of rowwise NOD random variables which is stochastically dominated by a random variable* X, *and let* $\{a_{ni}; i \ge 1, n \ge 1\}$ *be an array of real numbers satisfying* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ *for some* α *with* $0 < \alpha \le 2$. $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Then the following statements hold*:

(1) *If* $\alpha > \gamma$, *then* $E|X|^\alpha < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{1.6}$$

(2) *If* $\alpha = \gamma$, *then* $E|X|^\alpha \log^+ |X| < \infty$ *implies* (1.6).
(3) *If* $\alpha < \gamma$, *then* $E|X|^\gamma < \infty$ *implies* (1.6).

Theorem D. *Under the conditions of Theorem C,* $E|X|^\beta < \infty$ *for some* $\beta > \alpha + 2$ *and* $0 < \alpha \le 2$ *implies* (1.5).

The main purpose of this work is further to study complete convergence and complete moment convergence for END random variables. We establish some sharp results on these strong convergence properties for weighted sums of arrays of rowwise END random variables under some mild conditions, which extend and improve the known corresponding ones of Cai [2] for NA random variables, Wang et al. [17] and Shen [14] for NOD random variables to END cases, respectively.

Throughout this paper, the symbol C represents positive constant which may be different in various places, $a_n = O(b_n)$ stands for $a_n \le Cb_n$, $I(A)$ be the indicator function of a set A.

2. Preliminaries

In this section, we will state the definition of stochastical domination and some important lemmas, which are used to prove the main results.

Definition 2.1. An array of random variables $\{X_{ni}; i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists some positive constant C such that

$$\sup_{i\geq 1, n\geq 1} P\left(|X_{ni}| > x\right) \leq CP\left(|X| > x\right) \quad \text{for} \quad \forall x \geq 0. \tag{2.1}$$

Lemma 2.1 (Liu [10])**.** *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, and $\{f_n, n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing. Then $\{f_n(X_n), n \geq 1\}$ is a sequence of END random variables.*

Lemma 2.2 (Shen [13], Liu et al. [11])**.** *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with mean zero and $E|X_n|^M < \infty$ for $M \geq 2$. Then there exists a positive constant C depending only on M such that for all $n \geq 1$,*

$$E\left(\left|\sum_{i=1}^{n} X_i\right|^M\right) \leq C\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right), \tag{2.2}$$

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^M\right) \leq C(\log 2n)^M\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right). \tag{2.3}$$

Remark 2.1. For the case of NOD random variables, Lemmas 2.1 and 2.2 have been established in Ebrahimi and Ghosh [5], Asadian et al. [1] and Wu [24], respectively.

Lemma 2.3 (Wu [23])**.** *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For all $u > 0$ and $t > 0$, the following two statements hold*:

$$E|X_{ni}|^u I(|X_{ni}| \leq t) \leq C(E|X|^u I(|X| \leq t) + t^u P(|X| > t)), \tag{2.4}$$

$$E|X_{ni}|^u I(|X_{ni}| > t) \leq CE|X|^u I(|X| > t). \tag{2.5}$$

With the above Lemma 2.3 accounted for, the following results can been obtained immediately.

Lemma 2.4. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}; i \geq 1, n \geq 1\}$ be an array of real constants. Then for all $u > 0$ and $t > 0$, the following statements hold*:

$$\begin{aligned} E|a_{ni}X_{ni}|^\mu I\left(|a_{ni}X_{ni}| \leq t\right) \leq C\left(E|a_{ni}X|^\mu I\left(|a_{ni}X| \leq t\right)\right. \\ \left. + t^\mu P\left(|a_{ni}X| > t\right)\right), \end{aligned} \tag{2.6}$$

$$E|a_{ni}X_{ni}|^\mu I\left(|a_{ni}X_{ni}| > t\right) \leq CE|a_{ni}X|^\mu I\left(|a_{ni}X| > t\right). \tag{2.7}$$

