Bull. Korean Math. Soc. **56** (2019), No. 4, pp. 961–976
https://doi.org/10.4134/BKMS.b180740
pISSN: 1015-8634 / eISSN: 2234-3016

$\mathcal{F}_S$-MITTAG-LEFFLER MODULES AND GLOBAL DIMENSION RELATIVE TO $\mathcal{F}_S$-MITTAG-LEFFLER MODULES

Mingzhao Chen and Fanggui Wang

Abstract. Let R be any commutative ring and S be any multiplicative closed set. We introduce an S-version of $\mathcal{F}$-Mittag-Leffler modules, called $\mathcal{F}_S$-Mittag-Leffler modules, and define the projective dimension with respect to these modules. We give some characterizations of $\mathcal{F}_S$-Mittag-Leffler modules, investigate the relationships between $\mathcal{F}$-Mittag-Leffler modules and $\mathcal{F}_S$-Mittag-Leffler modules, and use these relations to describe noetherian rings and coherent rings, such as R is noetherian if and only if R_S is noetherian and every $\mathcal{F}_S$-Mittag-Leffler module is $\mathcal{F}$-Mittag-Leffler. Besides, we also investigate the $\mathcal{M}^{\mathcal{F}_S}$-global dimension of R, and prove that R_S is noetherian if and only if its $\mathcal{M}^{\mathcal{F}_S}$-global dimension is zero; R_S is coherent if and only if its $\mathcal{M}^{\mathcal{F}_S}$-global dimension is at most one.

1. Introduction

Throughout this paper, denote by R a commutative ring with identity and by S any multiplicative closed set. In addition, Mod-R and Mod-R_S denote the categories consisting of all R-modules and R_S-modules, respectively.

In recent years, S-versions of some classical notions have attracted the interest of some authors. For example, D. D. Anderson and T. Dumitrescu defined S-finite modules and S-Noetherian rings in [1], and generalized several well-known results including classical Cohen's result and Hilbert basis theorem; S. Bazzoni and L. Positselski defined S-strongly flat modules and S-almost perfect rings in [5], and proved that R is S-almost perfect if and only if every flat module is S-strongly flat; H. Kim, M. O. Kim and J. W. Lim [13] defined S-SM domains, S-factorial domains, and generalized some well-known results to these rings, such as a domain is S-factorial if and only if every prime ω-ideal

Received August 7, 2018; Revised February 25, 2019; Accepted March 8, 2019.
2010 *Mathematics Subject Classification.* 13B30, 13D05, 13E05.
Key words and phrases. S-finitely generated, S-exact, $\mathcal{F}$-Mittag-Leffler, $\mathcal{F}_S$-Mittag-Leffler.
Supported by the National Natural Science Foundation of China (Grant No.11671283).

is S-principal. In [16] and [17], Lim and Oh studied the S-Noetherian properties in terms of amalgamated algebra and composite ring extensions. Also, in [15], the author studied the Nagata ring of S-Noetherian rings. In [4], Baeck et al. introduced the noncommutative version of S-Noetherian rings. In [14], Kim and Lim generalized S-principal ideal domains by using star-operations.

In this article, according to [5] and [13], we also use these two S-versions to generalize the $\mathcal{F}$-Mittag-Leffler. For example, R is S-Noetherian if R_S is noetherian, which is not as same as S-Noetherian rings in [1]; R is S-coherent if R_S is coherent which is also unlike to the definition in [6]; M is S-finitely generated if M_S is finitely generated in Mod-R_S, which is also equivalent to saying that there is a finitely generated submodule N of M such that $M_S = N_S$; a map $\beta : M \to N$ is S-injective if β_S is injective.

Now, we recall some known notions and facts which are necessary for studying $\mathcal{F}$-Mittag-Leffler modules.

Clarke (1976) called an R-module M R-Mittag-Leffler if the natural map $M \otimes R^\Gamma \to M^\Gamma$ is injective for any set Γ, or called $\mathcal{F}$-Mittag-Leffler modules by some authors such as Rothmaler [19] and Herbera [2]. With the investigation of Goodearl (1972), an R-Mittag-Leffler module M is equivalently to say that, for any finitely generated submodule N of M, the conclusion $N \hookrightarrow M$ factors through a finitely presented R-module, and then we can easily see that every module is R-Mittag-Leffler if and only if R is noetherian. Besides, the notion of R-Mittag-Leffler modules was also called finitely pure-projective modules by Azumaya [3]. Rothmaler (1994) showed that Mittag-Leffler modules are as same as $\mathcal{F}$-Mittag-Leffler modules over von-Neumann rings [18, Theorem 6.7]. Simson (1977) showed that every modules is Mittag-Leffler if and only if R is pure-semisimple (or called an artinian principle ideal ring [11, Theorem 4.3]) [21, Theorem 6.3]. Cortés-Izurdiaga (2016) investigated $\mathcal{F}$-Mittag-Leffler modules and their relative dimension, and proved that R is coherent if and only if every ideal is $\mathcal{F}$-Mittag-Leffler. Besides, $\mathcal{F}$-Mittag-Leffler modules were also studied by other authors, such as [12].

In the article, we mainly study the properties of $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler modules, investigate the relative projective dimension and relative global dimension with respect to these modules, and compare these dimensions with the dimension relative to $\mathcal{F}$-Mittag-Leffler modules. Some results about Mittag-Leffler modules or $\mathcal{F}$-Mittag-Leffler modules are developed. Denote by $\mathcal{M}^{\mathcal{F}_S}$, $\mathcal{M}^{\mathcal{F}}$ and $\mathcal{ML}$ the class of all of $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler R-modules, $\mathcal{F}$-Mittag-Leffler R-modules and Mittag-Leffler R-modules respectively.

In Section 2, we introduce some main basic notations, such as S-finitely generated, S-torsion, S-exact and $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler. We note that there is a deference between the notion of S-finitely generated here and S-finite in [1, Definition 1]. Some examples of $\mathcal{F}$-Mittag-Leffler modules are given, and it is shown that every finitely generated S-torsion module is $\mathcal{F}$-Mittag-Leffler if and only if it is finitely presented Proposition 2.6.

In Section 3, we give several characterizations of $\mathcal{F}_S$-Mittag-Leffler modules (Theorem 3.1), and prove some results, such as R_S is noetherian if and only if every R-module is $\mathcal{F}_S$-Mittag-Leffler if and only if every finitely generated R-module is $\mathcal{F}_S$-Mittag-Leffler; R is noetherian if and only if R_S is noetherian and $\mathcal{M}^{\mathcal{F}_S} = \mathcal{M}^{\mathcal{F}}$. Besides, we also consider some cases $\mathcal{F}_S$-Mittag-Leffler modules is closed under direct products.

