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ON REVERSIBILITY RELATED TO IDEMPOTENTS

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Abstract. This article concerns a ring property which preserves the reversibility of elements at nonzero idempotents. A ring R shall be said to be *quasi-reversible* if $0 \neq ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$, where $I(R)$ is the set of all idempotents in R. We investigate the quasi-reversibility of 2 by 2 full and upper triangular matrix rings over various kinds of reversible rings, concluding that the quasi-reversibility is a proper generalization of the reversibility. It is shown that the quasi-reversibility does not pass to polynomial rings. The structure of Abelian rings is also observed in relation with reversibility and quasi-reversibility.

1. Quasi-reversible rings

Throughout every ring is an associative ring with identity unless otherwise stated. Let R be a ring. Use $I(R)$, $N^*(R)$, $N(R)$, and $J(R)$ to denote the set of all idempotents, the upper nilradical (i.e., the sum of all nil ideals), the set of all nilpotent elements, and the Jacobson radical in R, respectively. Note $N^*(R) \subseteq N(R)$. Write $I(R)' = \{e \in I(R) \mid e \neq 0\}$. $Z(R)$ denotes the center of R. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively. Let $n \geq 2$. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$), and $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere, and I_n denotes the identity matrix in $\mathrm{Mat}_n(R)$.

Following Cohn [4], a ring R (possibly without identity) is called *reversible* if $ab = 0$ for $a, b \in R$ implies $ba = 0$. Anderson and Camillo [1] used the term ZC_2 for the reversibility. A ring (possibly without identity) is usually said to be *reduced* if it has no nonzero nilpotent elements. Many commutative rings are not reduced (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and there exist many noncommutative reduced rings (e.g., direct products of noncommutative domains). It is easily checked that the class of reversible rings contains commutative rings

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and reduced rings. A ring (possibly without identity) is called *Abelian* if every idempotent is central. It is simple to check that reversible rings are Abelian. A ring R is usually called *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. Abelian rings are clearly directly finite.

Lemma 1.1. (1) *A ring R is reversible if and only if $ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$ if and only if $ab \in I(R)$ for $a, b \in R$ implies $ab = ba$.*

(2) *Let R be a reversible ring and suppose that $AB \in I(T_2(R))'$ for $A = \left(\begin{smallmatrix} a_1 & a_3 \\ 0 & a_2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_3 \\ 0 & b_2 \end{smallmatrix}\right) \in T_2(R)$. Then $b_1a_1(b_1a_3 + b_3a_2)b_2a_2 = 0$, and*

$$\begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)(b_1a_1 + b_2a_2) \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & b_1a_1(b_1a_3 + b_3a_2) \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in I(T_2(R))'.$$

(3) *Let R be a ring with $I(R) = \{0, 1\}$. If $AB \in I(D_2(R))'$ for $A, B \in D_2(R)$, then $AB = I_2 = BA$.*

Proof. (1) is obtained from [8, Proposition 1.4 and Corollary 1.5].

(2) From $0 \neq AB \in I(T_2(R))$, we have that $0 \neq a_1b_1 \in I(R)$ or $0 \neq a_2b_2 \in I(R)$. By (1), $a_1b_1 = b_1a_1$ and $a_2b_2 = b_2a_2$. Set $e = a_1b_1$ and $f = a_2b_2$. We use freely the fact that reversible rings are Abelian.

From $(AB)^2 = AB$, we get

$$a_1b_3 + a_3b_2 = (a_1b_3 + a_3b_2)e + (a_1b_3 + a_3b_2)f$$

and

$$\begin{aligned} &(a_1b_3 + a_3b_2)e = (a_1b_3 + a_3b_2)(1 - f), \\ &(a_1b_3 + a_3b_2)f = (a_1b_3 + a_3b_2)(1 - e), \\ &e(a_1b_3 + a_3b_2)f = 0. \end{aligned}$$

Next from $e(a_1b_3 + a_3b_2)f = 0$, we obtain

$$(*) \qquad a_1b_1a_1b_3a_2b_2 + a_1b_1a_3b_2a_2b_2 = 0.$$

Multiplying the equality $(*)$ b_1 on the left and a_2 on the right, we obtain

$$\begin{aligned} 0 &= b_1(a_1b_1a_1b_3a_2b_2 + a_1b_1a_3b_2a_2b_2)a_2 \\ &= b_1a_1b_1a_1b_3a_2b_2a_2 + b_1a_1b_1a_3b_2a_2b_2a_2 \\ &= (b_1a_1)(b_1a_1)b_3a_2(b_2a_2) + (b_1a_1)b_1a_3b_2(a_2b_2)a_2 \\ &= (a_1b_1)b_3a_2(b_2a_2) + (b_1a_1)b_1a_3(a_2b_2)(b_2a_2) \\ &= (b_1a_1)b_3a_2(a_2b_2) + (b_1a_1)b_1a_3(a_2b_2) \\ &= (b_1a_1)(b_1a_3 + b_3a_2)(a_2b_2) \\ &= e(b_1a_3 + b_3a_2)f. \end{aligned}$$

This result gives us

$$\begin{pmatrix} b_1a_1 & (b_1a_3+b_3a_2)(b_1a_1+b_2a_2) \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & b_1a_1(b_1a_3+b_3a_2) \\ 0 & b_2a_2 \end{pmatrix},$$
$$\begin{pmatrix} b_1a_1 & (b_1a_3+b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in I(T_2(R))'.$$

(3) Let R be a ring with $I(R) = \{0,1\}$. Then R is Abelian, and so $I(D_2(R)) = \{0, I_2\}$ by help of [6, Lemma 2]. Hence $D_2(R)$ is also Abelian. Suppose that $AB \in I(D_2(R))'$ for $A, B \in D_2(R)$. Then $AB = I_2$, and so $BA = I_2$ because $D_2(R)$ is directly finite. Thus $AB = BA$. □

We next consider the following notion, based on Lemma 1.1(1).

