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ON GENERALIZED GRADED CROSSED PRODUCTS AND KUMMER SUBFIELDS OF SIMPLE ALGEBRAS

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Abstract. Using generalized graded crossed products, we give necessary and sufficient conditions for a simple algebra over a Henselian valued field (under some hypotheses) to have Kummer subfields. This study generalizes some known works. We also study many properties of generalized graded crossed products and conditions for embedding a graded simple algebra into a matrix algebra of a graded division ring.

Introduction

Amitsur and Tignol determined in [12] necessary and sufficient conditions for Malcev-Neumann division algebras (under some hypotheses) to have Kummer subfields. This work was then extended by Morandi and Sethuraman in [7] where they showed that in fact these conditions are true for any (tame) division algebra of the form $D = S \otimes_E T$ over a Henselian valued field E, where S is an inertially split division algebra over E and T is a (tame) totally ramified division algebra over E. A second generalization of this work to arbitrary tame division algebra over a Henselian valued field was given by the second author in [8]. In the present article, we give a more general result showing that Amitsur and Tignol's conditions are also true for (tame) simple algebras over a Henselian valued field (see Corollaries 3.9 and 3.10). These results are based on a particular representation of (some) graded simple algebras as generalized graded crossed products satisfying some 'grading separation condition' (GSP) (see (1.6)). Also, they are based on canonical relations connecting graded simple algebras to simple algebras with tame gauges (as developed in [13] and [14]). Some necessary results concerning the possibility of embedding a graded simple algebra in a matrix algebra of a graded division ring, are shown in the second section of this work. The first section, which is independent of the rest, studies arbitrary generalized graded crossed products for their own right. We show that if $S = (A, H, (w, f))$ is a generalized graded crossed product

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satisfying the GSP, then many properties of S depend on analogous ones on A (see Proposition 1.14).

We precise that all rings are assumed to be associative with an identity element, and all free modules are assumed to be finite-dimensional. We recall below some necessary facts on graded algebras.

Let F be a ring and Γ a totally ordered abelian group. We say that F is a graded ring of type Γ if there are subgroups F_γ ($\gamma \in \Gamma$) of F such that $F = \oplus_{\gamma\in\Gamma} F_\gamma$ and $F_\gamma F_\delta \subseteq F_{\gamma+\delta}$ for all γ, $\delta \in \Gamma$. In this case, the set $\Gamma_F = \{\gamma \in \Gamma \mid F_\gamma \neq \{0\}\}$ is called the support of F.

If F is a graded ring of type Γ and $x \in F_\gamma$ for some $\gamma \in \Gamma_F$, then we say that x is a homogeneous element of F. In particular, if x is a nonzero element of F_γ, we say that x has grade γ and we write $\mathrm{gr}(x) = \gamma$. We denote by F^* the set of invertible homogeneous elements of F. We say that F is a graded field if F is a commutative graded ring and all nonzero homogeneous elements of F are invertible.

Let F be a graded field of type Γ and A be a (left) F-module such that $A = \oplus_{\gamma\in\Gamma} A_\gamma$, where A_γ are subgroups of A, and $F_\lambda A_\gamma \subseteq A_{\lambda+\gamma}$ for all $\lambda, \gamma \in \Gamma$, then we say that A is a graded (left) F-module (or a graded vector space over F). If in addition A is a ring and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$, then we say that A is a graded algebra over F. In this case, if I is an ideal of A such that $I = \oplus_{\gamma\in\Gamma}(I \cap A_\gamma)$, then we say that I is a graded ideal of A. If A has no graded ideals but 0 and A, then we say that A is graded simple. Graded algebras over F [resp., commutative graded algebras over F] for which nonzero homogeneous elements are invertible are called graded division algebras over F [resp., graded field extensions of F]. If F is the center of a graded division algebra [resp., a graded simple algebra] B, then we say that B is a graded central division algebra over F [resp., a graded central simple algebra over F].

Let A be a graded division algebra (of type Γ) over F. Since Γ is totally ordered and all nonzero homogeneous elements of A are invertible, then A is a domain, so we can consider its quotient algebra (i.e., the algebra of central quotients of A) that we denote by $q(A)$. It is clear that $q(A)$ coincides with the quotient field of A when A is a graded field extension of F.

Let F be a graded field, A and B be two graded F-algebras (of the same type Γ), and let $f : A \to B$ be an F-algebra homomorphism. We say that f is a graded F-algebra homomorphism if for any $\gamma \in \Gamma$, we have $f(A_\gamma) \subseteq B_\gamma$. If f is a bijective graded F-algebra homomorphism, then we say that f is a graded F-algebra isomorphism and we write $A \cong_g B$. If in addition $A = B$, then we say that f is a graded F-algebra automorphism of A.

A finite-dimensional graded field extension L of F is called tame over F if L_0 is separable over F_0 and Γ_L/Γ_F has no p-torsion, where $p = \mathrm{char}(F)$. We recall that L is tame over F if and only if $q(L)$ is a separable field extension of $q(F)$ (see [1, Theorem 4] or [4, Proposition 3.5]). We recall also that L is called Galois over F if it is Galois over F when both L and F are considered as commutative rings. By [4, Theorem 3.11] L is Galois over F if and only if

$q(L)$ is Galois over $q(F)$. In such a case, denoting by $\mathrm{Gal}(L/F)$ the group of graded automorphisms of L, which preserve 'elementwise' the elements of F, we have $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(q(L)/q(F))$.

Let E be a field and v be a (Krull) valuation on E, then the filtration of E induced by v yields a canonical graded field GE. Namely, let $E^{\gamma} = \{x \in E \mid v(x) \geq \gamma\}$ and $E^{>\gamma} = \{x \in E \mid v(x) > \gamma\}$, then $E^{>\gamma}$ is a subgroup of the additive group E^{γ}. So, we can define the quotient group $GE_{\gamma} = E^{\gamma}/E^{>\gamma}$. For $x \in E\backslash\{0\}$, we denote by x' the element $x + E^{>v(x)}$ of $GE_{v(x)}$. One can easily see that the additive group $GE = \oplus_{\gamma\in\Gamma} GE_{\gamma}$ endowed with the multiplication law defined by $x'y' = (xy)'$ is a graded field.

In the same way, if D is a valued division algebra over a field E, then the filtration of D by the principal fractional ideals yields a graded division algebra GD over GE (see [2, §4], or [5, §4]).

Let (E, v) be a Henselian valued field. We recall that there is a (canonical) bijective map, induced by the correspondence $K \mapsto GK$, between the set of isomorphism classes of finite-dimensional tame field extensions of E and the set of isomorphism classes of finite-dimensional tame graded field extensions of GE. Moreover, for any such field extension K of E, K is a Galois tame (finite-dimensional) field extension of E if and only if GK is a Galois (finite-dimensional) graded field extension of GE, and in such a case $\mathrm{Gal}(K/E)$ is isomorphic to $\mathrm{Gal}(GK/GE)$ (see [4, Theorem 5.2] or [10, Corollary 1.13]).

1. Generalities

Throughout this work all considered graded objects will be assumed to have the same grading type, which is a totally ordered (uniquely) divisible abelian group Γ.

(1.1) Let R be a commutative graded ring, A be a graded algebra over R, A^* be the group of invertible homogeneous elements of A, H be a finite group that acts on A by graded ring automorphisms, $\mathrm{GAut}(A)$ be the group of graded ring automorphisms of A, and consider two maps: $\omega : H \to \mathrm{GAut}(A)$ and $f : H \times H \to A^*$, which satisfy the following conditions (for all $a \in R$ and $\sigma, \tau, \mu \in H$):

(1) $\omega_{\sigma}(a) = \sigma(a)$,

(2) $\omega_{\sigma}\omega_{\tau} = \mathrm{Int}(f(\sigma, \tau))\omega_{\sigma\tau}$,

(3) $f(\sigma, \tau)f(\sigma\tau, \mu) = \omega_{\sigma}(f(\tau, \mu))f(\sigma, \tau\mu)$.

Then, we say that (ω, f) is a graded factor set of H in A. We define the corresponding generalized graded crossed product $(A, H, (\omega, f))$ to be the ring: $(A, H, (\omega, f)) := \oplus_{\sigma\in H} Ax_{\sigma}$, where x_{σ} are independent indeterminates on A, with the addition law defined componentwise, and the multiplication defined by (extension of) the equalities: $x_{\sigma}a = \omega_{\sigma}(a)x_{\sigma}$ and $x_{\sigma}x_{\tau} = f(\sigma, \tau)x_{\sigma\tau}$ for all $a \in A$ and $\sigma, \tau \in H$. We say that f is normalized if $f(\sigma, 1) = f(1, \sigma) = 1$ for any $\sigma \in H$. In this case, we identify A with Ax_1, especially we identify the identity element x_1 of S with 1_A.

If we suppose that f is normalized, then for any $\sigma \in H$, we have $w_\sigma w_1 = \mathrm{Int}(f(\sigma,1))w_\sigma = w_\sigma$, so $w_1 = \mathrm{id}_A$. It follows that $w_\sigma w_{\sigma^{-1}} = \mathrm{Int}(f(\sigma,\sigma^{-1}))w_1 = \mathrm{Int}(f(\sigma,\sigma^{-1}))\mathrm{id}_A$. One can then easily see that for a graded ideal I of A, the following two conditions: $w_\sigma(I) \subseteq I$ for any $\sigma \in H$, and $w_\sigma(I) = I$ for any $\sigma \in H$, are equivalent.

Let (ω, f) and (β, h) be two graded factor sets of H in A. We say that (ω, f) and (β, h) are cohomologous if there is a family $(a_\sigma)_{\sigma\in H}$ of elements of A^* such that for all $\sigma, \tau \in H$, $\beta_\sigma = \mathrm{Int}(a_\sigma)\omega_\sigma$ and $h(\sigma,\tau) = a_\sigma\omega_\sigma(a_\tau)f(\sigma,\tau)a_{\sigma\tau}^{-1}$. We write in this case $(\omega, f) \sim (\beta, h)$. One can easily see that the relation $\sim$ is an equivalence relation on the set of factor sets of H in A.