Lemma 2.5 (Wu et al. [25, 27]). *Suppose that* $\{a_{ni}; i \geq 1, n \geq 1\}$ *is an array of real constants satisfying* $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ *for some* $\alpha > 0$. *Let* X *be a random variable,* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^{\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n) \leq \begin{cases} CE|X|^{\alpha} & \text{for } \alpha > \gamma, \\ CE|X|^{\alpha}\log(1+|X|) & \text{for } \alpha = \gamma, \\ CE|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases}$$

Lemma 2.6 (Wu et al. [25, 27]). *Suppose that* $\{a_{ni}; i \geq 1, n \geq 1\}$ *is an array of real constants satisfying* $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ *for some* $\alpha > 0$. *Let* X *be a random variable,* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *If* $q > \max\{\alpha, \gamma\}$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^{q}} \sum_{i=1}^{n} E|a_{ni}X|^{q} I(|a_{ni}X| \leq b_n) \leq \begin{cases} CE|X|^{\alpha} & \text{for } \alpha > \gamma, \\ CE|X|^{\alpha}\log(1+|X|) & \text{for } \alpha = \gamma, \\ CE|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases}$$

3. Complete convergence

In this section, some sharp results on complete convergence of the maximum weighted sums for arrays of rowwise END random variables are established without assumption of identical distribution. The idea is mainly inspired by Shen [14], Shen and Wu [15].

Theorem 3.1. *Let* $0 < \delta < 1$, $0 < \alpha \leq 2$ *and* $\alpha p \geq 1$. *Suppose that* $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ *is an array of rowwise END random variables which is stochastically dominated by* X. *Let* $\{a_{ni}; i \geq 1, n \geq 1\}$ *be an array of real constants such that* $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n^{\delta})$. $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Assume further that* $EX_{ni} = 0$ *if* $1 < \alpha \leq 2$. *Then there exits some* $q > \max\{\alpha^2 p, \alpha+2, \alpha+\alpha(\alpha p-1)/(1-\delta), \alpha(\alpha p-1)+2\delta\}$ *such that* $E|X|^q < \infty$ *implies that*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \textit{for} \quad \forall\, \varepsilon > 0. \tag{3.1}$$

Proof. Without loss of generality, assume that $a_{ni} \geq 0$. For fixed $n \geq 1$ and all $1 \leq i \leq n$, define

$$Y_{ni} = -b_n I(X_{ni} < -b_n) + X_{ni} I(|X_{ni}| \leq b_n) + b_n I(X_{ni} > b_n),$$

$$T_{nj} = \sum_{i=1}^{j} (a_{ni}Y_{ni} - Ea_{ni}Y_{ni}),\ j = 1, 2, \ldots, n.$$

Obviously, for fixed $n \geq 1$, $\{Y_{ni}, i \geq 1\}$ and $\{Y_{ni} - EY_{ni}, i \geq 1\}$ are still sequences of END random variables by Lemma 2.1. For $\forall \varepsilon > 0$, noting that

$$\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \subset \left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}\right| > \varepsilon b_n\right) \bigcup \left(\bigcup_{i=1}^{n} (|X_{ni}| > b_n)\right),$$

which implies

$$(3.2)\quad P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right|>\varepsilon b_n\right)$$
$$\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}\right|>\varepsilon b_n\right)+P\left(\bigcup_{i=1}^{n}(|X_{ni}|>b_n)\right)$$
$$\le P\left(\max_{1\le j\le n}|T_{nj}|>\varepsilon b_n-\max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}Y_{ni}\right|\right)+P\left(\bigcup_{i=1}^{n}(|X_{ni}|>b_n)\right).$$

Really, noting that $\max\limits_{1\le i\le n}|a_{ni}|^{\alpha}\le\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O\left(n^{\delta}\right)$ for $0<\delta<1$. Then

$$(3.3)\quad \sum_{i=1}^{n}|a_{ni}|^{k}=\sum_{i=1}^{n}|a_{ni}|^{\alpha}|a_{ni}|^{k-\alpha}\le Cn^{\delta}n^{\delta(k-\alpha)/\alpha}=Cn^{\delta k/\alpha}\quad\text{for}\quad\forall k\ge\alpha.$$