Definition 1.2. A ring R (possibly without identity) is *quasi-reversible* provided that if $ab \in I(R)'$ for $a, b \in R$, then $ba \in I(R)$.

By Lemma 1.1(3), every ring R with $I(R) = \{0,1\}$ is quasi-reversible. Reversible rings are quasi-reversible by Lemma 1.1(1), but not conversely by Theorem 1.4 to follow. Recall that reversible rings are Abelian. But quasi-reversible rings need not be Abelian by Theorem 1.4 to follow.

Lemma 1.3. (1) *A ring R is quasi-reversible if and only if $ab \in I(R)'$ for $a, b \in R$ implies $ba \in I(R)'$.*

(2) *The class of quasi-reversible rings is closed under subrings* (*with or without identity*).

Proof. (1) It suffices to show the necessity. Let R be a quasi-reversible ring and suppose that $ab \in I(R)'$ for $a, b \in R$. Then $ba \in I(R)$. Assume $ba = 0$. Then $ab = abab = 0$, contrary to $ab \neq 0$. So $ba \in I(R)'$.

(2) Let R be a quasi-reversible ring and S be a subring (possibly without identity) of R. Suppose $ab \in I(S)'$ for $a, b \in S$. Note $I(S) = I(R) \cap S$ and $I(S)' = I(R)' \cap S$. Since R is quasi-reversible, $ba \in I(R)$. But $ba \in S$ and $ba \in I(S)$ follows. □

Following Marks [13], a ring R is called *NI* if $N(R) = N^*(R)$. It is obvious that a ring R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is reduced. If a ring R is NI, then R is directly finite by [7, Proposition 2.7(1)]. Reversible rings are easily shown to be NI. We use these facts freely.

Theorem 1.4. *A ring R is a reversible ring with $I(R) = \{0,1\}$ if and only if $T_2(R)$ is a quasi-reversible ring.*

Proof. Suppose that R is a reversible ring with $I(R) = \{0,1\}$. From $I(R) = \{0,1\}$, we get

$$I(T_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} \mid r, s \in R \right\},$$

through a simple computation. Suppose that $AB \in I(T_2(R))'$ for $A = \left(\begin{smallmatrix} a_1 & a_3 \\ 0 & a_2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_3 \\ 0 & b_2 \end{smallmatrix}\right) \in T_2(R)$. Then $a_1b_1 = 1$ or $a_2b_2 = 1$. If $AB = I_2$ (i.e., $a_1b_1 = 1$, $a_2b_2 = 1$, and $a_1b_3 + a_3b_2 = 0$), then $BA = I_2$ because $T_2(R)$ is NI and so directly finite. So, consider the cases of ($a_1b_1 = 1$, $a_2b_2 = 0$) and ($a_1b_1 = 0$, $a_2b_2 = 1$). Note that $BA = \left(\begin{smallmatrix} b_1a_1 & b_1a_3+b_3a_2 \\ 0 & b_2a_2 \end{smallmatrix}\right)$.

Assume $a_1b_1 = 1$ and $a_2b_2 = 0$. Then $b_1a_1 = a_1b_1 = 1$ and $b_2a_2 = a_2b_2 = 0$ by Lemma 1.1(1). This yields

$$BA = \begin{pmatrix} 1 & b_1a_3 + b_3a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1a_1 & b_1a_1(b_1a_3 + b_3a_2) \\ 0 & b_2a_2 \end{pmatrix} \in I(T_2(R))'.$$

Assume $a_1b_1 = 0$ and $a_2b_2 = 1$. Then $b_1a_1 = a_1b_1 = 0$ and $b_2a_2 = a_2b_2 = 1$ by Lemma 1.1(1). This yields

$$BA = \begin{pmatrix} 0 & b_1a_3 + b_3a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1a_1 & (b_1a_3 + b_3a_2)b_2a_2 \\ 0 & b_2a_2 \end{pmatrix} \in I(T_2(R))'.$$

Therefore $T_2(R)$ is quasi-reversible.

Conversely suppose that $T_2(R)$ is quasi-reversible. Let $ab \in I(R)$ for $a, b \in R$. Consider matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

in $T_2(R)$. Then $ab \in I(R)$ implies $0 \neq AB = \left(\begin{smallmatrix} ab & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(T_2(R))$. Since $T_2(R)$ is quasi-reversible, $BA = \left(\begin{smallmatrix} ba & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(T_2(R))$. This yields $ba \in I(R)$. Thus R is reversible by Lemma 1.1(1).

(Another proof) Let $ab = 0$ for $a, b \in R$. Consider matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

in $T_2(R)$. Then $0 \neq AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(T_2(R))$. Since $T_2(R)$ is quasi-invertible, $BA = \left(\begin{smallmatrix} ba & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(T_2(R))$. This implies $(ba)^2 = ba$, but $ab = 0$ yields $ba = baba = 0$. Thus R is reversible.