Lemma 1.2. *With the above notation in* (1.1)*, there exists a unique graded algebra structure on* $(A, H, (\omega, f))$ *which extends the grading of* A *and for which all elements* x_σ *are homogeneous.*

Proof. Consider the mapping $h : H \times H \to \Gamma$, defined by $h(\sigma,\tau) = \mathrm{gr}(f(\sigma,\tau))$. By condition (3) in (1.1), it is clear that h is a cocycle of $Z^2(H,\Gamma)$; moreover, since H is finite and Γ is uniquely divisible, then $\mathrm{H}^2(H,\Gamma) = \mathrm{H}^1(H,\Gamma) = 0$, where $\mathrm{H}^2(H,\Gamma)$ and $\mathrm{H}^1(H,\Gamma)$ are respectively the second and the first cohomological groups of H in Γ (the action of H on Γ being trivial). Therefore, there is a unique family $(\gamma_\sigma)_{\sigma\in H}$ of elements of Γ such that $h(\sigma,\tau) = \gamma_\sigma + \gamma_\tau - \gamma_{\sigma\tau}$ (the uniqueness follows from the fact that $\mathrm{H}^1(H,\Gamma) = 0$). The unique graded structure of $(A, H, (\omega, f))$ which extends the grading of A and for which all elements x_σ are homogeneous, is then defined by setting $\mathrm{gr}(x_\sigma) = \gamma_\sigma$. □

(1.3) Conversely to (1.1), graded factor sets can be constructed from graded algebras. Namely, in [9, Lemma 2.4] we proved that if A is a graded simple algebra over a graded field F such that A_0 is simple, then there is a natural graded factor set (w, f) of $H := \Gamma_A/\Gamma_F$ in $A_0.F$ (with f possibly chosen normalized) such that A is graded isomorphic to the generalized graded crossed product $(A_0.F, H, (w, f))$. We recall that in this case, we have $\Gamma_A = \Gamma_A^*$, where $\Gamma_A^* := \mathrm{gr}(A^*)$ (A^* being the multiplicative group of invertible homogeneous elements of A as seen above). Examples of such graded simple algebras (with simple 0-component) are given by matrix algebras $M_n(D)$ where D is an arbitrary (finite-dimensional) graded central division algebra over a graded field.

In the same way, if F is a graded field and B a finite-dimensional graded central F-algebra such that $\Gamma_B = \Gamma_B^*$ and $H := \Gamma_B/\Gamma_F$ is finite, then by chosen invertible homogeneous elements $x_\sigma \in B^*$ with $x_1 = 1$ and $\mathrm{gr}(x_\sigma) + \Gamma_F = \sigma$ for all $\sigma \in H$, then $B = \oplus_{\sigma\in H} B_0 F x_\sigma = (B_0F, H, (w, f))$ where (w, f) is the graded factor set of H in B_0F, defined by the conditions: $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$ and $x_\sigma a = w_\sigma(a)x_\sigma$, for all $\sigma, \tau \in H$ and $a \in B_0F$. It is clear that $\mathrm{gr}(x_\sigma) + \Gamma_{B_0F} (= \mathrm{gr}(x_\sigma) + \Gamma_F)$ are pairwise distinct (for $\sigma \in H$). This last condition will be needed in the main result of this section (see Proposition 1.14). We fix now the following notation:

(1.4) Notation: Throughout the rest of this section, R is a commutative graded ring, A is a graded algebra over R, H is a finite group that acts on A by graded ring automorphisms, (w, f) is a graded factor set of H in A with f normalized, and $S = (A, H, (w, f))$ is the corresponding generalized graded crossed product.

Lemma 1.5. *Let A and S be as in* (1.4). *If we suppose that $\Gamma_A = \Gamma_A^*$, then the following statements are equivalent*:

(1) $S_0 = A_0$,

(2) *for a representation $S := \oplus_{\sigma \in H} Ax_\sigma$ as in* (1.1), *we have* $\mathrm{gr}(x_\sigma) + \Gamma_F$ *pairwise distinct* (*for distinct* $\sigma \in H$).

Proof. (1) $\Rightarrow$ (2) Let $\sigma, \tau \in H$ and suppose that $\mathrm{gr}(x_\sigma) + \Gamma_A = \mathrm{gr}(x_\tau) + \Gamma_A$, then $\mathrm{gr}(x_\sigma x_\tau^{-1}) \in \Gamma_A^*(= \Gamma_A)$. Let $a \in A^*$ such that $\mathrm{gr}(x_\sigma x_\tau^{-1}) = \mathrm{gr}(a)$, then $\mathrm{gr}(x_\sigma x_\tau^{-1} a^{-1}) = 0$. So, $x_\sigma x_\tau^{-1} a^{-1} \in A_0(= S_0)$, which means $x_\sigma \in A_0.(ax_\tau) \subseteq Ax_\tau$. Therefore, $\sigma = \tau$.

(2) $\Rightarrow$ (1) Since $\mathrm{gr}(x_1) = 0$ and $\mathrm{gr}(x_\sigma) + \Gamma_A$ are pairwise distinct, then $S_0 \subseteq A_0$ (by identification of A_0 with its canonical image $A_0 x_1$ in S), so $S_0 = A_0$. □

(1.6) Let $A, H, (w, f)$ and S be as in (1.4). We will say that S satisfies the grading separation property (GSP) with respect to (w, f), if there is some representation of S as in (1.1), say $S := \oplus_{\sigma \in H} Ax_\sigma$, with $\mathrm{gr}(x_\sigma) + \Gamma_F$ pairwise distinct (for distinct $\sigma \in H$). Note that in this case, the homogeneous elements of S are the elements ax_σ, where a is a homogeneous element of A and $\sigma \in H$. Also, in this case, we have $S_0 = A_0$.

(1.7) We saw above in (1.3) that graded simple algebras with simple 0-component satisfy the graded separation property (GSP). We give here another example of generalized graded crossed products satisfying this property. This example is based on Malcev-Neumann construction for his power series division rings. Let $\Gamma_1 \subseteq \Gamma_2$ be an extension of totally ordered abelian groups with $H := \Gamma_2/\Gamma_1$ finite (one can take for example $\Gamma_1 = m_1\mathbb{Z} \times \cdots \times m_r\mathbb{Z}$ and $\Gamma_2 = \mathbb{Z}^r$, where $m_1, \ldots, m_r$ are nonnegative integers, and let Γ_1 and Γ_2 be ordered by the anti-lexicographic order). Consider a factor set (v, g) of Γ_2 in a ring B, with g normalized, and let $S := (B, \Gamma_2, (v, g)) = \oplus_{\mu \in \Gamma_2} Bx_\mu$, where x_μ are independent indeterminates over B satisfying the conditions: $x_\mu x_{\mu'} = g(\mu, \mu')x_{\mu+\mu'}$ and $x_\mu b = v_\mu(b)x_\mu$ for all $\mu, \mu' \in \Gamma_2$ and $b \in B$. Let $A := \oplus_{\mu \in \Gamma_1} Bx_\mu$ $(= (B, \Gamma_1, (v, g)))$, and for any element $\overline{\delta} \in H$, choose a representative δ of $\overline{\delta}$ in Γ_2 and let $y_{\overline{\delta}} := x_\delta$. Then, we have $S = \oplus_{\overline{\delta} \in H} Ay_{\overline{\delta}} = (A, H, (w, f))$, where (w, f) is the graded factor set of H in A defined by the equalities $y_{\overline{\delta}} a = w_{\overline{\delta}}(a) y_{\overline{\delta}}$ and $y_{\overline{\delta}} y_{\overline{\delta}'} = f(\overline{\delta}, \overline{\delta}') y_{\overline{\delta}+\overline{\delta}'}$ for any $\overline{\delta}, \overline{\delta}' \in H$ and $a \in A$. It is clear that $\mathrm{gr}(y_{\overline{\delta}}) + \Gamma_1 (= \overline{\delta})$ are pairwise distinct (for distinct $\overline{\delta}$ in H).

(1.8) Let $A, (w, f)$ and S as in (1.4). A graded ideal I of A will be called a graded w-ideal (or a w-invariant graded ideal) if for any $\sigma \in H$, we have

$w_\sigma(I) = I$. As seen in (1.1) this condition is equivalent to have $w_\sigma(I) \subseteq I$ for all $\sigma \in H$.

We will say that I is graded w-prime if for any graded w-ideals I_1, I_2 of A such that $I_1 I_2 \subseteq I$, we have $I_1 \subseteq I$ or $I_2 \subseteq I$. We will say that I is graded w-semiprime if for any graded w-ideal J of A such that $J^2 \subseteq I$, we have $J \subseteq I$.

A graded ideal I of S is called graded prime (resp., graded semiprime) if the condition above holds for graded ideals I_1, I_2 [resp., J] of S (without assuming that they are graded w-ideals).

We will say that A is a graded w-simple algebra if the only graded w-ideals of A are 0 and A.

The graded ring A is said to be graded w-prime (resp., graded w-semiprime) if 0 is graded w-prime (resp., graded w-semiprime). Similarly, S is called graded prime (resp., graded semiprime) if the graded ideal 0 is graded prime (resp., graded semiprime).

We will say that A is graded local if it has a unique maximal right graded ideal.

For a subset T consisting of homogeneous elements of A, we will write $\mathrm{Ann}^g_{A-l}(T)$ for the left annulator of T in A (which is a left graded ideal of A). We will say that A is w-compatible, if for any subset T consisting of homogeneous elements of A and any $\tau \in H$, we have $\mathrm{Ann}^g_{A-l}(T) = \mathrm{Ann}^g_{A-l}(w_\sigma(T))$.

A graded ring B is called graded Baer [resp., graded quasi-Baer] if the left annihilator of any nonempty subset consisting of homogeneous elements of B [resp., of any left graded ideal of B] is generated by a homogeneous idempotent.

We say that B is graded regular if for any homogeneous element x of B, there exists a homogeneous element y of B such that $x = xyx$.

Before giving some properties of the generalized graded crossed products, we show the following lemmas.

