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MITTAG-LEFFLER STABILITY OF SYSTEMS OF FRACTIONAL NABLA DIFFERENCE EQUATIONS

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Abstract. Mittag-Leffler stability of nonlinear fractional nabla difference systems is defined and the Lyapunov direct method is employed to provide sufficient conditions for Mittag-Leffler stability of, and in some cases the stability of, the zero solution of a system nonlinear fractional nabla difference equations. For this purpose, we obtain several properties of the exponential and one parameter Mittag-Leffler functions of fractional nabla calculus. Two examples are provided to illustrate the applicability of established results.

1. Introduction

Fractional nabla calculus deals with the generalization of nabla sums and nabla differences of arbitrary order. The concept of fractional nabla differences traces back to the works of Gray and Zhang [11], Anastassiou [5] and Atici and Eloe [6]. There has been an increasing interest in this area during the last two decades due to its applicability in various fields of science and engineering (for example, see [16]). As a result, researchers have developed a fairly strong basic theory of fractional nabla difference equations. For a detailed introduction to fractional nabla calculus, we refer the reader to the recent monograph on this topic [10].

Qualitative properties of dynamical systems assume importance in the absence of closed form solutions. One such property, which has wide applications, is the stability of solutions. The Lyapunov method is an efficient technique to analyze the stability of solutions of dynamical systems without explicitly solving the corresponding system of dynamical equations. Recently, Li, Chen and Podlubny [15] defined Mittag-Leffler stability of solutions of fractional differential equations and produced the first application of the Lyapunov direct method to fractional differential equations. More recently, Wyrwas and Mozyrska [18] defined Mittag-Leffler stability of solutions of systems of nonlinear fractional

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delta difference equations and applied the Lyapunov direct method in the discrete forward case. Motivated by these two works, in this article, we shall study Mittag-Leffler stability of solutions of systems of nonlinear fractional nabla difference equations and apply the Lyapunov direct method in the discrete backward case.

The paper is organized as follows. Section 2 contains preliminaries on fractional nabla calculus. Moreover, for easy reference, we summarize some properties of exponential and one parameter Mittag-Leffler functions in Theorem 2.4, and in Theorem 2.5, we summarize properties of the N-transform, which is the Laplace transform on the time scale $\mathbb{Z}$ [9]. In Section 3, we define Mittag-Leffler stability of solutions of systems of fractional nabla difference equations and establish sufficient conditions for Mittag-Leffler stability of the zero solution of fractional nabla difference systems. The presentation in Section 3 is modeled after [15] and [18]. In Section 4, we provide two illustrative examples to demonstrate the proposed stability notion.

2. Preliminaries

Throughout, we shall employ the following notations, definitions and known results of fractional nabla calculus [10]. Denote the set of all integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. Define $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1 (Rising Factorial Function)**.** For any $t, \alpha \in \mathbb{R}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0,$$

where $\Gamma(t)$ denotes the special gamma function. We use the convention that if $t \in \{\ldots, -2, -1, 0\}$ and $(t+\alpha) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then $t^{\overline{\alpha}} = 0$.

Lemma 2.1. *Assume* $\Re(z) > 0$. *Then*

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

Definition 2.2 ([10])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$. The α^{th}-order nabla sum of u is given by

$$\big(\nabla_a^{-\alpha} u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

Definition 2.3 ([10])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$.

(1) (Nabla Difference) The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

(2) (R-L Fractional Nabla Difference) The Riemann-Liouville α^{th}-order nabla difference of u is given by

$$\big(\nabla_a^\alpha u\big)(t) = \Big(\nabla^N\big(\nabla_a^{-(N-\alpha)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

(3) (Caputo Fractional Nabla Difference) The Caputo type α^{th}-order nabla difference of u is given by

$$\big(\nabla_{a*}^\alpha u\big)(t) = \Big(\nabla_a^{-(N-\alpha)}\big(\nabla^N u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

The following theorem provides an equivalent definition of the Riemann-Liouville type fractional nabla difference; it is obtained in [4] and is stated and proved as Theorem 3.62 in [10].

Theorem 2.1. *Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$. Then*

$$\big(\nabla_a^\alpha u\big)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{-\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

The following relation, proved in [1], holds between Riemann-Liouville and Caputo type fractional nabla differences.

Theorem 2.2. *For any $\alpha > 0$, using the convention that division at a pole yields zero, we have*

$$\big(\nabla_{a*}^\alpha u\big)(t) = \big(\nabla_a^\alpha u\big)(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \big(\nabla^k u\big)(a), \quad t \in \mathbb{N}_{a+N}.$$

In particular, for $0 < \alpha < 1$, we have

$$\big(\nabla_{a*}^\alpha u\big)(t) = \big(\nabla_a^\alpha u\big)(t) - \frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} u(a), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.2. *Let $0 < \alpha < 1$ and $u : \mathbb{N}_a \to \mathbb{R}$ with $u(a) \geq 0$. Then,*

$$\big(\nabla_{a*}^\alpha u\big)(t) \leq \big(\nabla_{a-1}^\alpha u\big)(t) \leq \big(\nabla_a^\alpha u\big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Proof. First, we prove

$$\big(\nabla_{a-1}^\alpha u\big)(t) \leq \big(\nabla_a^\alpha u\big)(t).$$

We know that

$$(t-a+1)^{\overline{-\alpha-1}} = \frac{\Gamma(t-a-\alpha)}{\Gamma(t-a+1)} > 0, \quad t \in \mathbb{N}_{a+1}.$$

Using Theorem 2.1, we obtain

$$\big(\nabla_{a-1}^{\alpha}u\big)(t)-\big(\nabla_{a}^{\alpha}u\big)(t)=-\alpha\frac{(t-a+1)^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)}u(a)\le 0,\quad t\in\mathbb{N}_{a+1}.$$

