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SOME EXTENSION RESULTS CONCERNING ANALYTIC AND MEROMORPHIC MULTIVALENT FUNCTIONS

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Abstract. Let $\mathscr{B}_{p,n}^{\eta;\mu}(\alpha)$; $(\eta,\mu\in\mathbb{R},n,p\in\mathbb{N})$ denote all functions f class in the unit disk $\mathbb{U}$ as $f(z)=z^p+\sum_{k=n+p}^{\infty}a_kz^k$ which satisfy:

$$\left|\left[\frac{f'(z)}{pz^{p-1}}\right]^{\eta}\left[\frac{z^p}{f(z)}\right]^{\mu}-1\right|<1-\frac{\alpha}{p};\quad (z\in\mathbb{U},\ 0\leq\alpha<p).$$

And $\mathscr{M}_{p,n}^{\eta;\mu}(\alpha)$ indicates all meromorphic functions h in the punctured unit disk $\mathbb{U}^*$ as $h(z)=z^{-p}+\sum_{k=n-p}^{\infty}b_kz^k$ which satisfy:

$$\left|\left[\frac{h'(z)}{-pz^{-p-1}}\right]^{\eta}\left[\frac{1}{z^ph(z)}\right]^{\mu}-1\right|<1-\frac{\alpha}{p};\quad (z\in\mathbb{U},\ 0\leq\alpha<p).$$

In this paper several sufficient conditions for some classes of functions are investigated. The authors apply Jack's Lemma, to obtain this conditions. Furthermore, sufficient conditions for strongly starlike and convex p-valent functions of order γ and type β, are also considered.

1. Introduction

Let $\mathbb{C}$, $\mathbb{R}=(-\infty,\infty)$ and $\mathbb{N}:=\{1,2,\dots\}$ be set of *complex, real* and *natural* numbers, respectively. Throughout this paper, by p,n it always means natural numbers.

Let $\mathcal{H}$ denote the class of *holomorphic functions* in the open unit disc $\mathbb{U}:=\{z\colon\ z\in\mathbb{C}$ and $|z|<1\}$ on the complex plane $\mathbb{C}$, and let $\mathcal{H}[a,n]$ denote the subclass of functions $\mathfrak{p}\in\mathcal{H}$ of the form:

$$\mathfrak{p}(z)=a+a_nz^n+\cdots;\qquad (a\in\mathbb{C},\ n\in\mathbb{N}).$$

Let $\mathcal{H}[1,n]$ denoted by $\mathcal{H}(n)$. A function $f(z)$ which is analytic in domain Ω is called *p-valent*, if

- for every complex number ω, the equation $f(z)=\omega$ has at most p roots in Ω, and

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- there exits a complex number w_0 such that the set $f^{-1}(\{\omega_0\})$ has exactly p elements in Ω.

Let $\mathcal{A}(p,n)$ denote the class of all functions $f \in \mathcal{H}$ of the following form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k; \qquad (p,n \in \mathbb{N}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}$. The class $\mathcal{A}(1,1)$ denoted by $\mathcal{A}$.

Let $\Sigma(p,n)$ be the class of *meromorphic* functions in the punctured open unit disk $\mathbb{U}^* := \{z \in \mathbb{C} : \ 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$ of the form:

$$h(z) = z^{-p} + \sum_{k=n-p}^{\infty} b_k z^k; \qquad (p,n \in \mathbb{N}), \tag{2}$$

with a pole of order p at the origin. The class $\Sigma(1,1)$ denoted by Σ.

Definition (Subclasses for $\mathcal{A}(p,n)$)**.** Let $\mathcal{S}_p^*(\alpha)$, $\mathcal{K}_p(\alpha)$, $\mathcal{R}_p(\alpha)$, $\widetilde{\mathcal{S}}_p^*(\gamma,\beta)$, and $\widetilde{\mathcal{K}}_p(\gamma,\beta)$ denote the subclasses of $\mathcal{A}(p,n)$ consisting of analytic functions which are, *p-valent starlike of order* α, *p-valent convex of order* α, *p-valent close-to-convex of order* α, *strongly starlike p-valent of order* γ *and type* β, and *strongly convex p-valent of order* γ *and type* β; respectively. Thus: (see, for details, [1,9,16])

$$\mathcal{S}_p^*(\alpha) := \left\{ f \in \mathcal{A}(p,n) \colon \ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{U}, \ 0 \le \alpha < p \right\},$$

$$\mathcal{K}_p(\alpha) := \left\{ f \in \mathcal{A}(p,n) \colon \ \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in \mathbb{U}, \ 0 \le \alpha < p \right\},$$