If $0<\alpha\le 1$, by (2.6) of Lemma 2.4, (3.3) (for $k=1$) and the Markov inequality, we have

$$(3.4)\qquad b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}Y_{ni}\right|$$
$$\le Cb_n^{-1}\sum_{i=1}^{n}|Ea_{ni}Y_{ni}|$$
$$\le Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|\,E|X|I(|X|\le b_n)+C\sum_{i=1}^{n}|a_{ni}|\,P(|X|>b_n)$$
$$\le Cb_n^{-1}n^{\delta/\alpha}E|X|I(|X|\le b_n)+Cn^{\delta/\alpha}P(|X|>b_n)$$
$$\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}E|X|I(b_{k-1}<|X|\le b_k)+Cn^{\delta/\alpha}b_n^{-q}E|X|^{q}$$
$$\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_kP(|X|>b_{k-1})+Cn^{\delta/\alpha}n^{-q/\alpha}(\log n)^{-q/\gamma}E|X|^{q}$$
$$\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_kE|X|^{q}b_{k-1}^{-q}+Cn^{\delta/\alpha}n^{-q/\alpha}(\log n)^{-q/\gamma}$$
$$\le Cn^{\delta/\alpha+1-q/\alpha}(\log n)^{-1/\gamma}+Cn^{\delta/\alpha-q/\alpha}(\log n)^{-q/\gamma}\to 0\qquad\text{as } n\to\infty.$$

In addition, $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O\left(n^{\delta}\right)$ and the Hölder inequality imply that

$$(3.5)\qquad \sum_{i=1}^{n}|a_{ni}|^{k}\le\left(\sum_{i=1}^{n}\left(|a_{ni}|^{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}}\left(\sum_{i=1}^{n}1\right)^{\frac{\alpha-k}{\alpha}}\le Cn\quad\text{for}\quad 1\le k<\alpha.$$

If $1 < \alpha \leq 2$, by $EX_{ni} = 0$, (2.7) of Lemma 2.4, (3.5) and $E|X|^q < \infty$, we have

$$
\begin{aligned}
(3.6) \qquad & b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} Ea_{ni}Y_{ni} \right| \\
& \leq Cb_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} Ea_{ni}X_{ni}I(|X_{ni}| \leq b_n) \right| + C\sum_{i=1}^{m} |a_{ni}|P(|X_{ni}| > b_n) \\
& \leq Cb_n^{-1} \sum_{i=1}^{n} |a_{ni}|E|X|I(|X| > b_n) + C\sum_{i=1}^{n} |a_{ni}|P(|X| > b_n) \\
& \leq Cb_n^{-1}nE|X|I(|X| > b_n) + CnP(|X| > b_n) \\
& = Cb_n^{-1}n\sum_{k=n}^{\infty} E|X|I(b_k < |X| \leq b_{k+1}) + Cn\frac{E|X|^q}{b_n^q} \\
& \leq Cb_n^{-1}n\sum_{k=n}^{\infty} b_{k+1}P(|X| > b_k) + Cnn^{-q/\alpha}(\log n)^{-q/\gamma} \\
& \leq Cb_n^{-1}n\sum_{k=n}^{\infty} b_{k+1}\frac{E|X|^q}{b_k^q} + Cn^{1-q/\alpha}(\log n)^{-q/\gamma} \\
& \leq Cn^{2-q/\alpha}(\log n)^{-1/\gamma} + Cn^{1-q/\alpha}(\log n)^{-q/\gamma} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, $\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} Ea_{ni}Y_{ni}\right| \leq \frac{\varepsilon b_n}{2}$ for all n large enough, which implies

$$
\begin{aligned}
(3.7) \qquad & P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \\
& \leq \sum_{i=1}^{n} P\left(|X_{ni}| > b_n\right) + P\left(\max_{1\leq j\leq n} |T_{nj}| > \frac{\varepsilon b_n}{2}\right).
\end{aligned}
$$