Now, we prove

$$\big(\nabla_{a*}^{\alpha}u\big)(t)\le\big(\nabla_{a-1}^{\alpha}u\big)(t).$$

We know that

$$(t-a+1)^{\overline{-\alpha}}=\frac{\Gamma(t-a+1-\alpha)}{\Gamma(t-a+1)}>0,\quad t\in\mathbb{N}_{a+1}.$$

Using Theorems 2.1 and 2.2, we have

$$\begin{aligned}\big(\nabla_{a*}^{\alpha}u\big)(t)&=\big(\nabla_{a}^{\alpha}u\big)(t)-\frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}u(a)\\&=\big(\nabla_{a-1}^{\alpha}u\big)(t)+\alpha\frac{(t-a+1)^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)}u(a)-\frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}u(a)\\&=\big(\nabla_{a-1}^{\alpha}u\big)(t)-\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}u(a)\\&\le\big(\nabla_{a-1}^{\alpha}u\big)(t),\end{aligned}$$

and the proof is complete. □

For completeness, we state a power rule that has been proved in [2].

Theorem 2.3 (Power Rule)**.** *Let $\alpha\in\mathbb{R}^+$ and $\mu\in\mathbb{R}$. Assume that the following factorial functions are well defined. Then,*

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}},\quad t\in\mathbb{N}_a.$$

Definition 2.4 ([7, 17])**.** For $t\in\mathbb{N}_a$, $\alpha>0$, and $|\lambda|<1$, define the one parameter Mittag-Leffler function of fractional nabla calculus by

$$F_\alpha(\lambda,(t-a)^{\overline{\alpha}})=\sum_{k=0}^{\infty}\lambda^k\frac{(t-a)^{\overline{\alpha k}}}{\Gamma(\alpha k+1)}.$$

Definition 2.5 ([3])**.** For $t\in\mathbb{N}_a$, $\alpha>0$, and $|\lambda|<1$, define the exponential function of fractional nabla calculus by

$$\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}})=(1-\lambda)\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+1)^{\overline{\alpha(k+1)-1}}}{\Gamma(\alpha(k+1))}.$$

Note that the ratio test implies that each of $F_\alpha(\lambda,(t-a)^{\overline{\alpha}})$ and $\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}})$ converge absolutely for $t\in\mathbb{N}_a$, $\alpha>0$, and $|\lambda|<1$.

We observe the following properties of exponential and one parameter Mittag-Leffler functions.

Theorem 2.4. *Let $0<\alpha<1$ and $|\lambda|<1$.*

(1) $F_\alpha(\lambda, 0) = \hat{e}_{\alpha,\alpha}(\lambda, 0) = 1$.

(2) $\nabla F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}}) = \frac{\lambda}{(1-\lambda)}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}})$, $t \in \mathbb{N}_{a+1}$.

(3) $\nabla_{a-1}^{-(1-\alpha)}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) = (1-\lambda)F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}})$, $t \in \mathbb{N}_a$.

(4) $\nabla_{a-1}^{\alpha}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) = \lambda\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}})$, $t \in \mathbb{N}_{a+1}$.

(5) $\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) \geq 0$, $t \in \mathbb{N}_a$.

(6) $\nabla_{a*}^{\alpha} F_\alpha(\lambda, (t-a)^{\overline{\alpha}}) = \lambda F_\alpha(\lambda, (t-a)^{\overline{\alpha}})$, $t \in \mathbb{N}_{a+1}$.

(7) $F_\alpha(\lambda, (t-a)^{\overline{\alpha}}) \geq 0$, $t \in \mathbb{N}_a$.

(8) $F_\alpha(0, (t-a)^{\overline{\alpha}}) = 1$, $t \in \mathbb{N}_a$.

(9) $\hat{e}_{\alpha,\alpha}(0, (t-a)^{\overline{\alpha}}) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$, $t \in \mathbb{N}_a$.

(10) *If* $0 \leq \lambda < 1$, *then* $\frac{\lambda}{(1-\lambda)}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) \leq F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}})$, $t \in \mathbb{N}_a$.

(11) *If* $0 \leq \lambda < 1$, *then* $F_\alpha(\lambda, (t-a)^{\overline{\alpha}})$ *and* $F_\alpha(-\lambda, (t-a)^{\overline{\alpha}})$ *are monotonically increasing and decreasing functions on* $\mathbb{N}_a$, *respectively.*

(12) *If* $0 < \lambda < 1$, *then* $\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) \to \infty$ *as* $t \to \infty$.

(13) *If* $0 \leq \lambda < 1$, *then* $\hat{e}_{\alpha,\alpha}(-\lambda, (t-a)^{\overline{\alpha}}) \to 0$ *as* $t \to \infty$.

(14) *If* $0 < \lambda < 1$, *then* $F_\alpha(\lambda, (t-a)^{\overline{\alpha}}) \to \infty$ *as* $t \to \infty$.

(15) *If* $0 < \lambda < 1$, *then* $F_\alpha(-\lambda, (t-a)^{\overline{\alpha}}) \to 0$ *as* $t \to \infty$.

Proof. The proof of Property 1 is clear.

To prove Property 2, consider

$$\begin{aligned}
\nabla F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}}) &= \nabla\Big[1 + \sum_{k=1}^{\infty}\lambda^k \frac{(t-a+1)^{\overline{\alpha k}}}{\Gamma(\alpha k+1)}\Big] \\
&= \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+1)}\nabla(t-a+1)^{\overline{\alpha k}} \\
&= \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k)}(t-a+1)^{\overline{\alpha k-1}} \\
&= \lambda\sum_{k=0}^{\infty}\lambda^k \frac{(t-a+1)^{\overline{\alpha(k+1)-1}}}{\Gamma(\alpha(k+1))} \\
&= \frac{\lambda}{(1-\lambda)}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}).
\end{aligned}$$

Refer to [3] for a proof of Property 3.