$$\mathcal{R}_p(\alpha) := \left\{ f \in \mathcal{A}(p,n) \colon \ \mathrm{Re}\left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha, \quad z \in \mathbb{U}, \ 0 \le \alpha < p \right\},$$

and for $0 \le \beta < 1$, $0 < \gamma \le 1$

$$\widetilde{\mathcal{S}}_p^*(\gamma,\beta) := \left\{ f \in \mathcal{A}(p,n) \colon \ \left| \arg\left\{ \frac{1}{p}\frac{zf'(z)}{f(z)} - \beta \right\} \right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U} \right\},$$

$$\widetilde{\mathcal{K}}_p(\gamma,\beta) := \left\{ f \in \mathcal{A}(p,n) \colon \ \left| \arg\left\{ \frac{1}{p}\left[1 + \frac{zf''(z)}{f'(z)} \right] - \beta \right\} \right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U} \right\}.$$

As usual, in the present investigation, we write: $\mathcal{S}^*(\alpha) := \mathcal{S}_1^*(\alpha)$; starlike functions of order α, $\mathcal{K}(\alpha) := \mathcal{K}_1(\alpha)$; convex functions of order α, $\mathcal{S}^* = \mathcal{S}_1^*(0)$; starlike functions, $\mathcal{K} := \mathcal{K}_1(0)$; convex functions, $\mathcal{R}(\alpha) := \mathcal{R}_1(\alpha)$; close-to-convex functions of order α, $\widetilde{\mathcal{S}}^*(\gamma) := \widetilde{\mathcal{S}}_1^*(\gamma,0)$; strongly starlike functions of order γ, and $\widetilde{\mathcal{K}}(\gamma) := \widetilde{\mathcal{K}}_1(\gamma,0)$; strongly convex functions of order γ.

Definition (Subclasses for $\Sigma(p,n)$)**.** Let $\mathcal{MS}_p^*(\alpha)$, $\mathcal{MK}_p(\alpha)$, $\mathcal{MR}_p(\alpha)$, $\mathcal{M}\widetilde{\mathcal{S}}_p^*(\gamma,\beta)$ and $\mathcal{M}\widetilde{\mathcal{K}}_p(\gamma,\beta)$ denote the subclasses of $\Sigma(p,n)$ consisting of meromorphic functions which are, *meromorphic p-valent starlike of order* α, *meromorphic p-valent convex of order* α, *meromorphic p-valent close-to-convex of order* α, *strongly starlike meromorphic p-valent of order* γ *and type* β, and *strongly convex meromorphic p-valent of order* γ *and type* β; respectively. Thus, we have: (see, for details, [15, 25])

$$\mathcal{MS}_p^*(\alpha) := \left\{ f \in \Sigma(p,n) : \; -\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{U}, \; 0 \le \alpha < p \right\},$$

$$\mathcal{MK}_p(\alpha) := \left\{ f \in \Sigma(p,n) : \; -\operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \; z \in \mathbb{U}, \, 0 \le \alpha < p \right\},$$

$$\mathcal{MR}_p(\alpha) := \left\{ f \in \Sigma(p,n) : \; -\operatorname{Re}\left\{ z^{p+1} f'(z) \right\} > \alpha, \quad z \in \mathbb{U}, \; 0 \le \alpha < p \right\},$$

and for $0 \le \beta < 1$, $0 < \gamma \le 1$

$$\mathcal{M}\widetilde{\mathcal{S}}_p^*(\gamma,\beta) := \left\{ f \in \Sigma(p,n) : \; \left| \arg\left\{ -\frac{1}{p} \frac{zf'(z)}{f(z)} - \beta \right\} \right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U} \right\},$$

$$\mathcal{M}\widetilde{\mathcal{K}}_p(\gamma,\beta) := \left\{ f \in \Sigma(p,n) : \; \left| \arg\left\{ -\frac{1}{p}\left[1 + \frac{zf''(z)}{f'(z)} \right] - \beta \right\} \right| < \frac{\pi}{2}\gamma, \; z \in \mathbb{U} \right\}.$$

As usual, we write: $\mathcal{MS}^*(\alpha) := \mathcal{MS}_1^*(\alpha)$; meromorphic starlike functions of order α, $\mathcal{MS}^* := \mathcal{MS}_1^*(0)$; meromorphic starlike functions, $\mathcal{MK}(\alpha) := \mathcal{MK}_1(\alpha)$; meromorphic convex functions of order α, $\mathcal{MK} := \mathcal{MK}_1(0)$; meromorphic convex functions, $\mathcal{MR}(\alpha) := \mathcal{MR}_1(\alpha)$; meromorphic close-to-convex functions of order α, $\mathcal{M}\widetilde{\mathcal{S}}^*(\gamma) := \mathcal{M}\widetilde{\mathcal{S}}_1^*(\gamma,0)$; strongly starlike meromorphic functions of order γ, and $\mathcal{M}\widetilde{\mathcal{K}}(\gamma) := \mathcal{M}\widetilde{\mathcal{K}}_1(\gamma,0)$; strongly convex meromorphic functions of order γ.