To prove (3.1), it suffices to show that

$$
(3.8) \qquad I \triangleq \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P\left(|X_{ni}| > b_n\right) < \infty,
$$

and

$$
(3.9) \qquad J \triangleq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq j\leq n} |T_{nj}| > \frac{\varepsilon b_n}{2}\right) < \infty.
$$

By the Markov inequality, $E|X|^q < \infty$ and some standard computations, we have

$$
(3.10) \qquad I \leq C\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P(|X| > b_n) \leq C\sum_{n=1}^{\infty} n^{\alpha p-1}\frac{E|X|^q}{b_n^q}
$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-1} n^{-q/\alpha}(\log n)^{-q/\gamma} < \infty.$$

For J, by the Markov inequality (for $M \geq 2$) and Lemma 2.2, we have

$$\begin{aligned}
J &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} E\left(\max_{1\leq j\leq n} |T_{nj}|^M\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \sum_{i=1}^{n} |a_{ni}|^M E|Y_{ni}|^M \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \left(\sum_{i=1}^{n} |a_{ni}|^2 E|Y_{ni}|^2\right)^{M/2} \\
&\triangleq J_1 + J_2.
\end{aligned} \tag{3.11}$$

Take some suitable constant M such that

$$\max\left\{2, \frac{\alpha(\alpha p-1)}{1-\delta}\right\} < M < \min\left\{q-\alpha, \frac{q-\alpha^2 p+\alpha}{\delta}\right\},$$

which implies

$$q > \alpha + M, \quad \frac{q}{\alpha} - \frac{M}{\alpha} > 1, \quad q > \alpha^2 p - \alpha + M\delta,$$

and

$$\frac{q}{\alpha} - \alpha p + 2 - \frac{M\delta}{\alpha} > 1, \quad \alpha p - 2 + \frac{M\delta}{\alpha} - \frac{M}{\alpha} < -1, \quad M > \alpha.$$

By (3.3), (2.4) of Lemma 2.3 and the Markov inequality, we have

$$\begin{aligned}
J_1 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \\
&\quad \left(\sum_{i=1}^{n} |a_{ni}|^M \left(E|X_{ni}|^M I\left(|X_{ni}| \leq b_n\right) + b_n^M P(|X_{ni}| > b_n)\right)\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \\
&\quad \left(\sum_{i=1}^{n} |a_{ni}|^M \left(E|X|^M I\left(|X| \leq b_n\right) + b_n^M P(|X| > b_n)\right)\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M n^{M\delta/\alpha} E|X|^M I\left(|X| \leq b_n\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log 2n)^M n^{M\delta/\alpha} P(|X| > b_n)
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha} b_n^{-M} (\log 2n)^M \sum_{k=2}^{n} E|X|^M I(b_{k-1}<|X|\le b_k) \\
&\qquad + C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha} (\log 2n)^M \frac{E|X|^q}{b_n^q} \\
&\le C\sum_{k=2}^{\infty} b_k^M P(|X|>b_{k-1}) \sum_{n=k}^{\infty} n^{\alpha p-2+M\delta/\alpha-M/\alpha} (\log n)^{-M/\gamma} (\log 2n)^M \\
&\qquad + C\sum_{n=1}^{\infty} \frac{n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M}{n^{q/\alpha}(\log n)^{q/\gamma}} \\
&\le C\sum_{k=2}^{\infty} b_k^M P(|X|>b_{k-1}) + C\sum_{n=1}^{\infty} \frac{n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M}{n^{q/\alpha}(\log n)^{q/\gamma}} \\
&\le C\sum_{k=3}^{\infty} \frac{k^{M/\alpha}(\log k)^{M/\gamma}}{(k-1)^{q/\alpha}(\log(k-1))^{q/\gamma}} + C\sum_{n=1}^{\infty} \frac{n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M}{n^{q/\alpha}(\log n)^{q/\gamma}} < \infty.
\end{aligned}
$$