To prove Property 4, consider

$$\begin{aligned}
&\nabla_{a-1}^{\alpha}\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) \\
&= (1-\lambda)\nabla_{a-1}^{\alpha}\Big[\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \sum_{k=1}^{\infty}\lambda^k \frac{(t-a+1)^{\overline{\alpha(k+1)-1}}}{\Gamma(\alpha(k+1))}\Big] \\
&= (1-\lambda)\sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha(k+1))}\nabla_{a-1}^{\alpha}(t-a+1)^{\overline{\alpha(k+1)-1}}
\end{aligned}$$

$$= (1-\lambda)\sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k)}(t-a+1)^{\overline{\alpha k-1}}$$
$$= \lambda(1-\lambda)\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+1)^{\overline{\alpha(k+1)-1}}}{\Gamma(\alpha(k+1))}$$
$$= \lambda\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}}).$$

For an alternate proof of Property 4, refer to [7].

Now, we prove Property 5. Let $u:\mathbb{N}_a\to\mathbb{R}$. We know that $\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}})$ is the unique solution of

$$\begin{aligned}\big(\nabla^{\alpha}_{a-1}u\big)(t) &= \lambda u(t), \quad t\in\mathbb{N}_{a+1},\\ u(a) &= 1.\end{aligned} \tag{2.1}$$

Expanding the left hand side of (2.1) using Theorem 2.1 and rearranging the terms, we get

$$u(t) = \frac{\alpha}{(1-\lambda)\Gamma(1-\alpha)}\sum_{s=a}^{t-1}(t-s+1)^{\overline{-\alpha-1}}u(s), \quad t\in\mathbb{N}_{a+1}.$$

Since

$$(t-s+1)^{\overline{-\alpha-1}} = \frac{\Gamma(t-s-\alpha)}{\Gamma(t-s+1)} \geq 0$$

for $a\leq s\leq (t-1)$, it successively follows that

$$\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}}) = u(t)\geq 0, \quad t\in\mathbb{N}_a.$$

To prove Property 6, we use Properties 2 and 3. Consider

$$\begin{aligned}\nabla^{\alpha}_{a*}F_\alpha(\lambda,(t-a)^{\overline{\alpha}}) &= \nabla_a^{-(1-\alpha)}\Big[\nabla F_\alpha(\lambda,(t-a)^{\overline{\alpha}})\Big]\\ &= \frac{\lambda}{(1-\lambda)}\nabla_a^{-(1-\alpha)}\hat{e}_{\alpha,\alpha}(\lambda,(t-a-1)^{\overline{\alpha}})\\ &= \lambda F_\alpha(\lambda,(t-a)^{\overline{\alpha}}).\end{aligned}$$

For an alternate proof of Property 6, refer to [17].

To prove Property 7, we use Properties 3 and 5. Replacing a by $(a+1)$ in Property 3 and rearranging the terms, we get

$$F_\alpha(\lambda,(t-a)^{\overline{\alpha}}) = \frac{1}{(1-\lambda)\Gamma(1-\alpha)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{-\alpha}}\hat{e}_{\alpha,\alpha}(\lambda,(s-a-1)^{\overline{\alpha}}), \ \ t\in\mathbb{N}_{a+1}.$$

We know that

$$(t-s+1)^{\overline{-\alpha}} = \frac{\Gamma(t-s+1-\alpha)}{\Gamma(t-s+1)} \geq 0$$

for $(a+1)\leq s\leq t$ and $\hat{e}_{\alpha,\alpha}(\lambda,(t-a-1)^{\overline{\alpha}})\geq 0$ for $t\in\mathbb{N}_{a+1}$, and so, Property 7 is verified.

The proofs of Properties 8 and 9 are clear. To prove Property 10, we use Properties 2 and 5. Applying ∇^{-1}-operator on both sides of Property 2 and rearranging the terms, we get

$$\begin{aligned} F_\alpha(\lambda, (t-a+1)^{\overline{\alpha}}) &- \frac{\lambda}{(1-\lambda)} \hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) \\ &= \frac{1}{(1-\lambda)} + \frac{\lambda}{(1-\lambda)} \sum_{s=a+1}^{t-1} \hat{e}_{\alpha,\alpha}(\lambda, (s-a)^{\overline{\alpha}}) \geq 0 \end{aligned}$$

for $t \in \mathbb{N}_a$.

The proof of Property 11 follows immediately from Properties 2 and 5. Recently, Jia et al. [12] have established the following results on asymptotic behavior of (2.1) with $u(a) > 0$.

(i) Assume that $0 < \lambda < 1$. The solutions of (2.1) satisfy

$$\lim_{t\to\infty} u(t) = \infty.$$

(ii) Assume that $\lambda \leq 0$. The solutions of (2.1) satisfy

$$\lim_{t\to\infty} u(t) = 0.$$

Since $\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}})$ is the unique solution of (2.1) with $u(a) = 1 > 0$, Properties 12 and 13 now follow. The proof of Property 14 now follows from Properties 10 and 12.