Let $\mathcal{B}(\mu,\alpha)$ be the class of functions $f \in \mathcal{A}$ which is in the following relations

$$\left| f'(z)\left(\frac{z}{f(z)} \right)^{\mu} \right| < 1 - \alpha; \qquad (z \in \mathbb{U}),$$

for some $\mu \in \mathbb{R}$ which $\mu \ge 0$, and some real number α with $0 \le \alpha < 1$. the class $\mathcal{B}(\mu,\alpha)$ has been investigated by Frasin and Jahangiri [7].

Motivated by the class $\mathcal{B}(\mu,\alpha)$, two differential operators are defined and then two new subclasses for multivalent analytic and multivalent meromorphic functions is introduced.

Definition. Let η and μ be real numbers not both zero. Defining the differential operators $\mathscr{F}_{p,n}^{\eta;\mu} \colon \mathcal{A}(p,n) \longrightarrow \mathcal{H}(n)$ and $\mathscr{G}_{p,n}^{\eta;\mu} \colon \Sigma(p,n) \longrightarrow \mathcal{H}(n)$ as follows:

$$\mathscr{F}_{p,n}^{\eta;\mu}[f](z) := \left[\frac{f'(z)}{pz^{p-1}} \right]^{\eta} \left[\frac{z^p}{f(z)} \right]^{\mu} = 1 + \left(\eta - \mu + \frac{n}{p}\eta \right) a_{n+p} z^n + \cdots$$

for some $f \in \mathcal{A}(p,n)$ given by (1) with $z \in \mathbb{U}$, and

$$\mathscr{G}_{p,n}^{\eta;\mu}[h](z) := \left[\frac{h'(z)}{-pz^{-p-1}} \right]^{\eta} \left[\frac{1}{z^p h(z)} \right]^{\mu} = 1 + \left(\eta - \mu - \frac{n}{p}\eta \right) b_{n-p} z^n + \cdots$$

for some $h \in \Sigma(p,n)$ given by (2) and $z \in \mathbb{U}$. Here and hereafter, all powers are mean as principal values.

Definition. Let η and μ be real numbers not both zero. A function $f \in \mathcal{A}(p, n)$ is a member of the class $\mathscr{B}_{p,n}^{\eta;\mu}(\alpha)$, if and only if

$$\left|\mathscr{F}_{p,n}^{\eta;\mu}[f](z) - 1\right| < 1 - \frac{\alpha}{p}; \quad (z \in \mathbb{U}) \quad \text{and} \quad \mathscr{F}_{p,n}^{\eta;\mu}[f](z)\Big|_{z=0} = 1 \tag{3}$$

for some α be real number within $0 \leq \alpha < p$.

Note that condition (3), implies that

$$\mathsf{Re}\left\{\mathscr{F}_{p,n}^{\eta;\mu}[f](z)\right\} > \frac{\alpha}{p}; \qquad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,.$$

The family $\mathscr{B}_{p,n}^{\eta;\mu}(\alpha)$ includes many classes of analytic functions as well as some very well-known ones. For example, $\mathscr{B}_{p,n}^{1;1}(\alpha) = \mathcal{S}_p^*(\alpha)$, $\mathscr{B}_{p,n}^{1;0}(\alpha) = \mathcal{R}_p(\alpha)$. Another interesting subclass is the special case $\mathscr{B}_{p,n}^{1;2}(\alpha)$ which introduced by Frasin and Darus [6]. Also, it is known that the class $\mathscr{B}_{1,1}^{1;\mu}(\alpha)$; $\mu > 1$ is the class of starlike functions [22].

Many important properties of certain subclasses of holomorphic p-valent functions study by several authors including: Irmak [12], Singh and Singh [27], Owa et al. [21], Goswami et al. [10].

Definition. Let η and μ be real numbers not both zero. A function $f \in \Sigma(p, n)$ is a member of the class $\mathcal{M}_{p,n}^{\eta;\mu}(\alpha)$, if and only if

$$\left|\mathscr{G}_{p,n}^{\eta;\mu}[f](z) - 1\right| < 1 - \frac{\alpha}{p}; \qquad (z \in \mathbb{U}) \quad \text{and} \quad \mathscr{G}_{p,n}^{\eta;\mu}[f](z)\Big|_{z=0} = 1 \tag{4}$$

for some α be real number with $0 \leq \alpha < p$.