Next assume on the contrary that there exists $e^2 = e \in R$ with $e \notin \{0, 1\}$. Consider two matrices

$$C = \begin{pmatrix} e & 1 \\ 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$$

in $T_2(R)$. Then

$$0 \neq CD = \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix} \in I(T_2(R)) \text{ but } DC = \begin{pmatrix} e & 1 \\ 0 & 0 \end{pmatrix} \notin I(T_2(R)),$$

contrary to $T_2(R)$ being quasi-reversible. Thus $I(R) = \{0, 1\}$. □

The existence of a reversible ring R, such that $I(R) = \{0, 1\}$ and R is not reduced, illuminates Theorem 1.4. Let R be a domain and consider $D_2(R)$. Then $D_2(R)$ is reversible by [11, Proposition 1.6], and $I(D_2(R)) = \{0, I_2\}$. $D_2(R)$ is clearly not reduced.

Theorem 1.4 yields the following results.

Corollary 1.5. (1) *If R is a domain, then $T_2(R)$ is quasi-reversible.*

(2) *Let R be a (quasi-)reversible ring such that $I(R)$ contains $\{0,1\}$ properly, and $n \geq 2$. Then $\mathrm{Mat}_n(R)$ and $T_n(R)$ need not be quasi-reversible.*

Proof. (1) is an immediate consequence of Theorem 1.4.

(2) By Theorem 1.4, $T_2(R)$ is not quasi-reversible, and thus $\mathrm{Mat}_n(R)$ and $T_n(R)$ need not be quasi-reversible for $n \geq 2$, by Lemma 1.3(2). $\square$

Considering Theorem 1.4 and Corollary 1.5, it is natural to ask whether $T_2(R)$ is quasi-reversible over a reduced ring R. But the answer is negative by the following. Furthermore this illuminates Lemma 1.1(2).

Example 1.6. (1) Let R_0 be a domain and $R = R_0 \times R_0$. Then R is a reduced ring but not a domain, and $I(R) = \{(0,0),(1,0),(0,1),(1,1)\}$. Then $T_2(R)$ is not quasi-reversible by Theorem 1.4.

(2) Note that the ring R in (1) is reversible with $\{0,1\} \subsetneq I(R)$, hence $T_2(R)$ is not a quasi-reversible ring by Theorem 1.4. Moreover, for $A = \begin{pmatrix} (1,0) & (1,1) \\ (0,0) & (0,0) \end{pmatrix}$ and $B = \begin{pmatrix} (1,1) & (1,1) \\ (0,0) & (1,0) \end{pmatrix} \in T_2(R)$, we have $AB \in I(T_2(R))$ and $BA \notin I(T_2(R))$.

The arguments in the following illuminate Theorem 1.4.

Example 1.7. Theorem 1.4 is not valid for $T_n(R)$ with $n \geq 3$. Let R be any ring and consider $T_3(R)$. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ in $T_3(R)$. Then $AB = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in I(T_3(A))$. But $BA = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \notin I(T_3(R))$. This argument can be applicable to the case of $n \geq 4$. Thus $T_n(R)$ cannot be quasi-reversible when $n \geq 3$.

In the following we see another kind of quasi-reversible rings in the class of simple Artinian rings.

Theorem 1.8. $\mathrm{Mat}_2(\mathbb{Z}_2)$ *is quasi-reversible.*

Proof. Let $R = \mathrm{Mat}_2(\mathbb{Z}_2)$. Then $I(R) = \{0, I_2, E_{11}, E_{22}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)\}$ by [12, Lemma 1.3]. Suppose that $AB \in I(R)'$ for $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \in R$. We proceed our argument, case by case.

Assume $AB = I_2$. Since R is Artinian, R is directly finite and so $AB = I_2$ implies $BA = I_2$.

Assume $AB = E_{11}$. Then $a_1b_1 + a_2b_3 = 1, a_1b_2 + a_2b_4 = 0, a_3b_1 + a_4b_3 = 0, a_3b_2 + a_4b_4 = 0$. From $a_1b_1 + a_2b_3 = 1$, we have the cases of $(a_1b_1 = 1, a_2b_3 = 0)$ and $(a_1b_1 = 0, a_2b_3 = 1)$.

Consider the case of $a_1b_1 = 1, a_2b_3 = 0$. Then $a_1 = 1 = b_1$; and $a_2 = 0$ or $b_3 = 0$. Let $a_2 = 0$. Then we get $b_2 = 0$ from $0 = a_1b_2 + a_2b_4 = a_1b_2 = b_2$. So $0 = a_3b_2 + a_4b_4 = a_4b_4$. These results provide us with

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_3 + b_4a_3 & 0 \end{pmatrix} \in I(R).$$

Let $b_3 = 0$. Then we get $a_3 = 0$ from $0 = a_3b_1 + a_4b_3 = a_3b_1$. So $0 = a_3b_2 + a_4b_4 = a_4b_4$. These results provide us with

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & a_2 + b_2a_4 \\ 0 & 0 \end{pmatrix} \in I(R).$$