In Section 4, we introduce the $\mathcal{M}^{\mathcal{F}_S}$-projective dimension and $\mathcal{M}^{\mathcal{F}_S}$-global dimension. We prove that $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_S}} M \leq n$ in Mod-R if and only if $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M_S \leq n$ in Mod-R_S for any R-module M. We also give a sufficient condition for $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$ and some examples to show that, although there is a bad relation between $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_S}} M$ and $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M$ in Mod-R, $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_S}} M \leq n$ if and only if $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M \leq n$ for any R-module M of flat dimension at most n (Example 4.3). In addition, we show that R_S is coherent if and only if $\mathrm{gl}_{\mathcal{M}^{\mathcal{F}}}.\mathrm{dim}(R) \leq 1$ if and only if every finitely generated R-module is $\mathcal{F}_S$-Mittag-Leffler (Corollary 4.9), and $\mathcal{ML} = \mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$ over a von-Neumann ring with S consisting of non-zero divisors.

2. Basic definitions

In this section, we introduce some notions with respect to the multiplicative closed set S, such as S-finitely generated modules and S-exact sequences, and some other definitions we need in this article. In addition, we give some examples of $\mathcal{F}_S$-modules and a property of finitely generated S-torsion modules.

Definition 2.1. Let M and N be R-modules. We say that

(1) M is S-finitely generated (respectively, cyclic, presented) if M_S is finitely generated (respectively, cyclic, presented) as an R_S-module.

(2) $\tau_S(M) = \{x \in M \mid sx = 0 \text{ for some } s \in S\}$ is called the total S-torsion submodule of M. If $\tau_S(M) = 0$, M is called S-torsion-free; If $\tau_S(M) = M$, M is called S-torsion.

(3) An ideal I of R is S-regular if it contains some elements of S. Thus, any S-regular ideal is S-cyclic.

(4) An R-homomorphism $f : M \to N$ is S-injective (respectively, S-surjective, S-isomorphic) if the induced R_S-homomorphism $f_S : M_S \to N_S$ is injective (respectively, surjective, isomorphic). In these case, we say that M is an S-submodule of N, N is an S-homomorphic image of M and M is S-isomorphic to N, respectively. We note that f is also injective if M and N are also R_S-modules under the natural ring homomorphism $R \to R_S$.

(5) A sequence $A \overset{f}{\to} B \overset{g}{\to} C$ is S-exact if it induces the exact sequence $0 \to A_S \overset{f_S}{\to} B_S \overset{g_S}{\to} C_S \to 0$.

(6) An R-homomorphism $f : M \to N$ can factor through a module L over R_S if there are two R_S-homomorphisms $\phi : M_S \to L_S$ and $\varphi : L_S \to N_S$ such that $f_S = \varphi\phi$.

Remark 2.2. The notion of S-finitely generated here is not the same as that of S-finite in [1]. For example, let R be not an S-Noetherian domain, where $S = R \setminus 0$. Then any ideal is S-finitely generated, but there is an ideal which is not S-finite.

Lemma 2.3. *If M is S-finitely generated, then there is a finitely generated submodule N of M such that $N_S = M_S$.*

Proof. By assumption, we set $M_S = \sum_{i=1}^{n} R_S \frac{m_i}{1}$, and the submodule N generated by $\{m_i\}$. Thus $N_S = M_S$. □

Recall the notion of $\mathcal{X}$-Mittag-Leffler with respect to a class $\mathcal{X}$ of modules; see [2] or [18].

Definition 2.4. Let M and N be R-modules, and $\mathcal{X}$ be a class of modules.

(a) We say that M is an $\mathcal{X}$-Mittag-Leffler module if the natural homomorphism $\phi : M \bigotimes_R \prod X_\alpha \to \prod(M \bigotimes_R X_\alpha)$ is injective for any family $\{X_\alpha\}_{\alpha\in\Gamma}$ of modules in $\mathcal{X}$. If $\mathcal{X}$ just consists of a single module X, then M is also called Q-Mittag-Leffler.

(1) If $\mathcal{X}$ =Mod-R, M is called Mittag-Leffler, and denote by $\mathcal{ML}$ the class of all Mittag-Leffler modules.

(2) If $\mathcal{X} = \mathcal{F}$, consisting of all of flat R-modules, M is called $\mathcal{F}$-Mittag-Leffler, and denote by $\mathcal{M}^{\mathcal{F}}$ the class of all $\mathcal{F}$-Mittag-Leffler modules. See [8].

(3) If $\mathcal{X} = \mathcal{F_S}$, consisting of all of flat R_S-modules, M is called $\mathcal{F_S}$-Mittag-Leffler, and denote by $\mathcal{M}^{\mathcal{F_S}}$ the class of all $\mathcal{F_S}$-Mittag-Leffler modules.

(b) We say that M is an $\mathcal{X}$-filtered module if there are an ordinal κ and a continuous chain of modules, $\{M_\alpha\}_{\alpha\leq\kappa}$, consisting of submodules of M, satisfying:

(1) $M = M_\kappa$, and

(2) each of the modules $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{X}$.

In the case, the chain $\{M_\alpha\}_{\alpha\leq\kappa}$ is called an $\mathcal{X}$-filtration of M, and $\mathcal{X}$ is closed under filtration if it contains all $\mathcal{X}$-filtered modules.

(c) We say that an (S-)exact sequence $0 \to A \to B \to C \to 0$ is $\mathcal{X}$-pure if $(0 \to A_S \bigotimes_R M \to B_S \bigotimes_R M \to C_S \bigotimes_R M \to 0)$ $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is also exact for any R-module $M \in \mathcal{X}$. In this case, A is called an (S-)pure submodule of B with respect to $\mathcal{X}$.

Example 2.5. (1) Any S-torsion R-module M is $\mathcal{F_S}$-Mittag-Leffler. In fact, for R_S-module X, we have $M \bigotimes_R X \cong M_S \bigotimes_R X = 0$.

(2) Let R be any non-coherent domain. Then there is an ideal I of R such that R/I is not an $\mathcal{F}$-Mittag-Leffler module. Taking S as any multiplicative closed set such that it intersects I non-empty. As a result, R/I is $\mathcal{F_S}$-Mittag-Leffler.

In the following, we give a characterization of an S-torsion R-module being $\mathcal{F}$-Mittag-Leffler.

Proposition 2.6. *Let M be a finitely generated and S-torsion R-module. Then M is $\mathcal{F}$-Mittag-Leffler if and only if it is finitely presented. In particular, if $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$, then any S-regular ideal is finitely generated.*

Proof. (⇐) Trivially.