Analogous to the proof of (3.12), for J_2, by (2.4) of Lemma 2.3 and (3.3),

$$
\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} (\log 2n)^M \\
&\quad \left(\sum_{i=1}^{n} |a_{ni}|^2 \left(E|X_{ni}|^2 I(|X_{ni}|\le b_n) + b_n^2 P(|X_{ni}|>b_n)\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} (\log 2n)^M \\
&\quad \left(\sum_{i=1}^{n} |a_{ni}|^2 \left(E|X|^2 I(|X|\le b_n) + b_n^2 P(|X|>b_n)\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} (\log 2n)^M n^{M\delta/\alpha} \\
&\quad \left(\left(E|X|^2 I(|X|\le b_n) + b_n^2 P(|X|>b_n)\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} (\log 2n)^M n^{M\delta/\alpha} \left(E|X|^2 I(|X|\le b_n)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} (\log 2n)^M n^{M\delta/\alpha} \left(P(|X|>b_n)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} (\log 2n)^M n^{M\delta/\alpha} E|X|^M I(|X|\le b_n)
\end{aligned} \tag{3.13}
$$

$$+ C \sum_{n=1}^{\infty} n^{\alpha p-2} (\log 2n)^M n^{M\delta/\alpha} P(|X| > b_n)$$
$$< \infty.$$

The proof of Theorem 3.1 is completed. □

The next theorem treats the case $\alpha p = 1$. Analogous to that in the proof of Theorem 3.1, here will omit the details.

Theorem 3.2. *Suppose that $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of rowwise END random variables which is stochastically dominated by X. Let $\{a_{ni}; i \ge 1, n \ge 1\}$ be an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some α with $0 < \alpha \le 2$. $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$ for some $\gamma > 0$. Assume further that $EX_{ni} = 0$ while $1 < \alpha \le 2$. Then there exits some $q > \alpha + 2$ such that $E|X|^q < \infty$, implies that*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni} X_{ni} \right| > \varepsilon b_n \right) < \infty \quad \text{for } \forall\, \varepsilon > 0. \tag{3.14}$$

Remark 3.1. The moment condition $E|X|^q < \infty$ in Theorems 3.1 and 3.2 is much weaker than the corresponding moment condition $E(\exp(h|X|^{\gamma})) < \infty$ of Wang et al. [17], Cai [2]. Since, NA sequences and NOD sequences are two special cases of END sequences, Theorems 3.1 and 3.2 hold for arrays of rowwise NA (and NOD) random variables. Therefore, Theorems 3.1 and 3.2 are extensions and improvements of the corresponding ones of Wang et al. [17], Cai [2] (by letting X_{ni} instead of X_i), respectively. In addition, it is worth pointing out that the method in the proof of Theorem 3.1 is different from those of Cai [2] and Wang et al. [17]. And also, the complete convergence results are obtained under the weighted sums satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for $0 < \alpha \le 2$.

4. Complete moment convergence

In this section, we will discuss the complete moment convergence for weighted sums of arrays of rowwise END random variables. The concept of complete moment convergence was introduced by Chow [4] as follows: let $\{X_n, n \ge 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \ge 0$,

$$\sum_{n=1}^{\infty} a_n E\left(b_n^{-1} |X_n| - \varepsilon\right)_+^q < \infty,$$

then the above result was called the complete moment convergence.

Theorem 4.1. *Suppose that $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of rowwise NOD random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}; i \ge 1, n \ge 1\}$ be an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some $0 < \alpha \le 2$. $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$ for some $\gamma > 0$.*

Assume further that $EX_{ni} = 0$ *if* $1 < \alpha \leq 2$. *Then the following statements hold*:

(i) *If* $\alpha > \gamma$, *then* $E|X|^{\alpha} < \infty$ *implies*

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon\right)_{+}^{\alpha} < \infty \quad \text{for } \forall\varepsilon > 0. \tag{4.1}$$

(ii) *If* $\alpha = \gamma$, *then* $E|X|^{\alpha}\log(1+|X|) < \infty$ *implies* (4.1).