Lemma 1.9. *Let S be a generalized graded crossed product as in* (1.4) *and suppose that S satisfies the GSP, then for any graded ideal J of S, we have $J = (J \cap A).S$. Conversely, let I be a graded w-ideal of A and let J_I be the graded ideal of S generated by the homogeneous elements of I, then $I = J_I \cap A$.*

Proof. It is clear that $(J \cap A).S \subseteq J$, so it suffices to prove that we have $J \subseteq (J \cap A).S$. Let $a_\gamma x_\sigma$ be a homogeneous element of J, then we have

$$\begin{aligned}(a_\gamma x_\sigma)(x_{\sigma^{-1}} f(\sigma, \sigma^{-1})^{-1}) &= a_\gamma f(\sigma, \sigma^{-1}) x_1 f(\sigma, \sigma^{-1})^{-1} \\ &= a_\gamma f(\sigma, \sigma^{-1}) w_1(f(\sigma, \sigma^{-1})^{-1}) x_1 \\ &= a_\gamma x_1.\end{aligned}$$

By identification of A with Ax_1 in S, we get $a_\gamma \in J \cap A$. So, $J \subseteq (J \cap A).S$.

Conversely, let I be a graded w-ideal of A and let J_I be the graded ideal of S generated by the homogeneous elements of I. For any $a, b \in A$, $c \in I$ and $\sigma, \tau \in H$, we have $(ax_\sigma)c(bx_\tau) = aw_\sigma(cb)f(\sigma, \tau)x_{\sigma\tau}$ with $aw_\sigma(cb)f(\sigma, \tau) \in I$. One can then easily deduce that $J_I \cap A = I$. □

Lemma 1.10. *Let $A, (w, f)$ and S be as in* (1.4)*. Then the following statements are equivalent*:

(1) *A is graded w-prime.*
(2) *For any nonzero graded w-ideal I of A, we have $\mathrm{Ann}^g_{A-l}(I) = 0$.*
(3) *For any homogeneous elements a, b in A such that $w_\sigma(a).A.w_\tau(b) = 0$ for all $\sigma, \tau \in H$, we have $a = 0$ or $b = 0$.*

Proof. (1) $\Rightarrow$ (2) Let I be a nonzero graded w-ideal of A. One can easily see that $\mathrm{Ann}^g_{A-l}(I)$ is a graded w-ideal of A. Indeed, let a be an arbitrary homogeneous element of I and take an element b of $\mathrm{Ann}^g_{A-l}(I)$. For any $\tau \in H$, we have $w_\tau(b)a = w_\tau(b)w_\tau(w_\tau^{-1}(a)) = w_\tau(bw_\tau^{-1}(a))$. Note that we have $w_\tau w_{\tau^{-1}} = \mathrm{Int}(f(\tau, \tau^{-1}))$, so $w_\tau^{-1} = w_{\tau^{-1}} \circ \mathrm{Int}(f(\tau, \tau^{-1})^{-1})$. Therefore $w_\tau^{-1}(a) \in I$ (because I is a graded w-ideal), hence $w_\tau(b)a(= w_\tau(bw_\tau^{-1}(a))) = 0$. It follows then that $w_\tau(b) \in \mathrm{Ann}^g_{A-l}(I)$, so $w_\tau(\mathrm{Ann}^g_A(I)) \subseteq \mathrm{Ann}^g_A(I)$. Now, we have $\mathrm{Ann}^g_{A-l}(I).I = 0$ with $I \neq 0$ and A graded w-prime, so necessarily $\mathrm{Ann}^g_{A-l}(I) = 0$.

(2) $\Rightarrow$ (3) Let a, b be two homogeneous elements of A and suppose that $w_\sigma(a).A.w_\tau(b) = 0$ for any $\sigma, \tau \in H$. Let I be the graded ideal of A generated by the elements $w_\tau(b)$, where τ describes H. Then, I is a graded w-ideal and $a \in \mathrm{Ann}^g_{A-l}(I)$. If $b \neq 0$, then $I \neq 0$, so necessarily $a = 0$.

(3) $\Rightarrow$ (1) Let I_1 and I_2 be two nonzero graded w-ideals of A and let a_i be a nonzero homogeneous element of I_i ($1 \leq i \leq 2$), then there exist $\sigma, \tau \in H$ such that $w_\sigma(a).A.w_\tau(b) \neq 0$. We then have $0 \neq w_\sigma(a).A.w_\tau(b) = w_\sigma(a).(A.w_\tau(b)) \subseteq I_1 I_2$. □

Analogously, using the same arguments, one can easily prove the following lemma.

Lemma 1.11. *Let $A, (w, f)$ and S be as in* (1.4)*. Then the following statements are equivalent*:

(1) *A is graded w-semiprime.*
(2) *For any homogeneous element a in A such that $w_\sigma(a).A.w_\tau(a) = 0$ for all $\sigma, \tau \in H$, we have $a = 0$.*

It is well known that for any ring B with (Jacobson) radical $\mathrm{rad}(B)$, if $b + \mathrm{rad}(B)$ is an idempotent of $B/\mathrm{rad}(B)$, then there exists an idempotent $a \in B$ such that $a - b \in \mathrm{rad}(B)$. Analogously, in the graded setting, if A is a graded ring with (Jacobson) graded radical $\mathrm{rad}^g(A)$ (i.e., $\mathrm{rad}^g(A)$ is the intersection of all maximal right graded ideals of A), then we have the following result[1].

Lemma 1.12. *Let F be a graded field and A a graded F-algebra. For any homogeneous idempotent $f + \mathrm{rad}^g(A)$ of $A/\mathrm{rad}^g(A)$ there is an idempotent e of A_0 such that $f - e \in \mathrm{rad}^g(A)$.*

[1] Note that all homogeneous idempotents of a graded ring R are in R_0 (because Γ is totally ordered).

Proof. One can easily see that $(\mathrm{rad}^g(A))_0 = \mathrm{rad}(A_0)$. Since $f + \mathrm{rad}^g(A)$ is a homogeneous idempotent of $A/\mathrm{rad}^g(A)$, then without loss of generality, we can assume that $f \in A_0$. It follows that $f + \mathrm{rad}(A_0)$ is an idempotent element of $A_0/\mathrm{rad}(A_0)$. So, by the above there is an idempotent element e in A_0 such that $f - e \in \mathrm{rad}(A_0)$, hence $f - e \in \mathrm{rad}^g(A)$. □

Lemma 1.13. *Let F be a graded field and A a finite-dimensional graded F-algebra. Then the following statements are equivalent*:

(1) *A is graded local.*
(2) *A has a unique maximal left graded ideal.*
(3) *The set of noninvertible homogeneous elements of A generate a proper two-sided graded ideal of A.*
(4) *For any element a of A_0, one of the elements a or $1 - a$ is invertible.*
(5) *A has only two homogeneous idempotents, 0 and 1.*
(6) *The graded F-algebra $A/\mathrm{rad}^g(A)$ is a graded division algebra.*
(7) *A_0 is a local algebra.*

Proof. (1) $\Rightarrow$ (3) By definition, $\mathrm{rad}^g(A)$ is the unique proper maximal (right) graded ideal of A. So, for any homogeneous element a of A, we have $a \in \mathrm{rad}^g(A)$ if and only if a has no right inverse. Let x be a nonzero homogeneous element of A with a right inverse y (in $A_{-\mathrm{gr}(x)}$), then we have $(1 - yx)y = y - yxy = y(1 - xy) = 0$. If y has no right inverse, then by the above $y \in \mathrm{rad}^g(A)$, so $1 - yx$ is invertible (this property of the Jacobson graded radical can be proved as in the ungraded case), hence $y = 0$ (because $(1 - yx)y = 0$ as seen above), but this is not true. Therefore, y has a right inverse, and so $1 - yx = 0$. This shows that x is invertible. We conclude that a homogeneous element a of A is in $\mathrm{rad}^g(A)$ if and only if a has no right inverse if and only if a is not invertible. Thus, $\mathrm{rad}^g(A)$ is the graded ideal of A generated by noninvertible homogeneous elements of A. Plainly, $\mathrm{rad}^g(A)$ is proper in A.

(2) $\Rightarrow$ (3) It follows in the same way.

(3) $\Rightarrow$ (4) Let a be a nonzero element of A_0 and let I be the (proper) graded ideal of A generated by noninvertible homogeneous elements of A. If both a and $1 - a$ are noninvertible, then $1 = a + (1 - a) \in I$, a contradiction.

(4) $\Rightarrow$ (5) If e is a homogeneous idempotent of A, then $e \in A_0$, and we have $e(1 - e) = 0$, so necessarily $e = 0$ or $e = 1$.

(5) $\Rightarrow$ (6) by Lemma 1.12, the homogeneous idempotents of $A/\mathrm{rad}^g(A)$ can be lifted modulo $\mathrm{rad}^g(A)$, so the graded semisimple algebra $C := A/\mathrm{rad}^g(A)$ has only two homogeneous idempotents 0 and 1. It follows by the graded version of Wedderburn's theorem that $A/\mathrm{rad}^g(A)$ is a graded division algebra (see [5, Proposition 1.3] for the graded version of Wedderburn's theorem on graded simple algebras).

(6) $\Rightarrow$ (1) [resp., (6) $\Rightarrow$ (2)] This is clear since in this case $\mathrm{rad}^g(A)$ is the unique maximal right [resp., left] graded ideal of A.

In the same way, we show that A_0 is a local algebra if and only if 0 and 1 are the only idempotent of A_0. Hence, (1) $\Leftrightarrow$ (7). □

The following proposition summarizes some facts relating properties of a generalized graded crossed product $S = (A, H, (w, f))$ which satisfies the grading separation property, to analogous ones on A.

Proposition 1.14. *Let $A, (w, f)$ and S be as in* (1.4) *and suppose that S satisfies the GSP. Then we have the following statements*:

(1) *S is graded simple if and only if A is graded w-simple.*
(2) *S is graded semiprime [resp., graded prime] if and only if A is graded w-semiprime [resp., graded w-prime].*
(3) *S is graded local if and only if A is so if and only if A_0 is local.*
(4) *If A is w-compatible, then S is graded Baer [resp. graded quasi-Baer] if and only if A is so.*

Proof. (1) Suppose that S is a graded simple algebra and let I be a graded w-ideal of A and J_I be the graded ideal of S generated by the homogeneous elements of I, then J_I is either 0 or S. By Lemma 1.9 we have $J_I \cap A = I$, so I is either 0 of A.