To prove Property 15, it is clear from Properties 7 and 11 that

$$\lim_{t\to\infty} F_\alpha(-\lambda, (t-a)^{\overline{\alpha}}) = A \geq 0.$$

Now, we prove $A = 0$. If possible, suppose $A > 0$. Since $u(t) = F_\alpha(-\lambda, (t-a)^{\overline{\alpha}})$ is the unique solution of

$$\begin{aligned} \big(\nabla^\alpha_{a*} u\big)(t) &= -\lambda u(t), \quad t \in \mathbb{N}_{a+1}, \\ u(a) &= 1, \end{aligned} \tag{2.2}$$

apply the $\nabla_a^{-\alpha}$-operator on both sides of (2.2) to obtain

$$u(t) = 1 - \frac{\lambda}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a. \tag{2.3}$$

Solve algebraically for $u(t)$ and we have

$$u(t) = \frac{1}{(1+\lambda)} - \frac{\lambda}{(1+\lambda)\Gamma(\alpha)} \sum_{s=a+1}^{t-1} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+1},$$

which implies

$$u(t) + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)} \sum_{s=a+1}^{t-1} (t-s+1)^{\overline{\alpha-1}} u(s) = \frac{1}{(1+\lambda)}, \quad t \in \mathbb{N}_{a+1}. \tag{2.4}$$

By Property 7, $u(t) \geq 0$ for all $t \in \mathbb{N}_a$ and

$$(t-s+1)^{\overline{\alpha-1}} = \frac{\Gamma(t-s+\alpha)}{\Gamma(t-s+1)} \geq 0$$

for $(a+1) \leq s \leq t$. Let $t-L \geq a+1$, $L \in \mathbb{N}_0$ in (2.4), and we have

$$u(t) + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)} \sum_{s=t-L}^{t-1} (t-s+1)^{\overline{\alpha-1}} u(s) \leq \frac{1}{(1+\lambda)}, \quad t \in \mathbb{N}_{a+1}. \tag{2.5}$$

Fix $L \in \mathbb{N}_2$ such that

$$\frac{A}{2}\Big(1 + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)}\Big[\Gamma(\alpha+1) + \frac{1}{2} + \cdots + \frac{1}{L}\Big]\Big) > \frac{1}{(1+\lambda)}. \tag{2.6}$$

Let $t_0 \in \mathbb{N}_a$ be such that if $t \in \mathbb{N}_a$, $t-L \geq t_0$, then $u(t) > \frac{A}{2}$. We observe that, for $k \geq 2$,

$$(t-(t-k)+1)^{\overline{\alpha-1}} = \frac{\Gamma(k+\alpha)}{\Gamma(k+1)} \geq \frac{1}{k}.$$

Then, from (2.5), we have

$$\begin{aligned} &u(t) + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)}\Big[\Gamma(\alpha+1)u(t-1) + \frac{1}{2}u(t-2) + \cdots + \frac{1}{L}u(t-L)\Big] \\ &\geq \frac{A}{2}\Big(1 + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)}\Big[\Gamma(\alpha+1) + \frac{1}{2} + \cdots + \frac{1}{L}\Big]\Big) > \frac{1}{(1+\lambda)}, \end{aligned}$$

which contradicts (2.5). Thus, $A = 0$ and the proof is complete. □

Related results on asymptotic behavior of solutions of discrete fractional difference equations are found in [8] and [12].

For the sake of clarity, we introduce the N-transform and summarize properties that will be employed in Section 3.

Definition 2.6 ([7])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$. The N-transform of u is defined by

$$N_a\big[u(t)\big] = \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} = U(z)$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 2.7 ([7])**.** Let u, $v : \mathbb{N}_a \to \mathbb{R}$. The convolution of u and v is defined by

$$\big(u \underset{a}{*} v\big)(t) = \sum_{s=a}^{t} u(t+a-\rho(s))v(s).$$

We observe the following properties of N-transform, which have been obtained in [7].

Theorem 2.5. *Assume the following N-transforms exist. Then*

(1) $N_a\Big[\big(u \underset{a}{*} v\big)(t)\Big] = N_1\big[u(t+a)\big]N_a\big[v(t)\big]$, $t \in \mathbb{N}_a$.

(2) $N_a\Big[(t-a+1)^{\overline{\alpha}}\Big] = (1-z)^{a-1}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}},\ \alpha \in \mathbb{R}\setminus\{\dots,-3,-2,-1\},\ t \in \mathbb{N}_a$.

(3) $N_a\Big[\big(\nabla_{a-1}^{-\alpha}u\big)(t)\Big] = z^{-\alpha}N_a\big[u(t)\big],\ \alpha > 0,\ t \in \mathbb{N}_a$.

(4) $N_a\Big[\big(\nabla_{a-1}^{\alpha}u\big)(t)\Big] = z^{\alpha}N_a\big[u(t)\big],\ 0 < \alpha < 1,\ t \in \mathbb{N}_{a+1}$.

(5) $N_{a+1}\Big[\big(\nabla_{a-1}^{\alpha}u\big)(t)\Big] = z^{\alpha}N_a\big[u(t)\big] - (1-z)^{a-1}u(a),\ 0 < \alpha < 1,\ t \in \mathbb{N}_{a+1}$.

(6) $N_{a+1}\Big[\big(\nabla_{a*}^{\alpha}u\big)(t)\Big] = z^{\alpha}N_{a+1}\big[u(t)\big] - (1-z)^{a}z^{\alpha-1}u(a),\ 0 < \alpha < 1,\ t \in \mathbb{N}_{a+1}$.

(7) $N_a\Big[F_\alpha(\lambda,(t-a+1)^{\overline{\alpha}})\Big] = (1-z)^{a-1}\frac{z^{\alpha-1}}{(z^{\alpha}-\lambda)},\ t \in \mathbb{N}_a$.

(8) $N_a\Big[\hat{e}_{\alpha,\alpha}(\lambda,(t-a)^{\overline{\alpha}})\Big] = (1-z)^{a-1}\frac{(1-\lambda)}{(z^{\alpha}-\lambda)},\ t \in \mathbb{N}_a$.