(iii) *If* $\alpha < \gamma$, *then* $E|X|^{\gamma} < \infty$ *implies* (4.1).

Proof. For $\forall\varepsilon > 0$, note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon\right)_{+}^{\alpha} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon > t^{1/\alpha}\right) dt \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{1} P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon + t^{1/\alpha}\right) dt \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon + t^{1/\alpha}\right) dt \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) dt \\
&\triangleq I + J.
\end{aligned} \tag{4.2}$$

To prove (4.1), it suffices to prove that $I < \infty$ and $J < \infty$. Follow the above result of Shen [14] referred, we directly have $I < \infty$.

Assume that $a_{ni} \geq 0$. For fixed $n \geq 1$, $i \geq 1$ and all $t \geq 1$, define the so-called *monotone truncation*:

$$\begin{aligned}
Y_{ni} = &-b_n t^{1/\alpha} I\left(a_{ni}X_{ni} < -b_n t^{1/\alpha}\right) + a_{ni}X_{ni} I\left(|a_{ni}X_i| \leq b_n t^{1/\alpha}\right) \\
&+ b_n t^{1/\alpha} I\left(a_{ni}X_{ni} > b_n t^{1/\alpha}\right), \\
Z_{ni} = &\left(a_{ni}X_{ni} + b_n t^{1/\alpha}\right) I\left(a_{ni}X_{ni} < -b_n t^{1/\alpha}\right) \\
&+ \left(a_{ni}X_{ni} - b_n t^{1/\alpha}\right) I\left(a_{ni}X_{ni} > b_n t^{1/\alpha}\right).
\end{aligned}$$

It is easy to check that for $\forall\varepsilon > 0$,

$$P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) \leq P\left(\left|\sum_{i=1}^{n} Y_{ni}\right| > b_n t^{1/\alpha}\right)$$

$$
\begin{aligned}
&+ P\left(\bigcup_{i=1}^{n}\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right)\right) \\
&\le P\left(\left|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})\right| > b_n t^{1/\alpha} - \left|\sum_{i=1}^{n} EY_{ni}\right|\right) \\
&+ \sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right).
\end{aligned}
$$

It is simple to see that

$$
\sup_{t\ge 1}\frac{1}{b_n t^{1/\alpha}}\left|\sum_{i=1}^{n} EY_{ni}\right| \to 0 \quad \text{as} \quad n\to\infty. \tag{4.3}
$$

Really, if $0 < \alpha \le 1$, by (2.6) of Lemma 2.4, the C_r inequality, the Markov inequality and $E|X|^{\alpha} < \infty$, we have that

$$
\begin{aligned}
&\sup_{t\ge 1}\frac{1}{b_n t^{1/\alpha}}\left|\sum_{i=1}^{n} EY_{ni}\right| \\
&\le C\sup_{t\ge 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}|EY_{ni}| \\
&\le C\sup_{t\ge 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n} E\left|a_{ni}X_{ni}\right| I\left(|a_{ni}X_{ni}| \le b_n t^{1/\alpha}\right) \\
&\quad + C\sup_{t\ge 1}\sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right) \\
&\le C\sup_{t\ge 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n} E\left|a_{ni}X\right| I\left(|a_{ni}X| \le b_n t^{1/\alpha}\right) \\
&\quad + C\sup_{t\ge 1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/\alpha}\right) \\
&\le C\sup_{t\ge 1}\frac{1}{b_n^{\alpha} t}\sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} I\left(|a_{ni}X| \le b_n t^{1/\alpha}\right) \\
&\quad + C\sup_{t\ge 1}\frac{1}{b_n^{\alpha} t}\sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{4.4}
$$

Noting that $0 < Z_{ni} = a_{ni}X_{ni} - b_n t^{1/\alpha} < a_{ni}X_{ni}$ if $a_{ni}X_{ni} > b_n t^{1/\alpha}$; $a_{ni}X_{ni} < Z_{ni} = a_{ni}X_{ni} + b_n t^{1/\alpha} < 0$ if $a_{ni}X_{ni} < -b_n t^{1/\alpha}$. Hence, $|Z_{ni}| < |a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right)$.