Conversely, suppose that A is a graded w-simple algebra, and let J be a graded ideal of S. One can easily see that $J \cap A$ is a graded w-ideal of A (indeed, for any homogeneous element $a \in J \cap A$ and any $\sigma \in H$, we have $w_\sigma(a) = x_\sigma a x_\sigma^{-1} \in J \cap A$). Therefore, $J \cap A$ is either 0 or A. So, by Lemma 1.9 J is then 0 or S.

(2) This follows easily from Lemmas 1.10 and 1.11.

(3) This is clear from Lemma 1.13 (since $S_0 = A_0$).

(4) We will show that S is graded quasi-Baer if and only if A is so. The fact that S is graded Baer if and only if A is so follows in a similar way. Suppose that A is graded quasi-Baer and let I be a graded left ideal of S. For a nonzero element p of S, we denote by $\mathrm{mincomp}(p)$ the homogeneous component $a_\gamma x_\sigma$ of minimal grade of p. Let's consider the left graded ideal J of A generated by the elements a_γ, where $a_\gamma x_\sigma = \mathrm{mincomp}(p)$ for some $\sigma \in H$ and p in I. Let's also consider the following sets: $T := \{a_\gamma \mid a_\gamma x_\sigma = \mathrm{mincomp}(p)$ for some $\sigma \in H$ and $p \in I\}$, and for $\tau \in H$, $T_\tau := w_\tau(T) = \{w_\tau(a_\gamma) \mid a_\gamma x_\sigma = \mathrm{mincomp}(p)$ for some $\sigma \in H$ and $p \in I\}$. By assumption we have $\mathrm{Ann}^g_{A-l}(T) = \mathrm{Ann}^g_{A-l}(T_\tau)$.

It is clear that the (left) annulator $\mathrm{Ann}^g_{A-l}(J)$ is contained in $\mathrm{Ann}^g_{A-l}(T)$. Conversely, let $x \in \mathrm{Ann}^g_{A-l}(T)$ and let c be a homogeneous element of A and $a_\gamma \in T$ with $a_\gamma x_\sigma = \mathrm{mincomp}(p)$ for some $\sigma \in H$ and $p \in I$. If $ca_\gamma \neq 0$, then $ca_\gamma x_\sigma = \mathrm{mincomp}(cp)$, so $x(ca_\gamma) = 0$. For an arbitrary element c of A, write $c = \sum c_\lambda$, where c_λ are homogeneous elements of A, then $x(ca_\gamma) = \sum x(c_\lambda a_\gamma) = 0$, so $x \in \mathrm{Ann}^g_{A-l}(J)$. Therefore, $\mathrm{Ann}^g_{A-l}(J) = \mathrm{Ann}^g_{A-l}(T)$.

Since A is graded quasi-Baer, then there is a homogeneous idempotent $e \in A$ (hence $e \in A_0$) such that $\mathrm{Ann}^g_{A-l}(J) = A.e$. Let q be an arbitrary nonzero element of $\mathrm{Ann}^g_{S-l}(I)$ and write $q = b_\lambda x_\tau + q_1$, where $b_\lambda x_\tau = \mathrm{mincomp}(q)$. For any $0 \neq p \in I$, we have $(b_\lambda x_\tau)\mathrm{mincomp}(p) = 0$. Write $\mathrm{mincomp}(p) = a_\gamma x_\sigma$ (for some $\gamma \in \Gamma$, $\sigma \in H$ and $a_\gamma \in A_\gamma$), then we have

$b_\lambda w_\tau(a_\gamma)f(\tau,\sigma)x_{\tau\sigma} = 0$, so $b_\lambda w_\tau(a_\gamma) = 0$. Therefore, $b_\lambda \in \mathrm{Ann}^g_{A-l}(T_\tau)(= \mathrm{Ann}^g_{A-l}(J))$, so there is a homogeneous element r_q of A such that $b_\lambda = r_q e$. We then have $\mathrm{mincomp}(q) = r_q e x_\tau = r_q x_\tau w_\tau^{-1}(e)$. We have $w_\tau(e) = x_\tau e x_\tau^{-1}$, so $e = w_\tau^{-1}(x_\tau)w_\tau^{-1}(e)w_\tau^{-1}(x_\tau)^{-1}$, which implies $w_\tau^{-1}(e) = w_\tau^{-1}(x_\tau^{-1})ew_\tau^{-1}(x_\tau)$, hence $w_\tau^{-1}(e) \in \mathrm{Ann}^g_{A-l}(J)$, so $w_\tau^{-1}(e) = s_q e$ for some homogeneous element s_q of A, hence $\mathrm{mincomp}(q) = r_q x_\tau s_q e$. On the other hand, one can easily see that $e \in \mathrm{Ann}^g_{S-l}(I)$. Indeed, for any $p \in I$, if $ep \neq 0$, then $0 = e\mathrm{mincomp}(p) = \mathrm{mincomp}(ep) \neq 0$, a contradiction.

Let $q_1 := q - \mathrm{mincomp}(q) = q - r_q x_\tau s_q e$, then $q_1 \in \mathrm{Ann}^g_{S-l}(I)$. If we continue in this way, we get $q \in S.e$. This shows that S is graded quasi-Baer.

Conversely, suppose that S is graded quasi-Baer and let J be a left graded ideal of A. Plainly, SJ is a left graded ideal of S. Therefore, there is a homogeneous idempotent e in S such that $\mathrm{Ann}^g_{S-l}(SJ) = S.e$. The grading group being totally ordered, then $e \in A_0(= S_0)$. It is clear that $A.e \subseteq \mathrm{Ann}^g_{A-l}(J)$. Conversely, let r be a homogeneous element of $\mathrm{Ann}^g_{A-l}(J)$, b be a homogeneous element of A, a be a homogeneous element of J and $\tau \in H$, then $rb \in \mathrm{Ann}^g_{A-l}(J)$, so $rbw_\tau(a) = 0$ (because A is graded w-compatible), thus $r((bx_\tau)a) = (rb)w_\tau(a)x_\tau = 0$. Consequently, for any $p \in S$, we have $r(pa) = 0$, which shows that $r \in \mathrm{Ann}^g_{S-l}(SJ)$, hence $r = se$ for some homogeneous element s of S. Since r is a homogeneous element of A and $e \in A_0$, then $s = tx_1$ for some homogeneous element t of A. Therefore, by identification of A with Ax_1, $r \in A.e$. Thus, $\mathrm{Ann}^g_{A-l}(J) \subseteq A.e$, so $\mathrm{Ann}^g_{A-l}(J) = A.e$. □

Remark 1.15. Under the hypotheses of Proposition 1.14, we show also that S is graded semisimple if and only if A is so. In fact, as in the ungraded case S [resp., A] is graded semismiple if and only if it is graded regular and every subset consisting of orthogonal idempotents of $S_0(= A_0)$ is finite. So, it is sufficient to show that S is graded regular if and only if A is so. This follows easily by computations. Indeed, suppose that A is graded regular, and let y be a homogeneous element of S, then we can write $y = a_\gamma x_\sigma$ for some $\gamma \in \Gamma$, $\sigma \in H$ and $a_\gamma \in A_\gamma$. Since A is graded regular, then there is a homogeneous element e of A such that $a_\gamma = a_\gamma e a_\gamma$. Let $z = w_\sigma^{-1}(ef(\sigma,\sigma^{-1})^{-1})x_{\sigma^{-1}}$, then we have

$$\begin{aligned} yzy &= (a_\gamma x_\sigma)[w_\sigma^{-1}(ef(\sigma,\sigma^{-1})^{-1})x_{\sigma^{-1}}](a_\gamma x_\sigma) \\ &= a_\gamma ef(\sigma,\sigma^{-1})^{-1}f(\sigma,\sigma^{-1})x_1 a_\gamma x_\sigma \\ &= a_\gamma e a_\gamma x_\sigma = a_\gamma x_\sigma = y, \end{aligned}$$

which shows that S is graded regular.

Conversely, suppose that S is graded regular and let a be a homogeneous element of A, then there is a homogeneous element cx_τ of S, where c is a homogeneous element of A, such that $a(cx_\tau)a = a$. So, $acw_\tau(a)x_\tau = a$, hence $\tau = 1$ and $aca = a$. This shows that A is graded regular.

2. Embedding of graded simple algebras

(2.1) Let F be a graded field of type Γ and λ be an element of Γ. We recall that the shifted graded F-space $F_{s(\lambda)}$ is obtained from F by shifting homogeneous elements by λ, i.e., $F_{s(\lambda)}$ equals to F as a set and $(F_{s(\lambda)})_\gamma = F_{\gamma+\lambda}$ for all $\gamma \in \Gamma$. For a positive integer n and elements $\delta_1, \dots, \delta_n$ of Γ, we let $M_n(F)(\delta_1, \dots, \delta_n)$ denote the following split graded central simple algebra (i.e., matrix graded algebra):

$$M_n(F)(\delta_1, \dots, \delta_n) = \begin{pmatrix} F_{s(\delta_1-\delta_1)} & \cdots & F_{s(\delta_1-\delta_n)} \\ \vdots & \ddots & \vdots \\ F_{s(\delta_n-\delta_1)} & \cdots & F_{s(\delta_n-\delta_n)} \end{pmatrix}.$$

This means that a nonzero homogeneous element of grade γ of $M_n(F)(\delta_1, \dots, \delta_n)$ is a matrix with ij-entry in $(F_{s(\delta_i-\delta_j)})_\gamma (= F_{\delta_i-\delta_j+\gamma})$. If A is a graded F-algebra (of type Γ), we define $M_n(A)(\delta_1, \dots, \delta_n)$ in a similar way. We will also denote $M_n(F)(\delta_1, \dots, \delta_n)$ simply by $M_n(F)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$.

(2.2) Now, let R be a ring and M an abelian group with endomorphism ring $\mathrm{End}(M)$ (acting on M on the right with the multiplication law in $\mathrm{End}(M)$ being the opposite of the usual composition law), then a right R-module action on M is equivalent to a ring homomorphism $\phi : R \to \mathrm{End}(M)$. The two conditions are related by the equation:

$$m.\phi(r) = m.r$$

for $m \in M$, $r \in R$, which defines the module action when ϕ is given and conversely define ϕ when the module action is given. In the graded setting, if F is a graded field of type Γ, $\delta_1, \dots, \delta_n$ are elements of Γ, and M is the graded F-vector space $F_{s(\delta_1)} \oplus \cdots \oplus F_{s(\delta_n)}$ (where $F_{s(\lambda)}$ is the shifted of F by λ as seen above), then $\mathrm{End}(M)$ is a graded F-algebra that we will denote by $\mathrm{GEnd}(M)$, and we have $\mathrm{GEnd}(M) \cong_g \mathrm{M}_n(F)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$ (see e.g., [14, Proposition 2.9, p. 41 (see also Proposition 2.8, p. 39)]). Thus, if A is a graded F-algebra, the fact that we have a graded F-algebra homomorphism $\phi : A \to \mathrm{M}_n(F)(\overline{\delta})$, is equivalent to having a graded right A-module structure on M (compatible with the action of F on M).