(⇒) Let M be $\mathcal{F}$-Mittag-Leffler, and consider the identity map $I : M \to M$. Then there is a finitely presented R-module F such that the map I can factor through F by [10, Theorem 1], and hence M is a direct summand of F. Therefore, M is finitely presented. □

3. $\mathcal{F_S}$-Mittag-Leffler modules

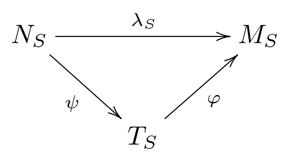
We begin this section by generalizing, we generalize the result [10, Theorem 1] to $\mathcal{F_S}$ (or R_S)-Mittag-Leffler modules, and give some properties of $\mathcal{F_S}$-Mittag-Leffler modules and a characterization of noetherian rings, etc.

Theorem 3.1. *For any R-module M, the following statements are equivalent*:

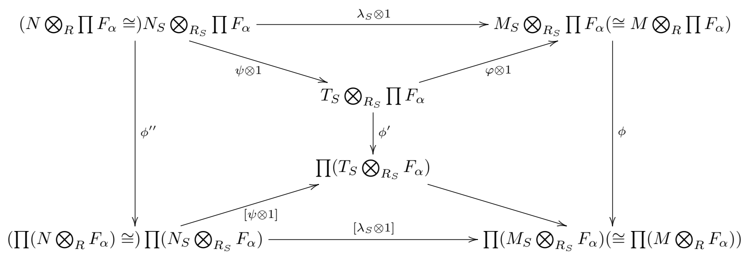
(a) *M is $\mathcal{F_S}$-Mittag-Leffler.*
(b) *The natural homomorphism $\phi : M \bigotimes_R (R_S)^\Gamma \to (M_S)^\Gamma$ is injective for any set Γ.*
(c) *The natural homomorphism $\phi : M \bigotimes_R (R_S)^I \to (M_S)^I$ is injective for some set I of cardinality at least Card(R).*
(d) *For any S-finitely generated submodule N of M, the inclusion $\lambda : N \hookrightarrow M$ can factor through a finitely presented module over R_S.*
(e) *For any finitely generated submodule N of M, the inclusion $\lambda : N \hookrightarrow M$ can factor through a finitely presented module over R_S.*
(f) *For any finitely generated submodule N of M, the inclusion $B \hookrightarrow A$ can factor through an S-finitely presented module over R_S.*

Proof. (a)⇒(b)⇒(c) and (d)⇒(e)⇒(f) are trivial. It only needs to show that (f)⇒(a) and (c)⇒(d).

(f)⇒(a) Let $x = \sum_{i=1}^n m_i \otimes \beta_i \in \mathrm{Ker}\phi$, and N be generated by $\{m_i\}_{i=1}^n$. Thus, there is an S-finitely presented R-module T such that the following diagram is commutative in Mod-R_S:



Now, for any family of flat R_S-modules $\{F_\alpha\}_{\alpha\in\Gamma}$, we consider the following commutative diagram in Mod-R_S:

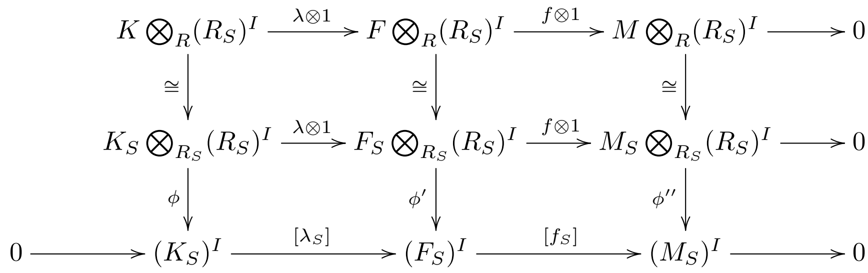


Without loss of generality, we view x as an element of $M_S \bigotimes_{R_S} \prod F_\alpha$. Then $[\lambda_S \otimes 1]$ is injective because all the F_α's are flat, and ϕ' is isomorphic for T_S being finitely presented. Therefore, $x = (\lambda_S \otimes 1)(x) = (\varphi \otimes 1)(\psi \otimes 1)(x) = (\varphi \otimes 1)(\phi')^{-1}([\psi \otimes 1])\phi''(x) = 0$ follows from $\phi''(x) \in \mathrm{Ker}[\lambda_S \otimes 1]$.

(c)$\Rightarrow$(d) If $\mathrm{Card}(R)$ is finite, then R is noetherian and R_S is a finitely generated R-module. (d) holds by taking $T = N$. Thus we assume that $\mathrm{Card}(R)$ is infinite.

Let N be an S-finitely generated submodules of M, R-homomorphism $f : F \to M$ be an epimorphism with F_R free, and set $K = \mathrm{Ker} f$. Besides, we also assume that $\{x_i\}_{i \in \Gamma_M}$ (respectively, $\{y_j\}_{j \in \Gamma_N}$) is a generating system of M (respectively, N). Since N_S is finitely generated, we set $N_S = \sum_{k=1}^{p} R_S \frac{z_k}{1}$, where $z_k \in N$, and then we can also set $z_k = \sum_{t=1}^{p_k} r_t y_{tk}$ for all $k \leq p$. Thus, we have $N' = \sum_{k=1}^{p} R z_k \subseteq N'' = \sum_{t=1}^{p_1} R y_{t1} + \cdots + \sum_{t=1}^{p_p} R y_{tp} \subseteq N$. It's easy to see $N'_S = N''_S = N_S$. Let $q = \sum_{t=1}^{p} p_t$. Therefore, there is a submodule G generated by $\{g_{tk}\}_{k \leq p, t \leq p_k}$ of F, and $f(g_{tk}) = y_{tk}$. It follows that $f(G) = N''$.

Now, we consider the following commutative diagram with exact rows:



First, G can be embedded into a finitely generated free submodules F' of F, and $F' \hookrightarrow F$ splits. It follows that $G_S \hookrightarrow F'_S$ and $F'_S \hookrightarrow F_S$ split. In addition, we can get $G^I \hookrightarrow (F')^I \cong F' \bigotimes_R R^I \hookrightarrow F^I$, and then $(G_S)^I \hookrightarrow (F'_S)^I \cong F'_S \bigotimes_{R_S} (R_S)^I \hookrightarrow (F_S)^I$. Thus, $(G_S)^I \hookrightarrow \phi'(F_S \bigotimes_{R_S} (R_S)^I)$. By the assumption (c), ϕ'' is injective, and then $0 \to \frac{(K_S)^I}{\mathrm{Im}\phi} \to \frac{(F_S)^I}{\mathrm{Im}\phi'}$ is exact follows from "Snake Lemma". So $(G_S \cap K_S) \hookrightarrow \mathrm{Im}\phi$.