3. Mittag-Leffler stability

In this section, we define Mittag-Leffler stability of the zero solution of the following systems of fractional nabla difference equations and employ a Lyapunov's direct method that has been introduced in [15] in the continuous case and [18] in the discrete delta case. We consider

$$\big(\nabla_{a*}^{\alpha}\mathbf{u}\big)(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad t \in \mathbb{N}_{a+1}, \tag{3.1}$$

$$\big(\nabla_{a}^{\alpha}\mathbf{u}\big)(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad t \in \mathbb{N}_{a+1}, \tag{3.2}$$

where $0 < \alpha < 1$, $\mathbf{u} : \mathbb{N}_a \to \mathbb{R}^n$ and $\mathbf{f} : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$, for all $t \in \mathbb{N}_a$, so that each of (3.1) and (3.2) admit the trivial solution.

Let $\mathbf{u}(t;a,\mathbf{u}_0)$ denote a solution of (3.1) (or (3.2)) satisfying the initial condition $\mathbf{u}(a;a,\mathbf{u}_0) = \mathbf{u}_0$. We shall assume throughout this section that a solution, $\mathbf{u}(t;a,\mathbf{u}_0)$ exists on $\mathbb{N}_a$. We refer the reader to [13, 14] for results on the existence and uniqueness of solutions of initial value problems (3.1) and (3.2).

Let $||\cdot||$ denote any norm equivalent to the Euclidean norm on $\mathbb{R}^n$.

Definition 3.1. Assume $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$ and assume all solutions $\mathbf{u}(t;a,\mathbf{u}_0)$ of (3.1) (or (3.2)) exist on $\mathbb{N}_a$. The trivial solution of (3.1) (or (3.2)) is said to be

(1) stable, if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon,a) > 0$ such that

$$\|\mathbf{u}_0\| < \delta \text{ implies } \|\mathbf{u}(t)\| < \epsilon, \quad t \in \mathbb{N}_a,$$

for any solution $\mathbf{u}(t) = \mathbf{u}(t;a,\mathbf{u}_0)$ of (3.1) (or (3.2)).

(2) asymptotically stable, if it is stable and for all $t \in \mathbb{N}_a$ there exists $\eta = \eta(a) > 0$ such that

$$\|\mathbf{u}_0\| < \eta \text{ implies } \lim_{t\to\infty}\mathbf{u}(t) = \mathbf{0}.$$

Definition 3.2. Let $M > 0$ be an arbitrary constant. Define

$$D = \{(t,\mathbf{u}) : \|\mathbf{u}(t)\| \le M \text{ for all } t \in \mathbb{N}_a\} \subseteq \mathbb{N}_a \times \mathbb{R}^n.$$

A mapping $\mathbf{g} : D \to \mathbb{R}^n$ is said to be locally Lipschitz with respect to the second variable with a Lipschitz constant L, if the inequality

$$\|\mathbf{g}(t, \mathbf{u}) - \mathbf{g}(t, \mathbf{v})| \leq L\|\mathbf{u} - \mathbf{v}\|$$

holds whenever $(t,\ \mathbf{u})$, $(t,\ \mathbf{v}) \in D$. $\mathbf{g}$ is said to be globally Lipschitz if $D = \mathbb{N}_a \times \mathbb{R}^n$.

First, we observe the relation between the Lipschitz condition and the system (3.1).

Theorem 3.1. *If* $\mathbf{f}$ *is globally Lipschitz with respect to the second variable with Lipschitz constant* $0 \leq L < 1$*, then any solution* $\mathbf{u}(t) = \mathbf{u}(t; a, \mathbf{u}_0)$ *of* (3.1) *satisfies*

$$\|\mathbf{u}(t)\| \leq \|\mathbf{u}_0\| F_\alpha(L, (t-a)^{\overline{\alpha}}).$$

Proof. If $\mathbf{u}(t) = \mathbf{u}(t; a, \mathbf{u}_0)$ is a solution of (3.1), then

$$\mathbf{u}(t) = \mathbf{u}_0 + \nabla_a^{-\alpha} \mathbf{f}(t, \mathbf{u}(t)), \quad t \in \mathbb{N}_a.$$

Consider

$$\begin{aligned}
\|\mathbf{u}(t)\| &\leq \|\mathbf{u}_0\| + \big\|\nabla_a^{-\alpha} \mathbf{f}(t, \mathbf{u}(t))\big\| \\
&\leq \|\mathbf{u}_0\| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \big\|\mathbf{f}(s, \mathbf{u}(s))\big\| \\
&= \|\mathbf{u}_0\| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \big\|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{0}) + \mathbf{f}(s, \mathbf{0})\big\| \\
&\leq \|\mathbf{u}_0\| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \big\|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{0})\big\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \big\|\mathbf{f}(s, \mathbf{0})\big\| \\
&\leq \|\mathbf{u}_0\| + L \nabla_a^{-\alpha} \|\mathbf{u}(t)\|.
\end{aligned}$$

Then there exists a nonnegative function $x : \mathbb{N}_a \to \mathbb{R}^+ \cup \{0\}$ satisfying

$$\|\mathbf{u}(t)\| + x(t) = \|\mathbf{u}_0\| + L \nabla_a^{-\alpha} \|\mathbf{u}(t)\|, \quad t \in \mathbb{N}_a. \tag{3.3}$$

Apply the N_{a+1}-transform to (3.3) to obtain

$$U(z) = (1-z)^a \frac{z^{\alpha-1}}{\big(z^\alpha - L\big)} \|\mathbf{u}_0\| - \frac{z^\alpha}{\big(z^\alpha - L\big)} X(z), \tag{3.4}$$

where $N_{a+1}\big[\|\mathbf{u}(t)\|\big] = U(z)$ and $N_{a+1}\big[x(t)\big] = X(z)$. Apply the inverse N_{a+1}-transform to (3.4) to obtain

$$\|\mathbf{u}(t)\| = \|\mathbf{u}_0\| F_\alpha(L, (t-a)^{\overline{\alpha}}) - x(t) - \frac{L}{(1-L)} \Big[\hat{e}_{\alpha,\alpha}(L, (t-a-2)^{\overline{\alpha}}) \underset{a+1}{*} x(t)\Big].$$

Since $x(t)$ is nonnegative and $\hat{e}_{\alpha,\alpha}(L,(t-a-2)^{\overline{\alpha}}) \underset{a+1}{*} x(t)$ is nonnegative,

$$\|\mathbf{u}(t)\| \leq \|\mathbf{u}_0\| F_\alpha(L,(t-a)^{\overline{\alpha}}),$$

and the proof is complete. □

Now, we define the stability of solutions of (3.1) and (3.2) in the sense of the one parameter Mittag-Leffler function.