If $u_1, \dots, u_n$ is a base of A over F, consisting of homogeneous elements of A, then as a graded vector space over F, A is isomorphic to the graded F-vector space $M := F_{s(\delta_1)} \oplus \cdots \oplus F_{s(\delta_1)}$, where $\delta_i = -\mathrm{gr}(u_i)$ for all i $(1 \le i \le n)$. Since we have a natural right A-module structure on A (hence on M), then there is a graded F-algebra homomorphism $\phi : A \to \mathrm{M}_n(F)(\overline{\delta})$, which is clearly injective (for $\ker(\phi) = \mathrm{ann} A_A = 0$).

Corollary 2.3. *Let F be a graded field, A be a graded central division algebra over F, L be a finite-dimensional graded field extension of F and S be a graded central simple algebra over L. If $\Gamma_S \subseteq \Gamma_A$, then there is a graded monomorphim of graded F-algebras from S into $M_n(A)$ for some positive integer n.*

Proof. Let $u_1, \dots, u_n$ be a base consisting of homogeneous elements of S over F and let $\delta_i = -\mathrm{gr}(u_i)$. By the above there is a graded F-algebra monomorphism from S into $M_n(F)(\delta_1, \dots, \delta_n)$. Therefore, there is graded F-algebra monomorphism from S into $M_n(A)(\delta_1, \dots, \delta_n)$. Since $\delta_1, \dots, \delta_n$ belongs to Γ_A, then by [5, (ii), p. 78] $M_n(A)(\delta_1, \dots, \delta_n) \cong_g M_n(A)(= M_n(A)(0, \dots, 0))$. Therefore, there is a graded F-algebra monomorphism from S into $M_n(A)$. □

(2.4) Let F be a graded field, L a finite-dimensional graded field extension of F, A a graded central division algebra over F and S a graded central simple algebra over L. If $\Gamma_S \subseteq \Gamma_A$, then by Corollary 2.3 there is some positive integer t such that S is graded isomorphic to a graded subfield of $M_t(A)$. Suppose that $\Gamma_S \subseteq \Gamma_A$ and let s be the smallest positive integer such that S embeds in $M_s(A)$ (as a graded ring). Inspired by [6, Proposition 2.1], we will show here that $s = [L : F]\deg(S)\mathrm{ind}(A \otimes_F S)/\deg(A)$. For this consider the graded simple algebra $C_{M_s(A)}(S)$ (where S is considered as a subring of $M_s(A)$). Since S is simple, then by the graded version of the double centralizer theorem [5, Proposition 1.5] $C^{(S)}_{M_s(A)}$ is also graded simple, therefore by the graded version of the Wedderburn Theorem (see [5, Proposition 1.3]), there exists a graded division algebra R, a positive integer m and some $\overline{\delta} = (\delta_1, \dots, \delta_m) \in \Gamma^m$ such that $C_{M_s(A)}(S) \cong_g M_m(R)(\overline{\delta})$ (see [5, Proposition 1.3]). Consider the graded central simple F-algebra $C := M_m(F)(\overline{\delta})$. Obviously, C embeds in $M_m(R)(\overline{\delta})$, so C can be considered as a graded simple subalgebra of $C_{M_s(A)}(S)$, hence of $M_s(A)$. Let $B = C_{M_s(A)}(C)$, then again by the graded version of the double centralizer theorem, we have $M_s(A) \cong_g C \otimes_F B$. In particular, $\Gamma_B \subseteq \Gamma_{M_s(A)} = \Gamma_A$. Note that C is (graded) Brauer-equivalent to F, so B is (graded) Brauer-equivalent to A. It follows that $\Gamma_B = \Gamma_A$ (because $\Gamma_B \subseteq \Gamma_A$ and B is (graded) Brauer-equivalent to A). Therefore, $B \cong_g M_l(A)$ for some positive integer l (see [5, Proposition 1.3, p. 81 and (1.4), p. 78]). If $C \neq F$ (i.e., $m \neq 1$), then necessarily $l < s$, but this contradicts the fact that s is minimal (see that $C \subseteq C_{M_s(A)}(S)$, so $S \subseteq B$). Thus $m = 1$, so $C_{M_s(A)}(S)(\cong_g M_m(R)(\overline{\delta}) = R)$ is a graded division algebra. As in the ungraded case, $C_{M_s(A)}(S)$ is (graded) Brauer-equivalent to $M_s(A) \otimes_F S^{\mathrm{op}}$, hence Brauer-equivalent to $A \otimes_F S^{\mathrm{op}}$. Therefore, $\deg(C_{M_s(A)}(S)) = \mathrm{ind}(A \otimes_F S^{\mathrm{op}})$.

Note that by the graded version of the double centralizer theorem, we have $[C_{M_s(A)}(S) : F][S : F] = [M_s(A) : F] = s^2\deg(A)^2$. Therefore, $[C_{M_s(A)}(S) : L][S : L][L : F]^2 = s^2\deg(A)^2$, so $s = [L : F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$. We get then the following proposition.

Proposition 2.5. *Let F be a graded field, L a finite-dimensional graded field extension of F, A a graded central division algebra over F and S a graded central simple algebra over L with $\Gamma_S \subseteq \Gamma_A$ and n a positive integer. Then, S embeds (as a graded ring) in $M_n(A)$ if and only if n is a multiple of $[L : F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$.*

Proof. Let $s = [L : F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$. We saw in (2.4) that S embeds in $M_s(A)$, so for any multiple n of s, it embeds in $M_n(A)$.

Conversely, suppose that S embeds in $M_n(A)$, then again as in the ungraded case $C_{M_n(A)}(S)$ is (graded) Brauer-equivalent to $A \otimes_F S^{\mathrm{op}}$, so it is (graded) Brauer-equivalent to the graded division algebra $C_{M_s(A)}(S)$. It follows that $\deg(C_{M_n(A)}(S))$ is a multiple of $\deg(C_{M_s(A)}(S))$. As in (2.4), by using the graded version of the double centralizer theorem, we get $n = [L : F]\deg(S)\deg(C^{(S)}_{M_n(A)})/\deg(A)$, so n is a multiple of s. □

Corollary 2.6. *Let E be a Henselian valued field, D be a tame central division algebra over E, n a positive integer, and K a tame finite-dimensional field extension of E such that $\Gamma_K \subseteq \Gamma_D$, then the following statements are equivalent*:

(1) *K embeds in $M_n(D)$.*
(2) *G_wK embeds in $M_n(GD)$, where w is the extension of the valuation of E to K.*

Proof. Let s be the smallest positive integer such that K embeds in $M_s(D)$. Since ungraded algebras can be considered as (trivially) graded algebras, then by Proposition 2.5, we have $s = [K : E]\mathrm{ind}(D \otimes_E K)/\deg(D)$. Note that because K is tame over E, then it is defectless over E, so $[K : E] = [\overline{K} : \overline{E}](\Gamma_K : \Gamma_E) = [GK : GE]$ (because $\overline{K} = (GK)_0$ and $\Gamma_K = \Gamma_{GK}$). Moreover, by [5, Corollary 5.7] $\mathrm{ind}(D \otimes_E K) = \mathrm{ind}(GD \otimes_{GE} G_wK)$, and obviously we have $\deg(D) = \deg(GD)$. Therefore, s is also the smallest positive integer such that G_wK embeds in $M_s(GD)$. Thus, again by Proposition 2.5, for any positive integer n, K embeds in $M_n(D)$ if and only if G_wK embeds in $M_n(GD)$. □

Proposition 2.5 can be applied also to give graded versions, with alternative proofs, to many results in [3]. Namely, we have the following proposition.

Proposition 2.7 (Compare [3, Proposition 2])**.** *Let L/F be a finite-dimensional graded field extension, R be a graded central division algebra over L, A a graded central division algebra over F with $\Gamma_R = \Gamma_A$, and m, n be two positive integers and suppose that $\Gamma_R \subseteq \Gamma_A$. If $M_m(R)$ embeds, as a graded ring, in $M_n(A)$, then m divides n and R embeds, as a graded ring, in $M_k(A)$, where $k = n/m$.*

Proof. Let r be the minimal positive integer l such that R embeds in $M_l(A)$, then by Proposition 2.5, $r = \frac{\deg(R)[L:F]\mathrm{ind}(A \otimes_F R^{\mathrm{op}})}{\deg(A)}$. For a positive integer m, let $S := M_m(R)$, then $mr(= \frac{\deg(S)[L:F]\mathrm{ind}(A \otimes_F S^{\mathrm{op}})}{\deg(A)})$, is the smallest positive integer t such that S embeds in $M_t(A)$. If S embeds in $M_n(A)$ for some positive integer n, then again by Proposition 2.5, n is a multiple of mr, so a multiple of m. Plainly, in this case n/m is a multiple of r, so R embeds in $M_{n/m}(A)$. □