Consider the case of $a_1b_1 = 0, a_2b_3 = 1$. Then $a_2 = 1 = b_3$; and $a_1 = 0$ or $b_1 = 0$. Let $a_1 = 0$. Then we get $b_4 = 0$ from $0 = a_1b_2 + a_2b_4 = a_2b_4 = b_4$. So $0 = a_3b_2 + a_4b_4 = a_3b_2$. These results provide us with

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & b_1 + b_2a_4 \\ 0 & 1 \end{pmatrix} \in I(R).$$

Let $b_1 = 0$. Then we get $a_4 = 0$ from $0 = a_3b_1 + a_4b_3 = a_4b_3 = a_4$. So $0 = a_3b_2 + a_4b_4 = a_3b_2$. These results provide us with

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_1 + b_4a_3 & 1 \end{pmatrix} \in I(R).$$

Thus $BA \in I(R)$ in any case when $AB = E_{11}$.

The argument for the case of $AB = E_{22}$ is similar to one of $AB = E_{11}$.

Assume $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $a_1b_1 + a_2b_3 = 1$, $a_1b_2 + a_2b_4 = 1$, $a_3b_1 + a_4b_3 = 0$, $a_3b_2 + a_4b_4 = 0$. From $a_1b_1 + a_2b_3 = 1$, $a_1b_2 + a_2b_4 = 1$, we have the cases of ($a_1b_1 = 1$, $a_2b_3 = 0$, $a_1b_2 = 1$, $a_2b_4 = 0$), ($a_1b_1 = 1$, $a_2b_3 = 0$, $a_1b_2 = 0$, $a_2b_4 = 1$), ($a_1b_1 = 0$, $a_2b_3 = 1$, $a_1b_2 = 1$, $a_2b_4 = 0$), and ($a_1b_1 = 0$, $a_2b_3 = 1$, $a_1b_2 = 0$, $a_2b_4 = 1$).

Consider the case of $a_1b_1 = 1$, $a_2b_3 = 0$, $a_1b_2 = 1$, $a_2b_4 = 0$. Then $a_1 = b_1 = b_2 = 1$. From $a_2b_3 = 0$, $a_2b_4 = 0$, we have the cases of ($a_2 = 0$) and ($a_2 = 1$, $b_3 = 0 = b_4$).

Let $a_2 = 0$. From $a_3b_1 + a_4b_3 = 0$, $a_3b_2 + a_4b_4 = 0$, we get $a_3 = a_4b_3 = a_4b_4$. If $a_3 = 1$, then $b_3 = a_4 = b_4 = 1$ and hence

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in I(R).$$

If $a_3 = 0$, then $a_4b_3 = a_4b_4 = 0$. Here if $a_4 = 1$, then $b_3 = 0 = b_4$; hence

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in I(R).$$

Here if $a_4 = 0$, then

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_3 & 0 \end{pmatrix} \in I(R).$$

Let $a_2 = 1$. Then $b_3 = 0 = b_4$, and so $0 = a_3b_1 + a_4b_3 = a_3$. These yield

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 + a_4 \\ 0 & 0 \end{pmatrix} \in I(R).$$

Consider the case of $a_1b_1 = 1$, $a_2b_3 = 0$, $a_1b_2 = 0$, $a_2b_4 = 1$. Then $a_1 = b_1 = a_2 = b_4 = 1$. From $a_2b_3 = 0$, $a_1b_2 = 0$, we get $b_3 = 0 = b_2$. Hence

$0 = a_3b_1 + a_4b_3 = a_3$ and $0 = a_3b_2 + a_4b_4 = a_4b_4$. These yield

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in I(R).$$

The arguments for the cases of $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_1b_2 = 1,\ a_2b_4 = 0)$ and $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_1b_2 = 0,\ a_2b_4 = 1)$ are similar to the preceding ones. Thus $BA \in I(R)$ in any case when $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Assume $AB = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $a_1b_1 + a_2b_3 = 1$, $a_1b_2 + a_2b_4 = 0$, $a_3b_1 + a_4b_3 = 1$, $a_3b_2 + a_4b_4 = 0$. From $a_1b_1 + a_2b_3 = 1$, $a_3b_1 + a_4b_3 = 1$, we have the cases of $(a_1b_1 = 1,\ a_2b_3 = 0,\ a_3b_1 = 1,\ a_4b_3 = 0)$, $(a_1b_1 = 1,\ a_2b_3 = 0,\ a_3b_1 = 0,\ a_4b_3 = 1)$, $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_3b_1 = 1,\ a_4b_3 = 0)$, and $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_1b_2 = 0,\ a_4b_3 = 1)$.

Consider the case of $a_1b_1 = 1$, $a_2b_3 = 0$, $a_3b_1 = 1$, $a_4b_3 = 0$. Then $a_1 = b_1 = a_3 = 1$. From $a_2b_3 = 0$, $a_4b_3 = 0$, we have the cases of $(b_3 = 0)$ and $(a_2 = 0 = a_4, b_3 = 1)$.