If $1 < \alpha \leq 2$, by $EX_{ni} = 0$, (2.7) of Lemma 2.4, the C_r inequality and $E|X|^\alpha < \infty$, we have that

$$
\begin{aligned}
(4.5) \qquad & \sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \left| \sum_{i=1}^{n} EY_{ni} \right| \\
&= \sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \left| \sum_{i=1}^{n} EZ_{ni} \right| \\
&\leq C \sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} E\left|Z_{ni}\right| \\
&\leq C \sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} E\left|a_{ni}X_{ni}\right| I\left(\left|a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) \\
&\leq C \sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} E\left|a_{ni}X\right| I\left(\left|a_{ni}X\right| > b_n t^{1/\alpha}\right) \\
&\leq C \sup_{t\geq 1} \frac{1}{b_n^\alpha t} \sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\left(\left|a_{ni}X\right| > b_n t^{1/\alpha}\right) \\
&\leq C(\log n)^{-\alpha/\gamma} E|X|^\alpha \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Therefore, $|\sum_{i=1}^n EY_{ni}| \leq \frac{b_n t^{1/\alpha}}{2}$ holds uniformly for n large enough and all $t \geq 1$, which implies

$$
\begin{aligned}
& P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) \\
&\leq P\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right| > \frac{b_n t^{1/\alpha}}{2}\right) + \sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right).
\end{aligned}
$$

To prove $J < \infty$, it suffices to show that

$$
(4.6) \qquad J_1 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty P\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right| > \frac{b_n t^{1/\alpha}}{2}\right) dt < \infty,
$$

$$
(4.7) \qquad J_2 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) dt < \infty.
$$

By the Markov inequality and (2.2) of Lemma 2.2, we have

$$
\begin{aligned}
(4.8) \qquad J_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \frac{1}{b_n^M t^{M/\alpha}} E\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right|^M\right) dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \frac{1}{b_n^M t^{M/\alpha}} \left(\sum_{i=1}^{n} E|Y_{ni} - EY_{ni}|^M\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(\sum_{i=1}^{n} E|Y_{ni} - EY_{ni}|^2 \right)^{M/2} \Bigg) dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n b_n^M} \int_1^{\infty} \frac{1}{t^{M/\alpha}} \sum_{i=1}^{n} E|Y_{ni}|^M dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^M} \int_1^{\infty} \frac{1}{t^{M/\alpha}} \left(\sum_{i=1}^{n} E|Y_{ni}|^2 \right)^{M/2} dt \\
&\triangleq J_{11} + J_{12},
\end{aligned}
$$

where $\left(M > \max\left\{2, \frac{2\gamma}{\alpha}\right\}\right)$.

For J_{12}, by (2.6) of Lemma 2.4, the C_r inequality, $0 < \alpha \leq 2$ and $M > \frac{2\gamma}{\alpha}$, we have

$$
\begin{aligned}
J_{12} &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \left(\sum_{i=1}^{n} \frac{E|a_{ni}X|^{\alpha}}{b_n^{\alpha} t} I\left(|a_{ni}X| \leq b_n t^{1/\alpha}\right) \right. \\
&\qquad \left. + \sum_{i=1}^{n} \frac{E|a_{ni}X|^{\alpha}}{b_n^{\alpha} t} I\left(|a_{ni}X| > b_n t^{1/\alpha}\right) \right)^{M/2} dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} t^{-M/2} \left(b_n^{-\alpha} \sum_{i=1}^{n} \left(E|a_{ni}X|^{\alpha}\right) \right)^{M/2} dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n} (\log n)^{-\alpha M/2\gamma} (E|X|^{\alpha})^{M/2} < \infty.
\end{aligned} \tag{4.9}
$$