Since $\mathrm{Card}(G_S) \leq \mathrm{Card}(R_S) = \mathrm{Card}(R)$ because R is infinite, we have

$$\mathrm{Card}(G_s \cap K_S) \leq \mathrm{Card}(R_S).$$

Thus there exists a surjection $i \mapsto g_i$ of I onto $G_s \cap K_S$. Let $\{u_l\}_{l\in I'}$ be a generating system of R_S-module $G_S \cap K_S$, where $\mathrm{Card}(I') \leq \mathrm{Card}(I)$. On the other hand, for any $l \in I'$, we can assume that $u_l \in \sum_{i=1}^{n_l} R_S \frac{g_i}{1}$, where $g_i \in \{g_{tk}\}_{k\leq p, t\leq p_k}$. In follows, we consider the inverse image of the element $[u_i]_{i\in I} = \begin{cases} u_i = u_l, & i \in I'; \\ 0, & i \notin I'. \end{cases}$

Since of $(G_S \cap K_S)^I \subseteq \mathrm{Im}\phi'$, there is an element $\alpha \in F_S \bigotimes_{R_S} (R_S)^I$ such that $[\lambda_S]([u_i]_{i\in I}) = \phi'(\alpha)$. Since $[f_S]\phi'(\alpha) = [f_S][\lambda_S]([u_i]_{i\in I}) = 0$ and $[f_S]\phi'(\alpha) = \phi''(f\otimes 1)(\alpha)$, we have $\alpha \in \mathrm{Ker}(f \otimes 1)$ for ϕ'' being injective. Therefore, we can find an element $\beta \in K_S \bigotimes_{R_S} (R_S)^I$ and write it as $\beta = \sum_{j=1}^{n} \frac{h_j}{1} \otimes [r'_{ji}]_{i\in I}$, where $\{h_j\}_{j\leq n} \subseteq K$ and $r'_{ji} \in R_S$. Thus, $\phi(\beta) = [u_i]_{i\in I}$, and $u_i = \phi(\sum_{j=1}^{n} \frac{h_j}{1} \otimes r'_{ji})$, where r'_{ji} denotes the ith element of $[r'_{ji}]_{i\in I}$ for any i. Here, we view r'_{ji} as $[r'_i]_{k\in I} = \begin{cases} r'_{ji}, & k = i; \\ 0, & k \neq i. \end{cases}$

We claim that for any $[g_i]_{i\in I} \in (G_S \cap K_S)^I$, there is some element $\beta' = \sum_{j=1}^{n} \frac{h_j}{1} \otimes [\frac{r_{ji}}{s_{ji}}]_{i\in I} \in K_S \bigotimes_{R_S} (R_S)^I$ such that $[g_i]_{i\in I} = \phi(\beta')$. Let $g_i = \sum_{j=1}^{n_i} r'_{ij} u_j$ for any $i \in I$. Then $g_i = \sum_{j=1}^{n_i} r'_{ij} \phi(\sum_{k=1}^{n} \frac{h_k}{1} \otimes r'_{kj}) = \phi(\sum_{k=1}^{n} \frac{h_k}{1} \otimes (\sum_{j=1}^{n_i} r'_{ij}) r'_{kj})$. It follows that $[g_i]_{i\in I} = \phi(\sum_{k=1}^{n} \frac{h_k}{1} \otimes [(\sum_{j=1}^{n_i} r'_{ij}) r'_{kj}]_{i\in I})$.

Now, we set H to be the submodule generated by $\{h_j\}_{j\leq n}$ of K, and then the submodule H_S is generated by $\{\frac{h_j}{1}\}_{j\leq n}$ of K_S. It is easy to see that $G \cap H \subseteq G \cap K$ and $G_S \cap H_S = G_S \cap K_S$. In addition, $G + H$ is contained in some finitely generated free submodule F_0 of F, and then $G_S + H_S \subseteq (F_0)_S$.

Now, we obtain an exact sequence $0 \longrightarrow G_S \cap K_S \xrightarrow{\lambda_S} G_S \xrightarrow{f_S} N_S \longrightarrow 0$ from the short exact sequence $0 \longrightarrow K_S \xrightarrow{\lambda_S} F_S \xrightarrow{f_S} M_S \longrightarrow 0$. Consequently, f_S induces the following commutative diagram:

$$\begin{array}{ccc} (G_S/(G_S \cap H_S) \cong)(G_S + H_S)/H_S & \xrightarrow{\cong_{f'_S}} & N''_S = N_S \\ & \searrow^{\lambda} \quad \nearrow_{\varphi} & \\ & (F_0)_S/H_S & \end{array}$$

As a result, we finish the proof with F_0/H being finitely presented and the commutative diagram as follows:

$$\begin{array}{ccc} N_S & \xrightarrow{\lambda_S} & M_S \\ & \searrow^{\lambda\cdot(f'_S)^{-1}} \quad \nearrow_{\varphi} & \\ & (F_0)_S/H_S & \end{array}$$

□

Corollary 3.2. *The following statements or conditions are equivalent*:

(a) *R_S is noetherian.*

(b) *Every R-module is $\mathcal{F}_S$-Mittag-Leffler.*

(c) *Every finitely generated R-module is $\mathcal{F}_S$-Mittag-Leffler.*

Proof. (a)⇒(b) If R_S is noetherian, then M_S is finitely presented in Mod-R_S for any finitely generated module M.

(b)⇒(c) Trivially.

(c)⇒(a) We consider the R-module R/I for any ideal I of R. Then there is a finitely presented R_S-module T such that $(R/I)_S$ is a direct summand of T by Theorem 3.1. Therefore, I is S-finitely generated, and hence R_S is noetherian. □

Through Corollary 3.2, we give some examples, which show that there is an $\mathcal{F}_S$-Mittag-Leffler module but not an $\mathcal{F}$-Mittag-Leffler module.

Example 3.3. (1) Let R be a non-noetherian domain and $S = R \setminus \{0\}$ which means that S consists of all non-zero elements in R. Then any R-module is $\mathcal{F}_S$-Mittag-Leffler, but some modules are not $\mathcal{F}$-Mittag-Leffler.

(2) Let T be any noetherian ring, $R = T[\{X_i\}_{i=1}^{\infty}]$, and $S = \{X_i\}_{i>k}$ for any non-negative integral k. Then every R-module is $\mathcal{F}_S$-Mittag-Leffler, but some modules are not $\mathcal{F}$-Mittag-Leffler.