3. Kummer subfields of simple algebras

(3.1) Let F be a graded field and L be a finite-dimensional abelian graded field extension of F such that $\mathrm{char}(F)$ does not divide $[L : F]$. We recall that L is a Kummer graded field extension of F if F_0 contains a primitive m^{th} root of unity, where m is the exponent of $\mathrm{Gal}(L/F)$. In such a case, we have $L = F[a \mid a \in \mathrm{KUM}(L/F)]$, where $\mathrm{KUM}(L/F) = \{x \in L^* \mid x^m \in F\}$ (see [8, (2.1)]), so Γ_L/Γ_F is generated by $\{\mathrm{gr}(a) + \Gamma_F \mid a \in \mathrm{KUM}(L/F)\}$. Therefore, if we set $\mathrm{kum}(L/F) = \mathrm{KUM}(L/F)/F^*$, then the group homomorphism $\psi : \mathrm{kum}(L/F) \to \Gamma_L/\Gamma_F$, defined by $\psi(aF^*) = \mathrm{gr}(a) + \Gamma_F$, for $a \in \mathrm{KUM}(L/F)$, is surjective. Note that in this case the graded subfield L_0F of L is a Kummer graded field extension of F. Moreover, since L_0F is unramified over F, then by applying [4, Remark 3.1] L_0 is a Kummer field extension of F_0. Let $\phi : \mathrm{kum}(L_0/F_0) \to \mathrm{kum}(L/F)$ be the group homomorphism defined by $\phi(aF_0^*) = aF^*$, for every $a \in \mathrm{KUM}(L_0/F_0)$, then clearly ϕ is injective. Also, we have $\psi \circ \phi = 0$, and by comparing the cardinalities of the terms in the following sequence of trivial Γ_L/Γ_F-modules: $\alpha_L : 1 \to \mathrm{kum}(L_0/F_0) \xrightarrow{\phi} \mathrm{kum}(L/F) \xrightarrow{\psi} \Gamma_L/\Gamma_F \to 0$, we see that α_L is exact (we recall that $\mathrm{kum}(L_0/F_0)$ is isomorphic to $\mathrm{Gal}(L_0/F_0)$ and $\mathrm{kum}(L/F)$ is isomorphic to $\mathrm{Gal}(L/F)$, see [8, 2.1]). Plainly, α_L can be considered as a (symmetric) 2-cocycle of $Z^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$. If L is a Kummer graded subfield of a graded central simple algebra A over F, then one can see that $\mathrm{KUM}(L/F) \cap A_0 = \mathrm{KUM}(L_0/F_0)$. In what follows, we will denote by $e_* : H^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}} \to H^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$ the homomorphism of cohomology groups corresponding to the canonical surjective homomorphism $e : \mathrm{KUM}(L_0/F_0) \to \mathrm{kum}(L_0/F_0)$.

(3.2) Let F be a graded field, A a graded central simple algebra over F with A_0 simple and R a graded central division algebra over F Brauer-equivalent to A. As previously seen in (1.3) A can be written as a generalized graded crossed product $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$, where (ω, f) is a graded factor set of Γ_A/Γ_F in A_0F (see [9, Lemma 2.4]). We can assume that f is normalized (i.e., $f(0, \overline{\gamma}) = f(\overline{\gamma}, 0) = 1$ for all $\overline{\gamma}(= \gamma+\Gamma_F) \in \Gamma_A/\Gamma_F$). Indeed, as in the proof of [9, Lemma 2.4] for any $\gamma \in \Gamma_A(= \Gamma_R)$, fix nonzero homogeneous elements $z_{\bar{\gamma}}$ of R with $\mathrm{gr}(z_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$ and with $z_0 = 1$. Then, $A = \oplus_{\overline{\gamma} \in \Gamma_A/\Gamma_F} A_0F z_{\overline{\gamma}} \cong (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$, where (ω, f) is the graded factor set of Γ_A/Γ_F in A_0F, defined as follows: $\omega : \Gamma_A/\Gamma_F \to \mathrm{Aut}(A_0F)$, $a \mapsto \omega_{\bar{\gamma}}(a) = z_{\bar{\gamma}} a z_{\bar{\gamma}}^{-1}$, and $f : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to (A_0F)^*$, $(\bar{\gamma}, \bar{\delta}) \mapsto z_{\bar{\gamma}} z_{\bar{\delta}} z_{\bar{\gamma}+\bar{\delta}}^{-1}$. This representation of A will be used in what follows to generalize the statements of [8, Theorems 2.4 and 2.6]. We get then conditions under which A has Kummer graded subfields.

(3.3) Let $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$ with f normalized as in (3.2) and denote also by ω the map $\Gamma_A/\Gamma_F \to \mathrm{Aut}(A_0)$, defined by $\overline{\gamma} \mapsto \omega_{\overline{\gamma}|A_0}$, where $\omega_{\overline{\gamma}|A_0}$ is the restriction of $\omega_{\overline{\gamma}}$ to A_0. One can easily see that there is a mapping $d : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to A_0^*$ and a symmetric 2-cocycle $h \in Z^2(\Gamma_A/\Gamma_F, F^*)_{\mathrm{sym}}$

such that (ω, d) is a factor set of Γ_A/Γ_F in A_0 and for any $\overline{\gamma}, \overline{\gamma}' \in \Gamma_A/\Gamma_F$, we have $f(\overline{\gamma}, \overline{\gamma}') = d(\overline{\gamma}, \overline{\gamma}')h(\overline{\gamma}, \overline{\gamma}')$. Indeed, let $(\overline{\delta}_i := \delta_i + \Gamma_F)_{1\leq i\leq r}$ be a basis of Γ_A/Γ_F (i.e., $\Gamma_A/\Gamma_F = \langle\overline{\delta}_1\rangle \oplus \cdots \oplus \langle\overline{\delta}_r\rangle$), and $q_i = \mathrm{ord}(\overline{\delta}_i)$ (for $1 \leq i \leq r$). For any $\overline{\gamma} \in \Gamma_A/\Gamma_F$, there is a unique element $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$ with $0 \leq m_i < q_i$, such that $\overline{\gamma} = (\sum_{i=1}^r m_i\delta_i) + \Gamma_F$. Let $\overline{m}$, $\overline{n} \in \mathbb{N}^r$ with $0 \leq m_i$, $n_i < q_i$, and let $s(\overline{m}+\overline{n}) \in \mathbb{N}^r$ with $0 \leq s(\overline{m}+\overline{n})_i < q_i$ and $m_i + n_i - s(\overline{m}+\overline{n})_i \equiv 0 \bmod q_i$ for all i, $(1 \leq i \leq r)$. Let $t_i \in \mathbb{N}$ such that $m_i + n_i - s(\overline{m}+\overline{n})_i = t_iq_i$, and fix elements $y_i \in F^*$ with $\mathrm{gr}(y_i) = q_i\delta_i$ and $y_1 = 1$. Let $h : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to F^*$ be the map defined by $h(\sum_{i=1}^r m_i\delta_i, \sum_{i=1}^r n_i\delta_i) = \prod_{i=1}^r {y_i}^{t_i}$, where m_i, n_i and t_i satisfy the above conditions, then by simple computations, one can see that h is a normalized symmetric 2-cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)$. Let $d : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to A_0^*$, be the map defined by $d(\overline{\gamma}, \overline{\gamma}') = f(\overline{\gamma}, \overline{\gamma}').h(\overline{\gamma}, \overline{\gamma}')^{-1}$. The fact that (ω, f) is a graded factor set of Γ_A/Γ_F in A_0F (with f normalized) and h is a normalized symmetric 2-cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)$, imply that (ω, d) is a factor set of Γ_A/Γ_F in A_0 (with d normalized).

The following two Theorems generalize the statements of [8, Theorems 2.4 and 2.6] by using the same arguments. For the convenience of the reader we give the detailed proofs. For a Kummer graded subfield L of a graded simple algebra A and for a factor set (w, g) of Γ_A/Γ_F in A_0, we will denote by $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(w, g)$ the restriction of (w, g) when considering Γ_L/Γ_F instead of Γ_A/Γ_F. Also, for a cocycle $k \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ and the canonical embedding $i : \mathrm{KUM}(L_0/F_0) \to A_0^*$, we denote by i_*k the mapping $\Gamma_L/\Gamma_F \times \Gamma_L/\Gamma_F \to A_0^*$, defined by $(\overline{\gamma}, \overline{\gamma}') \mapsto i \circ k(\overline{\gamma}, \overline{\gamma}')$.

Theorem 3.4. *Let F be a graded field, A a graded central simple algebra over F with A_0 simple and* $\mathrm{char}(F)$ *not dividing* $\deg(A)$*, L be a Kummer graded subfield of A, and α_L be the cocycle of $Z^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$ defined in* (3.1)*. Write $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$ as in* (3.2) *and consider the factor set (ω, d) of Γ_A/Γ_F in A_0 as defined in* (3.3)*, then there exists a normalized cocycle $d' \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ (for the trivial action of Γ_L/Γ_F on* $\mathrm{KUM}(L_0/F_0)$*) and a map $\omega' : \Gamma_L/\Gamma_F \to \mathrm{Aut}(A_0)$ which satisfies $\omega'_{\overline{\gamma}}(a) = a$ for all $a \in L_0$ and $\overline{\gamma} \in \Gamma_L/\Gamma_F$, such that:*

(1) *(ω', i_*d') is a factor set of Γ_L/Γ_F in A_0, cohomologous to* $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(\omega, d)$, *and*

(2) *$e_*([d']) = [\alpha_L]$, where $[d']$ [resp., $[\alpha_L]$] denotes the class of d' in* $H^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))$ *[resp., of α_L in* $H^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))$*] (see* (3.1) *for the definition of e_*).*

Proof. Write $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f)) = \oplus_{\overline{\gamma}\in\Gamma_A/\Gamma_F} A_0Fx_{\overline{\gamma}}$, where $x_0 = 1$, $x_{\overline{\gamma}} \in A^*$, $\mathrm{gr}(x_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$, $x_{\overline{\gamma}}a = \omega_{\overline{\gamma}}(a)x_{\overline{\gamma}}$ and $x_{\overline{\gamma}}x_{\overline{\gamma}'} = f(\overline{\gamma}, \overline{\gamma}')x_{\overline{\gamma}+\overline{\gamma}'}$ and write $f(\overline{\gamma}, \overline{\gamma}') = d(\overline{\gamma}, \overline{\gamma}')h(\overline{\gamma}, \overline{\gamma}')$ as in (3.3). Since the map ψ in (3.1) is surjective, then for any $\gamma \in \Gamma_L$, we can choose $y_{\overline{\gamma}} \in \mathrm{KUM}(L/F)$ such that $\mathrm{gr}(y_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$. Write $y_{\overline{\gamma}} = a_{\overline{\gamma}}x_{\overline{\gamma}}$, where $a_{\overline{\gamma}} \in (A_0F)^*$. We have $x_0 = 1$ and we can choose $y_0 = 1$, so $a_0 = 1$. Let $b_{\overline{\gamma}} \in A_0^*$ and $c_{\overline{\gamma}} \in F^*$ be such that $a_{\overline{\gamma}} = b_{\overline{\gamma}}c_{\overline{\gamma}}$ (with