Let $b_3 = 0$. From $a_1b_2 + a_2b_4 = 0$, $a_3b_2 + a_4b_4 = 0$, we get $b_2 = a_2b_4 = a_4b_4$. If $b_2 = 1$, then $a_2 = a_4 = b_4 = 1$ and hence

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_3 & 1 \end{pmatrix} \in I(R).$$

If $b_2 = 0$, then $a_2b_4 = a_4b_4 = 0$. Here if $b_4 = 1$, then $a_2 = 0 = a_4$; hence

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_3 & 0 \end{pmatrix} \in I(R).$$

Here if $b_4 = 0$, then

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & a_2 \\ 0 & 0 \end{pmatrix} \in I(R).$$

Let $b_3 = 1$. Then $a_2 = 0 = a_4$, and so $0 = a_1b_2 + a_2b_4 = b_2$. These yield

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 + b_4 & 0 \end{pmatrix} \in I(R).$$

Consider the case of $a_1b_1 = 1$, $a_2b_3 = 0$, $a_3b_1 = 0$, $a_4b_3 = 1$. Then $a_1 = b_1 = a_4 = b_3 = 1$. From $a_2b_3 = 0$, $a_3b_1 = 0$, we get $a_2 = 0 = a_3$. Hence $0 = a_1b_2 + a_2b_4 = b_2$ and $0 = a_3b_2 + a_4b_4 = a_4b_4$. These yield

$$BA = \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in I(R).$$

The arguments for the cases of $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_3b_1 = 1,\ a_4b_3 = 0)$ and $(a_1b_1 = 0,\ a_2b_3 = 1,\ a_1b_2 = 0,\ a_4b_3 = 1)$ are similar to the preceding ones. Thus $BA \in I(R)$ in any case when $AB = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Next, we can obtain similar arguments for the cases of $AB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $AB = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ to ones above. Therefore $\mathrm{Mat}_2(\mathbb{Z}_2)$ is quasi-reversible. $\square$

$\mathrm{Mat}_2(\mathbb{Z}_2)$ is simple Artinian. So, considering Theorem 1.8, one may ask naturally whether semisimple Artinian rings are quasi-reversible. But the answer is negative by the following.

Example 1.9. (1) Let K be a division ring and $R = K \times K$. Since $\mathrm{Mat}_2(R)$ is isomorphic to $\mathrm{Mat}_2(K) \times \mathrm{Mat}_2(K)$, $\mathrm{Mat}_2(R)$ is semisimple Artinian. Consider next $T_2(R)$. Then $T_2(R)$ is not quasi-reversible by Theorem 1.4 (see Example 1.6(1) for more details). So $\mathrm{Mat}_2(R)$ is not quasi-reversible by Lemma 1.3(2).

(2) Let $K = \mathbb{Z}_2$ in (1). Then $\mathrm{Mat}_2(\mathbb{Z}_2)$ is quasi-reversible by Theorem 1.8. But $\mathrm{Mat}_2(\mathbb{Z}_2) \times \mathrm{Mat}_2(\mathbb{Z}_2)$ is not quasi-reversible by the argument in (1). So the class of quasi-reversible rings is not closed under direct products.

However we do not know of any example of a field K over which $\mathrm{Mat}_2(K)$ is not quasi-reversible. Considering Theorem 1.4, it is also natural to ask whether $\mathrm{Mat}_2(R)$ quasi-reversible over a reversible ring R with $I(R) = \{0, 1\}$. So we raise the next questions.

Question. (1) Let K be a field. Is $\mathrm{Mat}_2(K)$ quasi-reversible?

(2) Let R be a reversible ring with $I(R) = \{0, 1\}$. Is $\mathrm{Mat}_2(R)$ quasi-reversible?

By help of the argument in Example 2.4(2) to follow, we can conclude that the quasi-reversibility does not pass to polynomial rings.

Example 1.10. Let R_1 be a reduced ring and R_2 be the reversible ring R in Example 2.1 to follow. Set $R_3 = R_1 \times R_2$. Then R_3 is clearly reversible (hence quasi-reversible by Lemma 1.1(1)). Consider $R_3[x]$. Since $R[x]$ is not reversible, based on the argument in [11, Example 2.1], we take $f(x) = \sum_{i=0}^{m} \alpha_i x^i, g(x) = \sum_{j=0}^{n} \beta_j x^j \in R_2[x]$ such that $f(x)g(x) = 0$ but $g(x)f(x) \neq 0$, where we can let $m = n$ by using zero coefficients if necessary. Next consider two polynomials

$$a(x) = (1, \alpha_0) + \sum_{i=1}^{m} (0, \alpha_i) x^i \quad \text{and} \quad b(x) = (1, \beta_0) + \sum_{j=1}^{m} (0, \beta_j) x^j$$

in $R_3[x]$. Then $a(x)b(x) = (1, 0) \in I(R_3[x])'$. But

$$b(x)a(x) = ((1, \beta_0) + \sum_{j=1}^{m} (0, \beta_j) x^j)((1, \alpha_0) + \sum_{i=1}^{m} (0, \alpha_i) x^i) = (1, 0) + c(x)$$

for some $0 \neq c(x) = \sum_{k=1}^{l} (0, \gamma_k) x^k \in R_3[x]$ by the argument in [11, Example 2.1]. Here assume $b(x)a(x) \in I(R_3[x])$. Then we have

$$\begin{aligned}(1, 0) + c(x) &= b(x)a(x) = b(x)a(x)b(x)a(x) \\ &= (1, 0)b(x)a(x) = (1, 0)[(1, 0) + \sum_{k=1}^{l} (0, \gamma_k) x^k] = (1, 0),\end{aligned}$$

a contradiction. Therefore $R_3[x]$ is not quasi-reversible.