Corollary 3.4. *Let R_S be a noetherian ring. Then R is noetherian if and only if $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$.*

Proof. It follows from Corollary 3.2 and [18, Corollary 5.14]. □

Corollary 3.5. *Let R_S be a coherent ring. An R-module M is $\mathcal{F}_S$-Mittag-Leffler if and only if every finitely generated submodule of M is S-finitely presented.*

Proof. It follows from the condition (e) in Theorem 3.1. □

Corollary 3.6. *Suppose that R_S is von-Neumann regular. The following statements and conditions are equivalent for any R-module M:*

(a) *The natural homomorphism $\phi : M \bigotimes_R \prod F_\alpha \to \prod(M \bigotimes_R F_\alpha)$ is injective for any collection of R_S-modules $\{F_\alpha\}_{\alpha\in\Gamma}$.*

(b) *Every S-finitely generated submodule N of M such that N_S is projective in Mod-R_S.*

(c) *Every finitely generated submodule N of M such that N_S is projective in Mod-R_S.*

Proof. The equivalences between these conditions follow from the facts that finitely generated projective modules are finitely presented and finitely presented flat modules are projective. □

Proposition 3.7. (a) *Let M be an R-module. Then the following conditions are equivalent:*

(1) *M belongs to $\mathcal{M}^{\mathcal{F}_S}$.*

(2) *M_S belongs to $\mathcal{M}^{\mathcal{F}_S}$.*

(3) *M_S belongs to $\mathcal{M}^{\mathcal{F}}$ in Mod-R_S.*

(b) *If M is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler, then $M/\tau_S(M)$ is also $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*
(c) *$\mathcal{M}^{\mathcal{F}_S}$ is closed under direct summands and finite direct sums.*
(d) *Let $0 \to A \to B \to C \to 0$ be an S-exact sequence of R-modules. If A and C belong to $\mathcal{M}^{\mathcal{F}_S}$, then B also belongs to $\mathcal{M}^{\mathcal{F}_S}$.*
(e) *Every S-pure submodule of an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler module M is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*

Proof. (a) These equivalences follow from the following commutative diagram:

$$\begin{array}{ccccc} M\bigotimes_R(R_S)^I & \xrightarrow{\cong} & M_S\bigotimes_R(R_S)^I & \xrightarrow{\cong} & M_S\bigotimes_{R_S}(R_S)^I \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ (M_S)^I & \xrightarrow{=} & (M_S)^I & \xrightarrow{=} & (M_S)^I \end{array}$$

(b) The assertion follows for $(M/\tau_S(M))_S \cong M_S$ and (a).

(c) It is known that $\mathcal{M}^{\mathcal{F}}$ is closed under direct summands and finite direct sums, and then the assertion follows from (a).

(d) Considering the commutative diagram with exact rows as below:

$$\begin{array}{ccccccc} & & A\bigotimes_R(R_S)^I & \longrightarrow & B\bigotimes_R(R_S)^I & \longrightarrow & C\bigotimes_R(R_S)^I \\ & & \downarrow{\scriptstyle\phi'} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi''} \\ 0 & \longrightarrow & (A_S)^I & \longrightarrow & (B_S)^I & \longrightarrow & (C_S)^I \end{array}$$

A short diagram chase which use ϕ' and ϕ'' shows that ϕ is injective.

(e) It is easy to show the assertion by the similar argument as in (d). □

It was known that every direct product of Mittag-Leffler modules is Mittag-Leffler if and only if R is self-injective over a von-Neumann ring R. According to this property, we give a relative result below:

Corollary 3.8. *Let R_S be a von-Neumann regular ring. Then, R_S is self-injective if and only if $\prod_{\alpha\in\Gamma} M_\alpha$ is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler for any family of S-torsion-free and $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler R-modules $\{M_\alpha\}_{\alpha\in\Gamma}$.*

Proof. ($\Leftarrow$) Let $\{M_\alpha\}_{\alpha\in\Gamma}$ be any family of Mittag-Leffler R_S-modules. Then M_α is also an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler and S-torsion-free R-module for any $\alpha \in \Gamma$. Thus, $\prod_{\alpha\in\Gamma} M_\alpha$ is an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler R-module by assumption. Therefore, $\prod_{\alpha\in\Gamma} M_\alpha \cong (\prod_{\alpha\in\Gamma} M_\alpha)_S$ is an $\mathcal{F}$-Mittag-Leffler R_S-module by Proposition 3.7, and hence is also a Mittag-Leffler R_S-module [19, Fact 3.19].

($\Rightarrow$) Let $\{M_\alpha\}_{\alpha\in\Gamma}$ be any family of S-torsion-free and $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler R-modules. Then $0 \to (\prod_{\alpha\in\Gamma} M_\alpha)_S \to \prod_{\alpha\in\Gamma}(M_\alpha)_S$ is pure exact for R_S being von-Neumann regular. Thus, $(\prod_{\alpha\in\Gamma} M_\alpha)_S$ is a Mittag-Leffler R_S-module, and then $\prod_{\alpha\in\Gamma} M_\alpha$ is an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler R-module. □

4. Global dimension with respect to $\mathcal{F_S}$-Mittag-Leffler modules

Since the class $\mathcal{M}^{\mathcal{F}_S}$ is not hereditary, which means that we can not obtain $A \in \mathcal{M}^{\mathcal{F}_S}$ for an exact sequence $0 \to A \to B \to C \to 0$, where both B and C are $\mathcal{F_S}$-Mittag-Leffler, we can not define the projective dimension of an R-module with respect to $\mathcal{F_S}$-Mittag-Leffler modules by any resolution in which every module is $\mathcal{F_S}$-Mittag-Leffler. Now, we give a modified definition of the projective dimension of an R-module, and the global dimension of R with respect to $\mathcal{F_S}$-Mittag-Leffler modules [8, Definition 3.1].

Definition 4.1. Let n be a natural number and M be an R-module.

(1) We say that M has projective dimension with respect to $\mathcal{F_S}$-Mittag-Leffler modules (or $\mathcal{M}^{\mathcal{F}_S}$-projective dimension) at most n, denoted by $pd_{\mathcal{M}^{\mathcal{F}_S}} M \leq n$, if there is a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0$$

such that $\mathrm{Ker}(d_{n-1})$ is $\mathcal{F_S}$-Mittag-Leffler. If such n does not exist, $pd_{\mathcal{M}^{\mathcal{F}_S}} M = \infty$. In addition, if n is the least such integer, define $pd_{\mathcal{M}^{\mathcal{F}_S}} M = n$.