For J_{11}, by (2.6) of Lemma 2.4 and (2.1), we have that

(4.10)

$$
\begin{aligned}
J_{11} &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right) dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^M} \int_1^{\infty} \frac{1}{t^{M/\alpha}} \sum_{i=1}^{n} E|a_{ni}X_{ni}|^M I\left(|a_{ni}X_{ni}| \leq b_n t^{1/\alpha}\right) dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/\alpha}\right) dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^M} \int_1^{\infty} \frac{1}{t^{M/\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^M I\left(|a_{ni}X| \leq b_n t^{1/\alpha}\right) dt \\
&= C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/\alpha}\right) dt
\end{aligned}
$$

$$+ C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(|a_{ni}X|\le b_n\right)dt$$
$$+ C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_nt^{1/\alpha}\right)dt$$
$$\triangleq J_{111}+J_{112}+J_{113}.$$

For J_{111}, by $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ and Lemma 2.5, we have that

$$\begin{aligned} J_{111} &= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_nt^{1/\alpha}\right)dt \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_1^{\infty}P\left(|a_{ni}X|>b_nt^{1/\alpha}\right)dt \\ &= C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_1^{\infty}P\left(\frac{|a_{ni}X|^{\alpha}}{b_n^{\alpha}}>t\right)dt \\ &\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|>b_n\right)<\infty. \end{aligned} \tag{4.11}$$

For J_{112}, by Lemma 2.6 and $M>2\ge\alpha$, we have that

$$\begin{aligned} J_{112} &= C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E|a_{ni}X|^M}{b_n^M}I\left(|a_{ni}X|\le b_n\right)\int_1^{\infty}\frac{1}{t^{M/\alpha}}dt \\ &\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(|a_{ni}X|\le b_n\right)<\infty. \end{aligned} \tag{4.12}$$

Take $t=x^{\alpha}$. By $M>2\ge\alpha$ and Lemma 2.5, we have that

(4.13)
$$\begin{aligned} J_{113} &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_nt^{1/\alpha}\right)dt \\ &\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}x^{\alpha-M-1}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_nx\right)dx \\ &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{m=1}^{\infty}\int_m^{m+1}x^{\alpha-M-1}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_nx\right)dx \\ &\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{m=1}^{\infty}m^{\alpha-M-1}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_n(m+1)\right) \\ &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^M I\left(b_ns<|a_{ni}X|\le b_n(s+1)\right)\sum_{m=s}^{\infty}m^{\alpha-M-1} \end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^M I\left(b_n s<|a_{ni}X|\leq b_n\left(s+1\right)\right)s^{\alpha-M}$$
$$\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}\sum_{s=1}^{\infty}\left(s+1\right)^{M-\alpha}s^{\alpha-M}E|a_{ni}X|^\alpha I\left(b_n s<|a_{ni}X|\leq b_n\left(s+1\right)\right)$$
$$\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(|a_{ni}X|>b_n\right)<\infty.$$

The proof of Theorem 4.1 is completed. □

Remark 4.1. Under the conditions of Theorem 4.1, noting that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|-\varepsilon\right)_+^\alpha \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|-\varepsilon>t^{1/\alpha}\right)dt \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\varepsilon^\alpha} P\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|>b_n t^{1/\alpha}+b_n\varepsilon\right)dt \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{n}\int_{\varepsilon^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|>b_n t^{1/\alpha}+b_n\varepsilon\right)dt \\
&\geq C\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\varepsilon^\alpha} P\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|>2b_n\varepsilon\right)dt \\
&\geq C\varepsilon^\alpha\sum_{n=1}^{\infty}\frac{1}{n}P\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|>2b_n\varepsilon\right).
\end{aligned} \tag{4.14}$$

Since $\varepsilon>0$ is arbitrary. From (4.14), it is clear to see that the complete moment convergence implies the complete convergence. Hence, Theorem 4.1 improves the corresponding result of Shen [14] listed in references under the same conditions.

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