2. On a property of Abelian rings

In this section we study the structure of Abelian rings and NI rings. We first find an equivalent condition to Abelian rings. The work in this section is based on the structure of rings in the following example, and the fact that in any ring R, $ab \in I(R)$ for $a, b \in R$ implies $(ba)^2 \in I(R)$.

Example 2.1. We refer to the construction in [11, Example 2.1]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra generated by noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, c over $\mathbb{Z}_2$; and set $B = \{f \in A \mid$ the constant term of f is zero$\}$. Next let I be the ideal of A generated by a_0b_0, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, $a_1b_2 + a_2b_1$, a_2b_2, a_0rb_0, a_2rb_2, b_0a_0, $b_0a_1 + b_1a_0$, $b_0a_2 + b_1a_1 + b_2a_0$, $b_1a_2 + b_2a_1$, b_2a_2, b_0ra_0, b_2ra_2, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$, $(b_0 + b_1 + b_2)r(a_0 + a_1 + a_2)$, and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in B$. Set $R = A/I$.

R is reversible but $R[x]$ is not reversible by [11, Example 2.1]. Since R is Abelian, every idempotent in $R[x]$ is contained in R, i.e., $I(R) = I(R[x])$, by [9, Lemma 8]. Moreover we claim $I(R) = \{0, 1\}$.

Let $f \in I(R)'$. We can write $f = 1 + f_0$ with $f_0 \in \bar{B}$, noting $\bar{B}^4 = 0$, where $\bar{B} = B/I$. Then $1 + f_0 = (1 + f_0)^2 = 1 + 2f_0 + f_0^2 = 1 + f_0^2$, and $f_0 = f_0^2$ follows. So $f_0 \in I(R)$. But $f_0 \in \bar{B} \subseteq N(R)$, entailing $f_0 = 0$. This yields $f = 1$ and therefore $I(R) = \{0, 1\} = I(R[x])$. Now let $g(x)h(x) \in I(R[x])'$ for $g(x), h(x) \in R[x]$. Then $g(x)h(x) = 1$ because $I(R[x])' = \{1\}$. Notice that $R[x]$ is directly finite by help of [9, Lemma 8]. Thus we get $h(x)g(x) = 1$ from $g(x)h(x) = 1$. Hence we now have $h(x)g(x)h(x)g(x) = 1 = g(x)h(x)$.

The argument in Example 2.1 is extended to the general situation as follows.

Theorem 2.2. *For a ring R the following conditions are equivalent*:

(1) *R is Abelian*;

(2) *If $ab \in I(R)'$ for $a, b \in R$, then $(ba)^2 = ab$*;

(3) *If $ab \in I(R)$ for $a, b \in R$, then $(ba)^2 = ab$.*

Proof. (1) $\Rightarrow$ (2). Let R be an Abelian ring. Suppose $ab \in I(R)'$ for $a, b \in R$. Then $(ba)^2 \in I(R)$. Since ab and $(ba)^2$ are central in R, we have

$$(ba)^2 = b(ab)a = b(ab)^2a = b(ab)a(ab) = (ba)^2ab = a(ba)^2b = (ab)^3 = ab.$$

(2) $\Rightarrow$ (1). Suppose the condition (2). Assume on the contrary that there exist $e \in I(R)'$ and $a \in R$ such that $ea(1-e) \neq 0$. Then $e + ea(1-e) \in I(R)'$ (for, if $e + ea(1-e) = 0$, then $0 = (e + ea(1-e))e = e$, contrary to $e \neq 0$; and clearly $e + ea(1-e) \in I(R)$). By the condition, $e(1 + a(1-e)) = e + ea(1-e) \in I(R)'$ implies

$$e = e^2 = [(1 + a(1-e))e][(1 + a(1-e))e] = e(1 + a(1-e)) = e + ea(1-e),$$

entailing $ea(1-e) = 0$. This contradicts $ea(1-e) \neq 0$. Therefore R is Abelian.

(3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) is obtained from that $ab = 0$ for $a, b \in R$ implies $baba = 0 = ab$. $\square$

In the proof of Theorem 2.2, note that $ab \in I(R)'$ implies $ba \neq 0$. It is easily shown that reversible rings are Ableian. This fact is also obtained from Lemma 1.1(1) and Theorem 2.2. The polynomial ring $R[x]$ in Example 2.1 is Abelian but not reversible. As another example of Abelian ring that is not reversible, consider $R = D_n(A)$ for $n \geq 3$ over any Abelian ring A. Then R is Abelian by [6, Lemma 2]. However R is not reversible by [11, Example 1.5].

From Lemma 1.1(1) and Theorem 2.2, we can deduce the following.

Corollary 2.3. *Let R be an Abelian ring but not reversible. Then there exist $a, b \in R$ such that $ab \in I(R)$, $(ba)^2 = ab$, and $ba \notin I(R)$.*

The concepts of quasi-reversible and Abelian are independent of each other as follows. It is easily shown that the class of Abelian rings is closed under subrings and direct products. We use this fact freely.

Example 2.4. (1) $\mathrm{Mat}_2(\mathbb{Z}_2)$ is quasi-reversible by Theorem 1.8. But it is non-Abelian clearly.