(2) The global dimension with respect to $\mathcal{F_S}$-Mittag-Leffler modules of R, denoted by $gl_{\mathcal{M}^{\mathcal{F}_S}}.\dim(R)$, is the supremum of the set consisting of $pd_{\mathcal{M}^{\mathcal{F}_S}} M$ for all of R-modules M, that is,

$$gl_{\mathcal{M}^{\mathcal{F}_S}}.\dim(R) = \sup\{pd_{\mathcal{M}^{\mathcal{F}_S}} M \mid M \text{ is an } R\text{-module}\}.$$

Proposition 4.2. *Let n be a natural number and M be an R-module. Then*

(a) *The $\mathcal{M}^{\mathcal{F}_S}$-projective dimension of M does not depend on the chosen projective resolution.*

(b) *The following conditions are equivalent:*
  (1) $pd_{\mathcal{M}^{\mathcal{F}_S}} M \leq n$.
  (2) $pd_{\mathcal{M}^{\mathcal{F}}} M_S \leq n$ *in Mod-R_S.*
  (3) *There exists an S-exact sequence $0 \longrightarrow K \hookrightarrow Q_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} M \xrightarrow{\delta_{-1}} 0$ such that K is $\mathcal{M}^{\mathcal{F}_S}$-Mittag-Leffler, where $K = \mathrm{Ker}(\delta_{n-1})$ and $(Q_i)_S$ is projective in Mod-R_S for any $i \leq n-1$.*

(c) *For any S-exact sequence $0 \longrightarrow K \xrightarrow{f} P \xrightarrow{g} M \longrightarrow 0$, where P_S is a projective R_S-module, then $pd_{\mathcal{M}^{\mathcal{F}_S}} M \leq n+1$ if and only if $pd_{\mathcal{M}^{\mathcal{F}_S}} K \leq n$.*

Proof. (a) Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

and

$$\cdots \longrightarrow P'_n \xrightarrow{d'_n} P'_{n-1} \xrightarrow{d'_{n-1}} \cdots \longrightarrow P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} M \longrightarrow 0$$

be any two of projective resolutions of M. Set $K_n = \mathrm{Ker}(d_{n-1})$ and $K'_n = \mathrm{Ker}(d'_{n-1})$, and then $K_n \oplus F \cong K'_n \oplus F$ for some projective R-module F by [20, Proposition 8.5]. Thus, K_n is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler if and only if K'_n is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler by Proposition 3.7.

(b) (1) $\Leftrightarrow$ (2) Let $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Then $0 \to K_S \to (P_{n-1})_S \to \cdots \to (P_1)_S \to (P_0)_S \to M_S \to 0$ be a projective resolution of M_S in Mod-R_S. Therefore, it follows from Proposition 3.7 that $pd_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M \le n$ is equivalent to $pd_{\mathcal{M}^{\mathcal{F}}} M_S \le n$ in Mod-R_S.

(1) $\Rightarrow$ (3) Trivially.

(3) $\Rightarrow$ (2) With the assumption, we obtain an exact sequence of R_S-modules

$$0 \to K_S \hookrightarrow (Q_{n-1})_S \overset{(\delta_{n-1})_S}{\longrightarrow} \cdots \longrightarrow (Q_1)_S \overset{(\delta_1)_S}{\longrightarrow} (Q_0)_S \overset{(\delta_0)_S}{\longrightarrow} M_S \longrightarrow 0,$$

and K_S is $\mathcal{F}$-Mittag-Leffler by Proposition 3.7. Thus, $pd_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M_S \le n$ in Mod-R_S.

(c) By assumption, the sequence $0 \longrightarrow K_S \overset{f_S}{\longrightarrow} P_S \overset{g_S}{\longrightarrow} M_S \longrightarrow 0$ is exact in Mod-R_S. In addition, it is equivalent to show that $pd_{\mathcal{M}^{\mathcal{F}}} M_S \le n+1$ if and only if $pd_{\mathcal{M}^{\mathcal{F}}} K_S \le n$ in Mod-R_S by (b). Thus, the assertion holds. □

It is obvious that $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M \le \mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M$ for any R-module M, but there does not exist some integer n such that $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M \le \mathrm{pd}_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M + n$ for an R-module M in general. However, there is a good relationship between them for an R-module M of finite flat dimension and a special multiplicative closed set S. See the examples below:

Example 4.3. (1) Let R be a domain but not weak n-coherent for some integer n. We note that if R is weak n-coherent, then R is weak m-coherent for any integer $m \ge n$. Consequently, there is an ideal I of R such that $pd_{\mathcal{M}^{\mathcal{F}}} R/I > n+1$ [8, Theorem 4.2]. Set a multiplicative closed set S to intersect I non-empty. Then $pd_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} R/I = 0$.

(2) Let R be an arithmetical and locally IF ring, but not coherent. Set $S = R \backslash m$ for any maximal ideal m of R. Then $\mathrm{fd}_R R^{\Gamma} = \infty$ for any set Γ of cardinality at least $\mathrm{Card}(R)$ by [9, Theorem 1]. Consequently, for any integer n, there is an ideal I of R such that $pd_{\mathcal{M}^{\mathcal{F}}} I \ge n$, but $pd_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} I = 0$ by Proposition 4.2 and [8, Corollary 4.3].

(3) Let n be an integer, an R-module M be of flat dimension at most n and the multiplicative closed set S consist of non-zero divisors. Then $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M \le n$ if and only if $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M \le n$.

Proof. The sufficiency is obvious. If $\mathrm{pd}_{\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}} M \le n$, then the $(n-1)th$ syzygy K of any projective resolution of M is flat and $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler. Therefore,

$\mathrm{pd}_{\mathcal{M}^{\mathcal{F}}} M \le n$ follows from K being $\mathcal{F}$-Mittag-Leffler for the commutative diagram below:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & M \bigotimes_R R^{\Gamma} & \longrightarrow & M^{\Gamma} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & M \bigotimes_R (R_S)^{\Gamma} & \longrightarrow & (M_S)^{\Gamma}
\end{array}$$

□

Next, we give a sufficient condition such that $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_{\mathcal{S}}}$, that is, every $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler module is an $\mathcal{F}$-Mittag-Leffler module.