Definition 3.3. Assume $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$ and assume all solutions $\mathbf{u}(t;a,\mathbf{u}_0)$ of (3.1) (or (3.2)) exist on $\mathbb{N}_a$. The trivial solution of (3.1) (or (3.2)) is said to be Mittag-Leffler stable if there exist $0 \leq \lambda < 1$, $b > 0$, and $m : \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ with $m(\mathbf{0}) = 0$ and m is locally Lipschitz on $\mathbb{D}$ with Lipschitz constant m_0 such that

$$\|\mathbf{u}(t)\| \leq \left[m(\mathbf{u}_0) F_\alpha(-\lambda,(t-a)^{\overline{\alpha}})\right]^b \tag{3.5}$$

for any solution $\mathbf{u}(t) = \mathbf{u}(t;a,\mathbf{u}_0)$ of (3.1) (or (3.2)).

We extend the Lyapunov direct method to the case of systems of fractional nabla difference equations to obtain sufficient conditions for the Mittag-Leffler stability of solutions of (3.1) (or (3.2)). Without loss of generality, the initial time can be taken as $a = 0$.

Theorem 3.2. *Assume* $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$ *and assume all solutions* $\mathbf{u}(t;0,\mathbf{u}_0)$ *of* (3.1) *exist on* $\mathbb{N}_0$. *Let* $\mathbb{D} \subset \mathbb{R}^n$ *be a domain containing the origin. Let* $V : \mathbb{N}_0 \times \mathbb{D} \to \mathbb{R}$ *be a locally Lipschitz function with respect to the second variable such that*

$$\alpha_1 \|\mathbf{u}\|^p \leq V(t,\mathbf{u}(t)) \leq \alpha_2 \|\mathbf{u}\|^{pq}, \tag{3.6}$$

$$\nabla_{0*}^\alpha V(t,\mathbf{u}(t)) \leq -\alpha_3 \|\mathbf{u}\|^{pq}, \tag{3.7}$$

for any solution $\mathbf{u}(t) = \mathbf{u}(t;0,\mathbf{u}_0)$ *of* (3.1), *where* α_i, $i = 1,2,3$, p *and* q *are arbitrary positive constants with* $\alpha_3 < \alpha_2$. *Then the trivial solution of* (3.1) *is Mittag-Leffler stable.*

Proof. From (3.6) and (3.7) we have

$$\nabla_{0*}^\alpha V(t,\mathbf{u}(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t,\mathbf{u}(t)).$$

Then there exists a nonnegative function $w : \mathbb{N}_0 \to \mathbb{R}^+ \cup \{0\}$ satisfying

$$\nabla_{0*}^\alpha V(t,\mathbf{u}(t)) + w(t) = -\frac{\alpha_3}{\alpha_2} V(t,\mathbf{u}(t)).$$

Apply the N_1-transform to both sides to obtain

$$\bar{V}(z) = \frac{z^{\alpha-1}}{z^\alpha + \frac{\alpha_3}{\alpha_2}} V(0,\mathbf{u}_0) - \frac{1}{z^\alpha + \frac{\alpha_3}{\alpha_2}} W(z), \tag{3.8}$$

where $N_1\left[V(t,\mathbf{u}(t))\right] = \bar{V}(z)$, $N_1\left[w(t)\right] = W(z)$ and $V(0,\mathbf{u}_0)$ is a nonnegative constant. If $\mathbf{u}_0 = \mathbf{0}$, i.e., $V(0,\mathbf{u}_0) = 0$, the solution of (3.1) is $\mathbf{u} = \mathbf{0}$. If $\mathbf{u}_0 \neq \mathbf{0}$, then $V(0,\mathbf{u}_0) > 0$. Apply the inverse N_1-transform to both sides of (3.8) to obtain

$$V(t,\mathbf{u}(t)) = V(0,\mathbf{u}_0)F_\alpha\Big(-\frac{\alpha_3}{\alpha_2}, t^{\overline{\alpha}}\Big) - \frac{1}{(1+\frac{\alpha_3}{\alpha_2})}\Big[\hat{e}_{\alpha,\alpha}\Big(-\frac{\alpha_3}{\alpha_2},(t-2)^{\overline{\alpha}}\Big) \underset{1}{*} w(t)\Big].$$

Since $w(t)$ is nonnegative, apply Theorem 2.4, Property 5 and we have

$$V(t,\mathbf{u}(t)) \le V(0,\mathbf{u}_0)F_\alpha\Big(-\frac{\alpha_3}{\alpha_2}, t^{\overline{\alpha}}\Big). \tag{3.9}$$