$b_0 = c_0 = 1$), then we have:

$$\begin{aligned} y_{\overline{\gamma}} y_{\overline{\gamma}'} &= a_{\overline{\gamma}} \omega_{\overline{\gamma}}(a_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') a^{-1}_{\overline{\gamma}+\overline{\gamma}'} h(\bar{\gamma}, \overline{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'} \\ &= b_{\overline{\gamma}} \omega_{\overline{\gamma}}(b_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') b^{-1}_{\overline{\gamma}+\bar{\gamma}'} c_{\overline{\gamma}} c_{\overline{\gamma}'} c^{-1}_{\overline{\gamma}+\bar{\gamma}'} h(\bar{\gamma}, \bar{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'} \\ &= d'(\overline{\gamma}, \overline{\gamma}') h'(\overline{\gamma}, \bar{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'}, \end{aligned}$$

where $d'(\overline{\gamma}, \overline{\gamma}') = b_{\overline{\gamma}} \omega_{\overline{\gamma}}(b_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') b^{-1}_{\overline{\gamma}+\bar{\gamma}'}$ and $h'(\overline{\gamma}, \overline{\gamma}') = c_{\overline{\gamma}} c_{\overline{\gamma}'} c^{-1}_{\overline{\gamma}+\bar{\gamma}'} h(\bar{\gamma}, \bar{\gamma}')$. Since $y_{\overline{\gamma}}$, $y_{\overline{\gamma}'}$ and $y_{\overline{\gamma}+\bar{\gamma}'}$ are in $\mathrm{KUM}(L/F)$ and $h'(\overline{\gamma}, \overline{\gamma}') \in F^*$, then $d'(\overline{\gamma}, \overline{\gamma}') \in \mathrm{KUM}(L/F) \cap A_0$ ($= \mathrm{KUM}(L_0/F_0)$) (see (3.1)). Moreover, one can easily check that $d' \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ (this follows from the equality $(y_{\overline{\gamma}} y_{\overline{\gamma}'}) y_{\overline{\gamma}''} = y_{\overline{\gamma}}(y_{\overline{\gamma}'} y_{\overline{\gamma}''})$, the fact that h' which is cohomologous to $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(h)$, is a symmetric 2-cocycle, and the fact that $y_{\overline{\gamma}}$ are pairwise commuting for $\overline{\gamma} \in \Gamma_L/\Gamma_F$). Also, since $y_0 = 1$, then d' is normalized.

Now, let $\omega' : \Gamma_L/\Gamma_F \to \mathrm{Aut}(A_0)$ be the map defined by $\omega'_{\overline{\gamma}} = \mathrm{Int}(b_{\overline{\gamma}})\omega_{\overline{\gamma}}$ (i.e., $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}} \omega_{\overline{\gamma}}(a) b^{-1}_{\overline{\gamma}}$ for all $a \in A_0$ and $\overline{\gamma} \in \Gamma_L/\Gamma_F$), then for any $a \in L_0$ and any $\overline{\gamma} \in \Gamma_L/\Gamma_F$, we have $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}} x_{\overline{\gamma}} a x^{-1}_{\overline{\gamma}} b^{-1}_{\overline{\gamma}} = a_{\overline{\gamma}} x_{\bar{\gamma}} a x^{-1}_{\overline{\gamma}} a^{-1}_{\overline{\gamma}} = y_{\bar{\gamma}} a y^{-1}_{\overline{\gamma}} = a$ (because $y_{\overline{\gamma}} \in \mathrm{KUM}(L/F)$). One can easily see that $(\omega', i_* d')$ is a factor set of Γ_L/Γ_F in A_0, cohomologous to $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(\omega, d)$. Finally, the equality $y_{\overline{\gamma}} y_{\overline{\gamma}'} = d'(\overline{\gamma}, \overline{\gamma}') h'(\overline{\gamma}, \overline{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'}$ yields, by considering classes modulo F^* in $\mathrm{kum}(L/F)$, that we have $\overline{y}_{\overline{\gamma}} \overline{y}_{\overline{\gamma}'} = e(d'(\overline{\gamma}, \overline{\gamma}')) \overline{y}_{\overline{\gamma}+\overline{\gamma}'}$, where $e : \mathrm{KUM}(L_0/F_0) \to \mathrm{kum}(L_0/F_0)$ is the canonical surjective homomorphism (we identify here $\mathrm{kum}(L_0/F_0)$ with its canonical image in $\mathrm{kum}(L/F)$). Hence, $e_*([d']) = [\alpha_L]$. □

(3.5) Let F be a graded field, D a graded division algebra over F, S a finite abelian subgroup of D^*/F^* with exponent m, and for any $s \in S$, let d_s be a representative of s in D^*. Suppose that $\mathrm{char}(F)$ does not divide $\deg(D)$, F_0 contains a primitive m^{th} root of unity and let $F(S) = F[d_s \mid s \in S]$ be the subring of D generated by F and the elements d_s ($s \in S$). If d_s are pairwise commuting, then as in the ungraded case $F(S)$ is a Kummer graded field extension of F with $\mathrm{kum}(F(S)/F) = S$ (it suffices to observe that $F(S)$ is a graded field and that $q(F(S)) = q(F)(S)$ when S is identified with its canonical image in $q(D)^*/q(F)^*$).

In the proof of Theorem 3.6 below, we will have a graded central simple algebra A over F (with A_0 simple and $\mathrm{char}(F)$ not dividing $\deg(A)$) and we will consider D in above to be a graded subfield L of A. In such a case, analogously to [12, Remark 3.14], if we assume that F_0 contains a primitive $\deg(A)^{th}$ root of unity, then necessarily F_0 contains a primitive m^{th} root of unity, where as in above, m is the exponent of S. Indeed, in this case, S will turn to be isomorphic to $\mathrm{kum}(L/F)$, so m divides $[L : F]$, and obviously $[L : F]$ divides $\deg(A)$. A similar situation will occur in Corollaries 3.9 and 3.10.

Theorem 3.6. *Let F be a graded field, A a graded central simple algebra over F with A_0 simple and (ω, d) [resp., h] the factor set of Γ_A/Γ_F in A_0 [resp., the cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)_{\mathrm{sym}}$] seen in* (3.3). *Assume that* $\mathrm{char}(F)$ *does not* *divide* $\deg(A)$*,* F_0 *contains enough roots of unity* (*e.g.,* F_0 *contains a primitive* $\deg(A)^{th}$ *root of unity*[2]) *and that there are*:

(1) *a Kummer field extension* M *of* F_0 *in* A_0*, and a subgroup* R *of* Γ_A/Γ_F *acting trivially on* M*,*

(2) *a normalized cocycle* $d' \in Z^2(R, \mathrm{KUM}(M/F_0))_{\mathrm{sym}}$ *and a map* $\omega' : R \to \mathrm{Aut}(A_0)$ *such that* (ω', i_*d') *is a factor set of* R *in* A_0*, cohomologous to* $\mathrm{res}_R^{\Gamma_A/\Gamma_F}(\omega, d)$*, and such that* $\omega'_{\overline{\gamma}}(a) = a$ *for all* $a \in M$ *and* $\overline{\gamma} \in R$*.*

Then, there exists a Kummer graded subfield L *of* A *such that*

(1) $L_0 = M$*,* $\Gamma_L/\Gamma_F = R$ *and*

(2) $e_*([d']) = [\alpha_L]$*.*

Proof. Write $A = \oplus_{\overline{\gamma}\in\Gamma_A/\Gamma_F} A_0 F x_{\overline{\gamma}}$, where $x_0 = 1$, $x_{\overline{\gamma}} \in A^*$, $\mathrm{gr}(x_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$, $x_{\overline{\gamma}} a = \omega_{\overline{\gamma}}(a) x_{\overline{\gamma}}$ and $x_{\overline{\gamma}} x_{\overline{\gamma}'} = d(\overline{\gamma},\overline{\gamma}')h(\overline{\gamma},\overline{\gamma}')x_{\overline{\gamma}+\overline{\gamma}'}$ as in (3.3). The fact that (ω', i_*d') is cohomologous to $\mathrm{res}_R^{\Gamma_A/\Gamma_F}(\omega, d)$ means that there is a family $(b_{\overline{\gamma}})_{\overline{\gamma}\in R}$ of elements of A_0^* such that for all $a \in A_0$ and $\overline{\gamma}, \overline{\gamma}' \in R$, we have $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}}\omega_{\overline{\gamma}}(a)b_{\overline{\gamma}}^{-1}$ and $d'(\overline{\gamma},\overline{\gamma}') = b_{\overline{\gamma}}\omega_{\overline{\gamma}}(b_{\overline{\gamma}'})d(\overline{\gamma},\overline{\gamma}')b_{\overline{\gamma}+\overline{\gamma}'}^{-1}$. Let $y_{\overline{\gamma}} = b_{\overline{\gamma}} x_{\overline{\gamma}}$ for all $\overline{\gamma} \in R$. Then, we have $y_{\overline{\gamma}} y_{\overline{\gamma}'} = d'(\overline{\gamma},\overline{\gamma}')h(\overline{\gamma},\overline{\gamma}')y_{\overline{\gamma}+\overline{\gamma}'}$. Let $L = \oplus_{\overline{\gamma}\in R} MF y_{\overline{\gamma}} (\subseteq A)$. Since d' and h are symmetric, then $y_{\overline{\gamma}}$ are pairwise commuting. Moreover, by hypotheses $\omega'_{\overline{\gamma}}(a) = a$ for all $a \in M$ and $\overline{\gamma} \in R$, so L is a commutative graded subring of A with $L_0 = M$ and $\Gamma_L/\Gamma_F = R$.