Proposition 4.4. *Let Γ be any set of cardinality at least Card(R), and let the multiplicative closed set S consist of non-zero divisors. If the exact sequence $0 \to R^{\Gamma} \to (R_S)^{\Gamma} \to (R_S/R)^{\Gamma} \to 0$ is $\mathcal{M}^{\mathcal{F}_{\mathcal{S}}}$–pure, then $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_{\mathcal{S}}}$.*

Proof. Let M be an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler module. Considering the commutative diagram below:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
0 & \longrightarrow & M \bigotimes_R R^{\Gamma} & \xrightarrow{\delta} & M \bigotimes_R (R_S)^{\Gamma} \\
 & & \phi \downarrow & & \downarrow \phi' \\
 & & M^{\Gamma} & \xrightarrow{\eta} & (M_S)^{\Gamma}
\end{array}$$

By assumption, δ and ϕ' are injective, and hence it follows from $\eta\phi = \phi'\delta$ that ϕ is also injective. Thus M is $\mathcal{F}$-Mittag-Leffler. □

Definition 4.5. Let κ be an infinite cardinal and $\mathcal{X}$ a class of modules. We call a module M $(\kappa, \mathcal{X})$-free if M has a $(\kappa, \mathcal{X})$-dense system of submodules, that is, a direct family $\mathcal{S}$ of submodules of M, satisfying:

(1) $\mathcal{S} \subseteq \mathcal{X}$;

(2) $\mathcal{S}$ is closed under well order ascending chains of length smaller than κ, and

(3) Any subset of M of cardinality smaller than κ is contained in an element of $\mathcal{S}$.

In this case, we say that $\mathcal{X}$ is closed under κ-free modules if every $(\kappa, \mathcal{X})$-free module belongs to $\mathcal{X}$.

If an R-module M is $(\mathcal{N}_0, \mathcal{X})$-free, then $M = \bigcup_{M_\alpha \in \mathcal{S}} M_\alpha$ follows from the condition (3) in Definition 4.5, for any $(\mathcal{N}_0, \mathcal{X})$-dense system $\mathcal{S} = \{M_\alpha\}_{\alpha \in \Gamma}$.

Lemma 4.6 ([8]). *Let n be a non-zero natural number and $\mathcal{X}$ a class of R-modules closed under filtration that contains all projective modules. The following statements and conditions are equivalent*:

(a) $gl_{\mathcal{X}}.\dim(R) \leq n$.

(b) $pd_{\mathcal{X}} I \leq n-1$ *for any ideal I of R.*

If in addition, $\mathcal{X}$ is closed under direct summands, finite direct sums and κ-free modules for some infinite regular cardinal κ, then these conditions are also equivalent to:

(c) $pd_{\mathcal{X}} I \leq n-1$ *for any ideal I of R, which can be generated by a set of cardinality smaller than κ.*

Corollary 4.7. *Let n be a non-zero natural number. Then $gl_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}}.\dim(R) \leq n$ if and only if $pd_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}} I \leq n-1$ for any finitely generated ideal I of R.*

Proof. According to Proposition 3.7 and Lemma 4.6, it only needs to show that $\mathcal{M}^{\mathcal{F}_\mathcal{S}}$ is closed under filtration and κ-free modules for some infinite regular cardinal κ.

First, let M be an $\mathcal{M}^{\mathcal{F}_\mathcal{S}}$-filtered module. Replace $\mathcal{S}$ and $\mathcal{Q}$ by $\mathcal{M}^{\mathcal{F}_\mathcal{S}}$ and $\mathcal{F}_\mathcal{S}$ respectively, then M belongs to $\mathcal{M}^{\mathcal{F}_\mathcal{S}}$ by [2, Proposition 1.9].

Finally, assume that M is $(\mathcal{N}_0, \mathcal{M}^{\mathcal{F}_\mathcal{S}})$-free, that is, $M = \bigcup_{M_\alpha \in \mathcal{S}} M_\alpha$ for some $(\mathcal{N}_0, \mathcal{X})$-dense system $\mathcal{S} = \{M_\alpha\}_{\alpha\in\Gamma}$. For any submodule N of M generated by elements $\{x_1, \ldots, x_n\}$, there is an element $M_\alpha \in \mathcal{S}$ such that $\{x_1, \ldots, x_n\} \subseteq M_\alpha$. Consequently, $N \subseteq M_\alpha$ and $M \in \mathcal{M}^{\mathcal{F}_\mathcal{S}}$ by Theorem 3.1. □

Theorem 4.8. *Let n be a non-zero natural number. Then the following statements and conditions are equivalent*:

(a) $gl_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}}.\dim(R) \leq n$.

(b) $gl_{\mathcal{M}^{\mathcal{F}}}.\dim(R_S) \leq n$.

(c) $pd_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}} M \leq n$ *for any S-cyclic R-module M.*

(d) $pd_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}} R/I \leq n$ *for any finitely generated ideal I of R.*

(e) *There is a set Γ of cardinality at least Card(R) such that $fd_R(R_S)^\Gamma \leq n-1$.*

Proof. (a)⇔(b) follows from Proposition 4.2.

(a)⇒(c)⇒(d) Trivially.

(d)⇒(a) follows from Proposition 4.2 and Corollary 4.7.

(a)⇔(e) Let M be an R-module. We claim that $pd_{\mathcal{M}^{\mathcal{F}_\mathcal{S}}} M \leq n$ if and only if $\mathrm{Tor}_n^R(M, (R_S)^\Gamma) = 0$, and hence the equivalence between (a) and (c) holds.

We proceed by induction on n. Let $0 \to K \to P \to M \to 0$ be a projective resolution of M. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & K\bigotimes_R (R_S)^\Gamma & \xrightarrow{\varphi} & P\bigotimes_R (R_S)^\Gamma & \longrightarrow & M\bigotimes_R (R_S)^\Gamma \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} \\ 0 & \longrightarrow & (K_S)^\Gamma & \longrightarrow & (P_S)^\Gamma & \longrightarrow & (M_S)^\Gamma \end{array}$$

In the case $n = 1$, since $\mathrm{Tor}_1^R(P, (R_S)^\Gamma) = 0$, we can easily obtain that $\mathrm{Tor}_1^R(M, (R_S)^\Gamma) = 0$ if and only if φ is injective if and only if ϕ' is injective if and only if K is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler by Theorem 3.1, if and only if $pd_{\mathcal{M}^{\mathcal{F}_S}} M \leq 1$ by Proposition 4.2.