Substituting (3.9) into (3.6), we obtain

$$\|\mathbf{u}(t)\| \le \Big[\frac{V(0,\mathbf{u}_0)}{\alpha_1}F_\alpha\Big(-\frac{\alpha_3}{\alpha_2}, t^{\overline{\alpha}}\Big)\Big]^{\frac{1}{p}}, \tag{3.10}$$

where $\frac{V(0,\mathbf{u}_0)}{\alpha_1} > 0$. Let $m(\mathbf{u}_0) = \frac{V(0,\mathbf{u}_0)}{\alpha_1}$. Then, we have

$$\|\mathbf{u}(t)\| \le \Big[m(\mathbf{u}_0)F_\alpha\Big(-\frac{\alpha_3}{\alpha_2}, t^{\overline{\alpha}}\Big)\Big]^{\frac{1}{p}}, \tag{3.11}$$

where $m(\mathbf{u}_0) = 0$ holds if and only if $\mathbf{u}_0 = \mathbf{0}$. Since $V(t,\mathbf{u})$ is Lipschitz with respect to $\mathbf{u}$ and $V(0,\mathbf{u}_0) = 0$ if and only if $\mathbf{u}_0 = \mathbf{0}$, it follows that $m(\mathbf{u}_0)$ is also Lipschitz with respect to $\mathbf{u}_0$ and $m(\mathbf{0}) = 0$. Thus, the trivial solution of (3.1) is Mittag-Leffler stable. □

Theorem 3.3. *Assume $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$ and assume all solutions $\mathbf{u}(t;0,\mathbf{u}_0)$ of* (3.2) *exist on $\mathbb{N}_0$. Let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : \mathbb{N}_0 \times \mathbb{D} \to \mathbb{R}$ be a locally Lipschitz function with respect to the second variable such that*

$$\alpha_1\|\mathbf{u}\|^p \le V(t,\mathbf{u}(t)) \le \alpha_2\|\mathbf{u}\|^{pq}, \tag{3.12}$$

$$\nabla_0^\alpha V(t,\mathbf{u}(t)) \le -\alpha_3\|\mathbf{u}\|^{pq}, \tag{3.13}$$

for any solution $\mathbf{u}(t) = \mathbf{u}(t;0,\mathbf{u}_0)$ of (3.2)*, where α_i, $i = 1,2,3$, p and q are arbitrary positive constants with $\alpha_3 < \alpha_2$. Then the trivial solution of* (3.2) *is Mittag-Leffler stable.*

Proof. From Lemma 2.3, we have

$$\nabla_{0*}^\alpha V(t,\mathbf{u}(t)) \le \nabla_0^\alpha V(t,\mathbf{u}(t)).$$

Combining this inequality with (3.13), we have

$$\nabla_{0*}^\alpha V(t,\mathbf{u}(t)) \le -\frac{\alpha_3}{\alpha_2}V(t,\mathbf{u}(t)). \tag{3.14}$$

So, replace (3.7) by (3.14) and apply Theorem 3.2 to obtain the Mittag-Leffler stability of the trivial solution of (3.2). □

Theorem 3.4. *Assume $\mathbf{f}(t,\mathbf{0}) = \mathbf{0}$ and assume all solutions $\mathbf{u}(t;0,\mathbf{u}_0)$ of* (3.2) *exist on $\mathbb{N}_0$. If $\mathbf{f}$ satisfies a Lipschitz condition with respect to the second variable with Lipschitz constant $L > 0$ and there exists a Lyapunov candidate $V(t,\mathbf{u})$ satisfying*

$$\alpha_1\|\mathbf{u}\|^r \le V(t,\mathbf{u}(t)) \le \alpha_2\|\mathbf{u}\|, \tag{3.15}$$

$$\nabla V(t, \mathbf{u}(t)) \leq -\alpha_3 \|\mathbf{u}\|, \tag{3.16}$$

for any solution $\mathbf{u}(t) = \mathbf{u}(t; 0, \mathbf{u}_0)$ *of* (3.2)*, where* α_i*,* $i = 1, 2, 3$ *and* r *are arbitrary positive constants with* $\alpha_3 < L\alpha_2$*. Then*

$$\|\mathbf{u}(t)\| \leq \Big[\frac{V(0, \mathbf{u}_0)}{\alpha_1} F_{1-\alpha}\Big(-\frac{\alpha_3}{L\alpha_2}, t^{\overline{1-\alpha}}\Big)\Big]^{\frac{1}{r}}. \tag{3.17}$$

Proof. Using Definition 2.3, (3.15) and (3.16), we have

$$\begin{aligned}
\nabla_{0*}^{(1-\alpha)} V(t, \mathbf{u}(t)) &= \nabla_0^{-\alpha}\big[\nabla V(t, \mathbf{u}(t))\big] \leq -\alpha_3 \nabla_0^{-\alpha}\|\mathbf{u}(t)\| \\
&= \frac{-\alpha_3}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-s+1)^{\overline{\alpha-1}} \|\mathbf{u}(s)\| \\
&\leq \frac{-\alpha_3}{L\Gamma(\alpha)} \sum_{s=1}^{t} (t-s+1)^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s))\| \\
&\leq \frac{-\alpha_3}{L} \Big\| \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-s+1)^{\overline{\alpha-1}} \mathbf{f}(s, \mathbf{u}(s)) \Big\| \\
&= \frac{-\alpha_3}{L} \Big\| \nabla_0^{-\alpha} \mathbf{f}(t, \mathbf{u}(t)) \Big\| = \frac{-\alpha_3}{L} \Big\| \nabla_0^{-\alpha} \nabla_0^{\alpha} \mathbf{u}(t) \Big\| = \frac{-\alpha_3}{L} \|\mathbf{u}(t)\|;
\end{aligned}$$

in particular,

$$\nabla_{0*}^{(1-\alpha)} V(t, \mathbf{u}(t)) \leq \frac{-\alpha_3}{L} \|\mathbf{u}(t)\|. \tag{3.18}$$

Now apply Theorem 3.2 with (3.7) replaced by (3.18) and with $p = r$ and $pq = 1$. Then (3.10) reduces to (3.17). ☐

Remark 3.1. Assume, in addition to the hypotheses of Theorem 3.2 (or Theorem 3.3), that $\frac{\alpha_3}{\alpha_2} < 1$. Then, (3.10) coupled with statement (15) of Theorem 2.4, and Theorem 3.2 (or Theorem 3.3) implies the trivial solution of (3.1) (or (3.2)) is asymptotically stable. For the system (3.2), choose a Lipschitz constant $L > 0$ such that $\frac{\alpha_3}{L\alpha_2} < 1$. Then, Theorem 3.4 implies the asymptotic stability of the trivial solution of (3.2) independent of the size of $\frac{\alpha_3}{\alpha_2}$.