(2) Let R_1 be an Abelian ring, and R_2 be an Abelian ring that is not reversible (see Example 2.1 for example). Set $R = R_1 \times R_2$. Let $e \in I(R_1)'$ and $a, b \in R_2$ such that $ab = 0$ but $ba \neq 0$. Set $f = (e, a)$ and $g = (e, b)$ in R. Then $fg = (e, 0) \in I(R)'$. But $gf = (e, ba)$ and $(gf)^2 = (e, 0) \neq gf$, entailing $gf \notin I(R)$. Thus R is not quasi-reversible but R is Abelian.

Next we extend [8, Proposition 1.4 and Corollary 1.5] in the sprit of Theorem 2.2. Due to Bell [3], a ring R is said to be *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. It is easily checked that reversible rings are IFP, IFP rings are Abelian, and IFP rings are NI. We use this fact freely.

Proposition 2.5. (1) *A ring R is reversible if and only if $ab \in I(R)$ for $a, b \in R$ implies $bra = braab$ for all $r \in R$.*

(2) *A ring R is IFP if and only if $ab \in I(R)$ for $a, b \in R$ implies $arb = arb(ba)^2$ for all $r \in R$.*

Proof. (1) Let R be a reversible ring and suppose that $ab \in I(R)$ for $a, b \in R$. Then $ab = ba$ by Lemma 1.1(1). So, from $ba = (ba)^2$, we get $0 = ba(1 - ba) = ba(1 - ab)$. Since R is IFP, $ba(1 - ab) = 0$ implies $bra(1 - ab) = 0$ for all $r \in R$. Thus $bra = braab$.

Conversely suppose that $ab \in I(R)$ for $a, b \in R$ implies $bra = braab$ for all $r \in R$. Letting $r = 1$ here, we get $ba = baab$. Assume $ab = 0$. Then $ba = baab = 0$, and so R is reversible.

(2) Let R be an IFP ring and suppose that $ab \in I(R)$ for $a, b \in R$. Then $ab = (ba)^2$ by Theorem 2.2. Since R is Abelian, we get furthermore $ab = baba = b(ab)aba = (ab)baba = ab(ba)^2$ and $ab(1 - (ba)^2) = 0$. Since R is IFP, we get $arb(1 - (ba)^2) = 0$ for all $r \in R$. Thus $arb = arb(ba)^2$.

Conversely suppose that $ab \in I(R)$ for $a, b \in R$ implies $arb = arb(ba)^2$ for all $r \in R$. Assume $ab = 0$. Then $arb = arb(ba)^2 = arbb(ab)a = 0$, and so R is IFP. $\square$

By help of Proposition 2.5, we get a way to show that reversible rings are IFP, through idempotents. Let R be a ring and suppose $ab \in I(R)$ for $a, b \in R$. Let R be reversible. Then $ab = ba$ by Lemma 1.1(1); hence $arb = arbba$ for all $r \in R$ by Proposition 2.5(1). Moreover $arb = arbba$ implies $arb = arb(ba)^2$ because $ab = ba$. Thus R is IFP by Proposition 2.5(2). By [11, Example 1.5], $D_3(A)$ is IFP but not reversible when A is a reduced ring. Let $R_2 = D_3(A)$ and R_1 be any IFP ring. Next set $R = R_1 \times R_2$. Then R is clearly IFP. But R is not quasi-reversible by Example 2.4(2). Thus the concepts of IFP and quasi-reversible are independent of each other, considering the non-Abelian quasi-reversible $\mathrm{Mat}_2(\mathbb{Z}_2)$.

Following Neumann [15], a ring R is said to be *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Every regular ring R is clearly semiprimitive (i.e., $J(R) = 0$). It is shown by [5, Theorem 1.1] that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent. Furthermore we have the following equivalences by [5, Theorem 3.2] and the fact of semiprime NI rings being reduced.

For a regular ring R, the following conditions are equivalent: (1) R is reduced; (2) R is reversible; (3) R is IFP; (4) R is NI; (5) R is Abelian.

However the quasi-reversibility need not be equivalent to the reversibility when given rings are regular. Consider $\mathrm{Mat}_2(\mathbb{Z}_2)$. $\mathrm{Mat}_2(\mathbb{Z}_2)$ is regular by [5, Theorem 1.7], but it is non-Abelian (hence not reversible).

Following McCoy [14], a ring R is said to be π*-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$ and $b \in R$ such that $a^n = a^n b a^n$. A regular ring is clearly π-regular, but the converse need not hold as can be seen by $R = U_n(A)$ ($n \geq 2$) over a division ring A. The Jacobson radicals of π-regular rings are nil by [10, Lemma 5].

Let R be an Abelian π-regular ring. Then $R/J(R)$ is a regular ring with $J(R) = N^*(R) = N(R)$ by [2, Theorem 3], entailing that $R/J(R)$ is reduced. This implies that R is an NI ring. But R need not be a reversible ring, being compared with the fact above for regular rings. Consider $R = D_n(A)$ for $n \geq 3$ over a division ring. Then R is Abelian by [6, Lemma 2] and clearly π-regular. However R is not reversible by [11, Example 1.5]. For NI rings, we have the following result related to idempotents.