Since both d' and $\mathrm{res}_R^{\Gamma_A/\Gamma_F} h$ are normalized and $y_{\overline{\gamma}} (= b_{\overline{\gamma}} x_{\overline{\gamma}})$ is invertible for any $\overline{\gamma} \in R$, then $y_0 = 1$ (it suffices to see that $y_0 y_{\overline{\gamma}} = d'(0,\overline{\gamma})h(0,\overline{\gamma})y_{\overline{\gamma}} = y_{\overline{\gamma}}$). For any $\overline{\gamma} \in R$, we have $y_{\overline{\gamma}} y_{-\overline{\gamma}} = d'(\overline{\gamma},-\overline{\gamma})h(\overline{\gamma},-\overline{\gamma})y_0$, hence $y_{\overline{\gamma}}$ is invertible in L. One can easily see that nonzero homogeneous elements of L are the elements of the form $a y_{\overline{\gamma}}$, where $a \in (MF)^*$, and $\overline{\gamma} \in R$, so all nonzero homogeneous elements of L are invertible. This shows that L is a graded subfield of A.

Let S be the subgroup of L^*/F^* generated by $\mathrm{kum}(M/F_0)$ and the set $\{\overline{y}_{\overline{\gamma}}\}_{\overline{\gamma}\in R}$, where $\overline{y}_{\overline{\gamma}}$ is the class of $y_{\overline{\gamma}}$ in L^*/F^* (and where as in above, we identify $\mathrm{kum}(M/F_0)$ with its canonical image in $\mathrm{kum}(MF/F)$). One can easily see that up to a graded isomorphism we have $L = F(S)$. Therefore, by (3.5) L is a Kummer graded field extension of F with $\mathrm{kum}(L/F) = S$. Considering classes in $\mathrm{kum}(L/F)$, we have $\overline{y}_{\overline{\gamma}}\overline{y}_{\overline{\gamma}'} = e(d'(\overline{\gamma},\overline{\gamma}'))\overline{y}_{\overline{\gamma}+\overline{\gamma}'}$, where $e : \mathrm{KUM}(M/F_0) \to \mathrm{kum}(M/F_0)$ is the canonical surjective homomorphism (we identify here $\mathrm{kum}(M/F_0)$ with its canonical image in $\mathrm{kum}(L/F)$), so $\mathrm{kum}(L/F)$ is the extension of $\mathrm{kum}(M/F_0)$ by R with cocycle $e_*([d'])$. This shows that $e_*([d']) = [\alpha_L]$. □

(3.7) For the rest we will need some facts from the gauge theory developed by Tignol and Wadsworth in [13] and [14]. We recall here the following notions: Let D be a division ring, Γ be a totally ordered abelian group, $v : D \to \Gamma \cup \{\infty\}$ a valuation, and let M be a (right) D-vector space. A function $y : M \to \Gamma \cup \{\infty\}$

[2] See the last paragraph in (3.5) above.

is called a D-value function (or a v-value function) if it satisfies the following conditions (for all $m, n \in M$ and $d \in D$):

$y(m) = \infty$ if and only if $m = 0$;

$y(md) = y(m) + v(d)$;

$y(m+n) \geq \min\{y(m), y(n)\}$.

In a similar way as in the construction of GD (see the preliminaries), if y is a D-value function on M, then we associate to M a graded GD-module that we denote by G_yM (or simply GM). In this case a base $(m_i)_{i=1}^n$ of M over D is called a splitting base if for any elements $d_1, \dots, d_n$ of D, we have $y(\sum_{i=1}^n m_i d_i) = \min\{y(m_i) + v(d_i) \mid 1 \leq i \leq n\}$. If such a base exists, we say that y is a D-norm (or a v-norm). We recall that y is a D-norm if and only if $[GM : GD] = [M : D]$; furthermore, if this occurs, then $(m_i')_{i=1}^n$ is a base of GM over GD, where $m_i' = m_i + GM_{y(m_i)}$ (see [11, Corollary 2.3] or [13, Proposition 1.1]).

Let (E, v) be a valued field and let A be an E-algebra. A function $\alpha : A \to \Gamma \cup \{\infty\}$, is called a surmultiplicative E-value function if it satisfies the following conditions (for all $a, b \in A$ and $e \in E$):

$\alpha(a) = \infty$ if and only if $a = 0$;

$\alpha(1) = 0$;

$\alpha(ea) = v(e) + \alpha(a)$;

$\alpha(a+b) \geq \min\{\alpha(a), \alpha(b)\}$;

$\alpha(ab) \geq \alpha(a) + \alpha(b)$.

If α is a surmultiplicative E-value function, then the graded GE-module GA is a graded GE-algebra for the multiplication law defined (for all nonzero elements a, b of A) by (extension of): $a'b' = (ab)'$ if $\alpha(ab) = \alpha(a) + \alpha(b)$ and $a'b' = 0$ otherwise (see [13, (1.5), p. 691]).

If α is a surmultiplicative E-value function on A, then α is called an E-gauge if it is an E-norm and GA is a graded semisimple GE-algebra. If in addition $Z(GA) = G(Z(A))$ and $Z(GA)$ is separable over GE, then we say that α is a tame E-gauge. We say that a gauge α on A is residually simple if the 0-component $(G_\alpha A)_0$ of $G_\alpha A$ is simple.

(3.8) Let E be a Henselian field, D be a tame central division algebra over E, v be the extension of the valuation of E to D, $B = M_n(D)$ where n is a positive integer, $\Gamma = v(D^*)$ and define the map $\beta : B \to \Gamma \cup \{\infty\}$ by $\beta((d_{ij})_{1 \leq i,j \leq n}) = \min\{v(d_{ij}) \mid 1 \leq i, j \leq n\}$. One can easily see that β is a surmultiplicative E-value function and an E-norm on B. For $b = (d_{ij})_{1 \leq i,j \leq n} \in B$ and $\gamma \in \Gamma$, we have $\beta(b) \geq \gamma$ [resp., $\beta(b) > \gamma$] if and only if $v(d_{ij}) \geq \gamma$ [resp., $v(d_{ij}) > \gamma$] for all i, j $(1 \leq i, j \leq n)$, so the correspondence $b' \mapsto (d_{ij} + (GD)^{>\gamma})_{1 \leq i,j \leq n}$, where $\gamma = \beta(b)$, induces a graded isomorphism $GB \to M_n(GD)$. Therefore, β is a tame E-gauge. Note that we have $\overline{B}(:= GB_0) \cong M_n(\overline{D})$, so $\overline{B}$ is simple.

Corollary 3.9. *Let E be a Henselian valued field, n be a positive integer, D be a tame central division algebra over E, $B = M_n(D)$, and suppose that* $\text{char}(\overline{E})$ *does not divide* $\deg(B)$*, that there exists a Kummer subfield K of B*

with $\Gamma_K \subseteq \Gamma_D$ *and that* $\overline{E}$ *contains enough roots of unity* (*e.g.,* $\overline{E}$ *contains a primitive* $\deg(B)^{th}$ *root of unity*)*, then there is a normalized cocycle* $d' \in Z^2(\Gamma_K/\Gamma_E, \mathrm{KUM}(\overline{K}/\overline{E}))_{\mathrm{sym}}$ (*for the trivial action of* Γ_K/Γ_E *on* $\mathrm{KUM}(\overline{K}/\overline{E})$) *and a map* $w' : \Gamma_K/\Gamma_E \to \mathrm{Aut}(M_n(\overline{D}))$*, which satisfies* $\omega'_{\overline{\gamma}}(a) = a$ *for all* $a \in \overline{K}$ *and* $\overline{\gamma} \in \Gamma_K/\Gamma_E$*, such that*:

(a) (ω', i_*d') *is a factor set of* Γ_K/Γ_E *in* $\overline{B}$*, cohomologous to* $\mathrm{res}^{\Gamma_B/\Gamma_E}_{\Gamma_K/\Gamma_E}(\omega, d)$*, where* (ω, d) *is the factor set corresponding to a representation of* $M_n(GD)$ *as in* (3.3)*, and*

(b) $e_*([d']) = [\alpha_{GK}]$.

Proof. Indeed, take the residually simple tame E-gauge β on B as defined in (3.8), then we have $G_\beta B \cong_g M_n(GD)$. We have also $\Gamma_{GK} = \Gamma_K \subseteq \Gamma_B = \Gamma_{M_n(D)} = \Gamma_D$, so by Corollary 2.6 GK embeds in $M_n(GD)$. Moreover since K is a tame Kummer field extension of E and $\overline{E}$ contains enough roots of unity, then GK is a Kummer graded field extension of E. Our corollary follows then by Theorem 3.4. □

Similarly, the following corollary follows by applying Theorem 3.6, Corollary 2.6 and the fact that isomorphism classes of tame (abelian) field extensions of E are in one-to-one correspondence with the isomorphism classes of (abelian) graded field extensions of GE (as seen in the preliminaries).

Corollary 3.10. *Let* E *be a Henselian valued field,* n *be a positive integer,* D *be a tame central division algebra over* E*,* $B = M_n(D)$*, and suppose that* $\mathrm{char}(\overline{E})$ *does not divide* $\deg(B)$*, that* $\overline{E}$ *contains enough roots of unity* (*e.g.,* $\overline{E}$ *contains a primitive* $\deg(B)^{th}$ *root of unity*)*, and that there are*:

1. (1) *a Kummer field extension* M *of* $\overline{E}$ *in* $M_n(\overline{D})$*, and a subgroup* R *of* Γ_D/Γ_E *acting trivially on* M*,*
2. (2) *a normalized cocycle* $d' \in Z^2(R, \mathrm{KUM}(M/\overline{E}))_{\mathrm{sym}}$ *and a map* $\omega' : R \to \mathrm{Aut}(M_n(\overline{D}))$ *such that* (ω', i_*d') *is a factor set of* R *in* $M_n(\overline{D})$*, cohomologous to* $\mathrm{res}^{\Gamma_D/\Gamma_E}_R(\omega, d)$ (*where* (ω, d) *is the factor set corresponding to a representation of* $M_n(GD)$ *as in* (3.3)) *and such that* $\omega'_{\overline{\gamma}}(a) = a$ *for all* $a \in M$ *and* $\overline{\gamma} \in R$*.*

Then, there exists a Kummer subfield K *of* B *with* $\Gamma_K \subseteq \Gamma_D$*, such that*:

1. (1) $\overline{K} = M$*,* $\Gamma_K/\Gamma_E = R$ *and*
2. (2) $e_*([d']) = [\alpha_{GK}]$.

Remark 3.11. One can easily see that Corollaries 3.9 and 3.10 restrict to [8, Corollaries 2.11 and 2.12] when $n = 1$.

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