4. Examples

Example 4.1. Consider the fractional nabla difference equation

$$\big(\nabla_0^{\alpha} u\big)(t) = -\lambda u(t), \quad t \in \mathbb{N}_1, \tag{4.1}$$

where $0 < \alpha, \lambda < 1$.

The condition $0 < \lambda < 1$ implies all solutions $u(t; 0, u_0)$ of (4.1) exist on $\mathbb{N}_0$. To see this, rewrite (4.1) as

$$u(t) + \frac{1}{\Gamma(-\alpha)} \sum_{s=1}^{t-1} (t-s+1)^{\overline{-\alpha-1}} u(s) = -\lambda u(t)$$

which can be solved uniquely for $u(t)$ for all $t \in \mathbb{N}_1$. Note that if $u(0) > 0$, then $u(t) > 0$ for all $t \in \mathbb{N}_0$. To see this, if, for the sake of contradiction, there exists $t_1 \in \mathbb{N}_1$ with $u(t) > 0$ on $\mathbb{N}_0^{t_1-1}$ and $u(t_1) \le 0$, then

$$\begin{aligned}\big(\nabla_0^{\alpha} u\big)(t_1) &= \frac{1}{\Gamma(-\alpha)} \sum_{s=1}^{t_1} (t_1 - s + 1)^{\overline{-\alpha-1}} u(s) \\ &= u(t_1) - \frac{\alpha}{\Gamma(1-\alpha)} \sum_{s=1}^{t_1-1} \frac{\Gamma(t_1 - s - \alpha)}{\Gamma(t_1 - s + 1)} u(s) \\ &< 0.\end{aligned}$$

But from (4.1) we have

$$\big(\nabla_0^{\alpha} u\big)(t_1) = -\lambda u(t_1) \ge 0$$

which is a contradiction.

Similarly, $-u(t)$ is also a solution of (4.1), and if $u(0) < 0$, then $u(t) < 0$ for all $t \in \mathbb{N}_0$. Thus, it is the case that if u is a solution of (4.1), then

$$\nabla_0^{\alpha} |u(t)| = -\lambda |u(t)|, \quad t \in \mathbb{N}_1.$$

Choose $V(t,u) = |u|$. Then, V is Lipschitz with Lipschitz constant $L = 1$. Take $\alpha_1 = \alpha_2 = p = q = 1$ and $0 < \alpha_3 \le \lambda < 1$ such that $\alpha_3 < \alpha_2$. Then, V satisfies (3.12) of Theorem 3.3. Now,

$$\nabla_0^{\alpha} V(t, u(t)) = \nabla_0^{\alpha} |u(t)| = -\lambda |u(t)| \le -\alpha_3 |u(t)|$$

implies V satisfies (3.13) of Theorem 3.3. Hence, by Theorem 3.3, the trivial solution of (4.1) is Mittag-Leffler stable.

Example 4.2. Let $m \ge 3$ denote a positive odd integer. Consider the fractional nabla difference equation

$$\big(\nabla_0^{\alpha} u\big)(t) = -u^m(t), \quad t \in \mathbb{N}_1, \tag{4.2}$$

where $0 < \alpha < 1$.

As in Example 1 all solutions $u(t; 0, u_0)$ of (4.2) exist on $\mathbb{N}_0$. To see this, write

$$u(t) + \frac{1}{\Gamma(-\alpha)} \sum_{s=1}^{t-1} (t - s + 1)^{\overline{-\alpha-1}} u(s) = -u^m(t).$$

If $f(u) = u + u^m$, then $f'(u) \ge 1$ and so, (4.2) can be solved uniquely for $u(t)$ for all $t \in \mathbb{N}_1$. Moreover, $u(0) > 0$ implies $u(t) > 0$ for all $t \in \mathbb{N}_0$. To see this, note that

$$\begin{aligned}\big(\nabla_0^{\alpha} u\big)(t_1) &= \frac{1}{\Gamma(-\alpha)} \sum_{s=1}^{t_1} (t_1 - s + 1)^{\overline{-\alpha-1}} u^m(s) \\ &= u^m(t_1) - \frac{\alpha}{\Gamma(1-\alpha)} \sum_{s=1}^{t_1-1} \frac{\Gamma(t_1 - s - \alpha)}{\Gamma(t_1 - s + 1)} u^m(s)\end{aligned}$$

implies a contradiction analogous to the one produced in Example 1.

Since m is odd, $-u(t)$ is also a solution of (4.2) and if $u(0) < 0$, then $u(t) < 0$ for all $t \in \mathbb{N}_0$. Thus, we have

$$\nabla_0^\alpha |u(t)| = -|u(t)|^m, \quad t \in \mathbb{N}_1.$$

Choose $V(t,u) = |u|^m$. Then, V is locally Lipschitz. Take $\alpha_1 = \alpha_2 = q = 1$, $p = m$, and $0 < \alpha_3 < \alpha_2 < 1$. Then, V satisfies (3.12) and (3.13) of Theorem 3.3. Hence, by Theorem 3.3, the trivial solution of (4.2) is Mittag-Leffler stable.

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