Theorem 2.6. *Let R be an NI ring and suppose $ab \in I(R)$ for $a, b \in R$.*

(1) $\{ab-(ba)^n, ba-(ba)^2, ab-ab(ba)^l, ba-(ba)^l ab, ab-arbba, ba-braab\} \subseteq N^*(R)$ *for all $r \in R$, $n \geq 1$, and $l \geq 2$.*

(2) $\{e \in I(R) \mid e - ab \in N^*(R)$ *and* $e(ba)^2 = (ba)^2 e\} = \{(ba)^2\}$.

Proof. (1) Since R is NI, $\bar{R} = R/N^*(R)$ is a reduced ring. Suppose $ab \in I(R)$ for $a, b \in R$. Then $(ba)^2 \in I(R)$ and $(ba)^2 = (ba)^k$ for all $k \geq 3$. By Lemma 1.1(1), we get $\bar{a}\bar{b} = \bar{b}\bar{a}$ (i.e., $ab - ba \in N^*(R)$) because $\bar{a}\bar{b} \in I(\bar{R})$. Since $(\bar{b}\bar{a})^2 = \bar{b}\bar{a}$, we have that $ab - (ba)^n \in N^*(R)$ for all $n \geq 1$. From $\bar{a}\bar{b} = \bar{b}\bar{a}$, we also obtain

$$\bar{a}\bar{b}(1-(\bar{b}\bar{a})^2) = \bar{b}\bar{a}(1-\bar{b}\bar{a}) = 0 \text{ and } 0 = (1-(\bar{b}\bar{a})^2)\bar{a}\bar{b} = \bar{a}\bar{b}-(\bar{b}\bar{a})^2\bar{a}\bar{b} = \bar{b}\bar{a}-(\bar{b}\bar{a})^2\bar{a}\bar{b},$$

entailing $ab - ab(ba)^2$, $ba - (ba)^2ab \in N^*(R)$. It then follows that $ab - ab(ba)^l$, $ba - (ba)^l ab \in N^*(R)$ for all $l \geq 2$.

Moreover, by Proposition 2.5(1), we obtain

$$\bar{a}\bar{b} = \bar{a}\bar{r}\bar{b}\bar{b}\bar{a} \text{ and } \bar{b}\bar{a} = \bar{b}\bar{r}\bar{a}\bar{a}\bar{b}$$

for all $r \in R$ because reduced rings are reversible and $\bar{a}\bar{b} = \bar{b}\bar{a} \in I(\bar{R})$. This implies $ab - arbba, ba - braab \in N^*(R)$ for all $r \in R$.

(2) Let $e \in I(R)$ with $e - ab \in N^*(R)$ and $e(ba)^2 = (ba)^2e$. Then

$$e(1 - (ba)^2) = 0 = (1 - (ba)^2)e.$$

From this result, we also obtain

$$(e - (ba)^2)^2 = e - 2e(ba)^2 + (ba)^2$$

and

$$(e - (ba)^2)^4 = [(e - 2e(ba)^2) + (ba)^2]^2 = (e - 2e(ba)^2) + (ba)^2,$$

entailing $(e - 2e(ba)^2) + (ba)^2 \in I(R)$. But $e - (ba)^2 = (e - ab) + (ab - (ba)^2) \in N^*(R)$ by (1), and $(e - (ba)^2)^4 \in N^*(R)$ follows. Thus we must have $(e - 2e(ba)^2) + (ba)^2 = 0$. Here $(-e + 2e(ba)^2)^2 = e$ and this yields

$$(ba)^2 = ((ba)^2)^2 = (-e + 2e(ba)^2)^2 = e. \qquad \square$$

Assume that R is Abelian in Theorem 2.6. Then, by Theorem 2.2, we can let $e = ab$ and $ab = e = (ba)^2$. But when R is not Abelian, this result need not hold. As an example, consider the NI ring $T_2(R)$ in Example 1.6(1). Recall $AB = \begin{pmatrix} (1,0) & (2,0) \\ (0,0) & (0,0) \end{pmatrix} \in I(T_2(R))$, $BA = \begin{pmatrix} (1,0) & (1,1) \\ (0,0) & (0,0) \end{pmatrix} \notin I(T_2(R))$, and $(BA)^2 = \begin{pmatrix} (1,0) & (1,0) \\ (0,0) & (0,0) \end{pmatrix}$, where $A = \begin{pmatrix} (1,0) & (1,1) \\ (0,0) & (0,0) \end{pmatrix}$ and $B = \begin{pmatrix} (1,1) & (1,1) \\ (0,0) & (1,0) \end{pmatrix}$. Here $e = (BA)^2 = \begin{pmatrix} (1,0) & (1,0) \\ (0,0) & (0,0) \end{pmatrix} \neq \begin{pmatrix} (1,0) & (2,0) \\ (0,0) & (0,0) \end{pmatrix} = AB$.

Abelian π-regular rings are NI as mentioned above. The condition "π-regular" is not superfluous here. To see this, we refer to the construction of Smoktunowicz [16]. In fact, Smoktunowicz showed in [16, Corollary 13] that there exists a nil algebra N over a countable field K such that $N[x]$ is not nil. Set $R = K + N$. Then R is a local ring with $J(R) = N = N(R)$, entailing $I(R) = \{0, 1\}$. This yields that $R[x]$ is Abelian by [9, Lemma 8]. But $R[x]$ is not NI because $N[x]$ is not nil (i.e., $N^*(R[x]) \subsetneq N(R[x])$). Clearly $R[x]$ is not π-regular.

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