Suppose we have shown the claim for some integral n. Then we have $\mathrm{Tor}_{n+1}^R(M, (R_S)^\Gamma) = 0$ if and only if $\mathrm{Tor}_n^R(K, (R_S)^\Gamma) = 0$ if and only if $pd_{\mathcal{M}^{\mathcal{F}_S}} K \leq n$, by induction assumption, if and only if $pd_{\mathcal{M}^{\mathcal{F}_S}} M \leq n + 1$. □

In the case of $n = 1$ in Theorem 4.8, we have a new characterization of S-coherent rings:

Corollary 4.9. *The following conditions are equivalent*:

(a) *R_S is coherent.*
(b) *For any family $\{F_\alpha\}_{\alpha\in\Gamma}$ of S-divisible and flat R-modules, $\prod_{\alpha\in\Gamma} F_\alpha$ is flat.*
(c) *For any non-zero integral m, and any family $\{F_\alpha\}_{\alpha\in\Gamma}$ of R_S-modules of flat dimension at most m, then $fd_R \prod_{\alpha\in\Gamma} F_\alpha \leq m$.*
(d) *$gl_{\mathcal{M}^{\mathcal{F}_S}}.\dim(R) \leq 1$.*
(e) *Every ideal of R is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*
(f) *Every finitely generated ideal of R is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*
(g) *Every submodule of an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler module is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*
(h) *Every S-submodule of an $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler module is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler.*

Proof. (a)⇔(b)⇔(c) follow from [7, Theorem 2.1].

(a)⇔(d)⇔(e)⇔(f) follow from Theorem 4.8 and [8, Corollary 4.3].

(a)⇔(g) follows from Proposition 4.2 and [8, Corollary 4.3].

(h)⇒(g) Trivially.

(g)⇒(h) follows from Proposition 3.7. □

Corollary 4.10. *Let R_S be coherent. If $\mathcal{M}^{\mathcal{F}_S} = \mathcal{M}^{\mathcal{F}}$, then*

(a) *R is coherent.*
(b) *any S-regular ideal of R is finitely presented.*

Proof. (a) The conclusion holds by Corollary 4.9 and [8, Corollary 4.3].

(b) Let I be an S-regular ideal. Then R/I is $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler. Thus, I is finitely presented by Proposition 2.6 and (a). □

Finally, we give some examples to show the relationship among Mittag-Leffler modules, $\mathcal{F}$-Mittag-Leffler modules and $\mathcal{F}_{\mathcal{S}}$-Mittag-Leffler modules.

Example 4.11. (1) Let R be a domain and $S = R\backslash\{0\}$, or R_S be noetherian for some S. Then $\mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$ if and only if R is noetherian; $\mathcal{M}^{\mathcal{F}_S} = \mathcal{ML}$ if and only if R is a field or an artinian principle ideal ring [18, Corollary 4.8] and [11, Theorem 4.3].

(2) Let R be a von-Neumann regular ring, and the multiplicative closed set S consist of non-zero divisors. Then we have $\mathcal{ML} = \mathcal{M}^{\mathcal{F}} = \mathcal{M}^{\mathcal{F}_S}$ by Example 4.3 and [19, Fact 3.19].

Acknowledgements. The authors wish to thank the referee for the valuable suggestions and comments in shaping the article into its present form.

References

[1] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra **30** (2002), no. 9, 4407–4416. https://doi.org/10.1081/AGB-120013328

[2] L. Angeleri Hügel and D. Herbera, *Mittag-Leffler conditions on modules*, Indiana Univ. Math. J. **57** (2008), no. 5, 2459–2517. https://doi.org/10.1512/iumj.2008.57.3325

[3] G. Azumaya, *Finite splitness and finite projectivity*, J. Algebra **106** (1987), no. 1, 114–134. https://doi.org/10.1016/0021-8693(87)90024-X

[4] J. Baeck, G. Lee, and J. W. Lim, *S-Noetherian rings and their extensions*, Taiwanese J. Math. **20** (2016), no. 6, 1231–1250. https://doi.org/10.11650/tjm.20.2016.7436

[5] S. Bazzoni and L. Positselski, *$\mathcal{S}$-almost perfect commutative rings*, eprint arXiv: 1801.04820, (2018).

[6] D. Bennis and M. El Hajoui, *On S-coherence*, J. Korean Math. Soc. **55** (2018), no. 6, 1499–1512.

[7] S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. https://doi.org/10.2307/1993382

[8] M. Cortés-Izurdiaga, *Products of flat modules and global dimension relative to $\mathcal{F}$-Mittag-Leffler modules*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4557–4571. https://doi.org/10.1090/proc/13059

[9] F. Couchot, *Finitistic weak dimension of commutative arithmetical rings*, Arab. J. Math. (Springer) **1** (2012), no. 1, 63–67. https://doi.org/10.1007/s40065-012-0023-4

[10] K. R. Goodearl, *Distributing tensor product over direct product*, Pacific J. Math. **43** (1972), 107–110. http://projecteuclid.org/euclid.pjm/1102959646

[11] P. Griffith, *On the decomposition of modules and generalized left uniserial rings*, Math. Ann. **184** (1969/1970), 300–308. https://doi.org/10.1007/BF01350858

[12] D. Herbera and J. Trlifaj, *Almost free modules and Mittag-Leffler conditions*, Adv. Math. **229** (2012), no. 6, 3436–3467. https://doi.org/10.1016/j.aim.2012.02.013

[13] H. Kim, M. O. Kim, and J. W. Lim, *On S-strong Mori domains*, J. Algebra **416** (2014), 314–332. https://doi.org/10.1016/j.jalgebra.2014.06.015

[14] H. Kim and J. W. Lim, *S-$*_w$-principal ideal domains*, Algebra Colloq. **25** (2018), no. 2, 217–224. https://doi.org/10.1142/S1005386718000159

[15] J. W. Lim, *A note on S-Noetherian domains*, Kyungpook Math. J. **55** (2015), no. 3, 507–514. https://doi.org/10.5666/KMJ.2015.55.3.507

[16] J. W. Lim and D. Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra **218** (2014), no. 6, 1075–1080. https://doi.org/10.1016/j.jpaa.2013.11.003

[17] ______, *S-Noetherian properties of composite ring extensions*, Comm. Algebra **43** (2015), no. 7, 2820–2829. https://doi.org/10.1080/00927872.2014.904329

[18] Ph. Rothmaler, *Mittag-Leffler modules and positive atomicity*, Habilitationsschrift, Kiel, 1994.

[19] ______, *Torsion-free, divisible, and Mittag-Leffler modules*, Comm. Algebra **43** (2015), no. 8, 3342–3364. https://doi.org/10.1080/00927872.2014.918990

[20] J. J. Rotman, *An Introduction to Homological Algebra*, second edition, Universitext, Springer, New York, 2009. https://doi.org/10.1007/b98977

[21] D. Simson, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. **96** (1977), no. 2, 91–116. https://doi.org/10.4064/fm-96-2-91-116

Mingzhao Chen
College of Mathematics and Software Science
Sichuan Normal University
Chengdu, 610068, P. R. China
Email address: mingzhao.c@163.com

Fanggui Wang
College of Mathematics and Software Science
Sichuan Normal University
Chengdu, 610068, P. R. China
Email address: wangfg2004@163.com