Note that condition (4), implies that

$$\mathsf{Re}\left\{\mathscr{G}_{p,n}^{\eta;\mu}[f](z)\right\} > \frac{\alpha}{p}; \qquad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,.$$

Many important properties of certain *p-valent* subclasses meromorphic functions did the study by several researchers including: Singh et al. [26], Owa et al. [19], Goyal and Prajapat [11], Srivastava et al. [28], Ganigi and Uralegaddi [8].

Definition. For $\alpha > p$, let $\mathscr{N}_p(\alpha)$ be the subclass of $\mathcal{A}(p, n)$ consisting of functions $f(z)$ which satisfy

$$\mathsf{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha; \qquad (z \in \mathbb{U})\,.$$

The class $\mathscr{N}_1(\alpha)$ was introduced and studied by Owa and et al. [20].

Definition. Let η_i, μ_i be real numbers not both zero for all $i = 1, \ldots, m$; $(m \in \mathbb{N})$. Let $\mathscr{I}^{\eta_i,\mu_i} : \mathcal{A}^m(p, n) \to \mathcal{A}(p, n)$ be the integral operator define by

$$\mathscr{I}^{\eta_i,\mu_i}\left[f_1, \ldots, f_m\right](z) \coloneqq \int_0^z \prod_{i=1}^m \left[\frac{f_i'(\tau)}{p\tau^{p-1}}\right]^{\eta_i} \left[\frac{f_i(\tau)}{\tau^p}\right]^{\mu_i} \mathrm{d}\tau$$

$$(5) \qquad = \int_0^z \prod_{i=1}^{m} \mathscr{F}_{p,n}^{\eta_i,\mu_i}[f_i](z)\,\mathrm{d}\tau \quad (z \in \mathbb{U}),$$

for all $i = 1, \ldots, n$; $f_i \in \mathcal{A}(p,n)$. Note that this operator generalized by integral operators and have been investigated in some reports (see [2, 4]).

Lemma 1.1 ([24, Corollary 1.7])**.** *If $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ satisfies the condition*

$$\left|f'(z) - 1\right| < \frac{(n+1)\sin\left(\frac{\pi}{2}\alpha\right)}{\sqrt{1 + (n+1)^2 + 2(n+1)\cos\left(\frac{\pi}{2}\alpha\right)}}; \qquad (z \in \mathbb{U},\ 0 < \alpha \le 1),$$

then $f \in \widetilde{\mathcal{S}}^*(\alpha)$.

The structure of the paper is as follows. In Sections 2, at first, we get enough conditions for the functions in classes $\mathcal{A}(p,n)$ and $\Sigma(p,n)$ be *p-valent close-to-convex* and *p-valent starlike*. In the sequel, we get sufficient conditions for this functions being to the classes $\mathscr{B}_{p,n}^{\eta;\mu}(\alpha)$ or $\mathscr{M}_{p,n}^{\eta;\mu}(\alpha)$. Furthermore, we decide the *order of convexity* of $\mathscr{I}^{\eta_i,\mu_i}$. In Section 3, we consider sufficient conditions for the function f being to *p-valent strongly starlike and convex of order* γ *and type* β in classes $\mathcal{A}(p,n)$ or $\Sigma(p,n)$.

2. Properties of the classes $\mathscr{B}_{p,n}^{\eta;\mu}(\alpha)$ and $\mathcal{M}_{p,n}^{\eta;\mu}(\alpha)$

Before starting our main result, we need the following Lemma due to Jack.

Lemma 2.1 ([14] (See also [17, Lemma 2.2a]))**.** *Let the* (*non-constant*) *function $\omega(z) = a_n z^n + a_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ with $a_n \neq 0$. If $|\omega(z)|$ reaches its maximum value on the circle $|z| = r < 1$ at the point $z_0 \in \mathbb{U}$, then*

$$z_0\omega'(z_0) = m\omega(z_0),$$

where m is a real number and $m \ge n \ge 1$.

Theorem 2.2. *Let $\mathfrak{p} \in \mathcal{H}(n)$, and suppose that*

$$(6) \qquad \operatorname{Re}\left\{\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}\right\} > \frac{n(\alpha - p)}{2\alpha}; \qquad \left(z \in \mathbb{U},\ \frac{p}{2} \le \alpha < p\right).$$

Then

$$\operatorname{Re}\{\mathfrak{p}(z)\} > \frac{\alpha}{p}; \qquad \left(z \in \mathbb{U},\ \frac{p}{2} \le \alpha < p\right).$$

Proof. We define the analytic function $\omega(z)$ in unit disk $\mathbb{U}$ by

$$(7) \qquad \mathfrak{p}(z) = \frac{p + (2\alpha - p)\,\omega(z)}{p[1 + \omega(z)]}; \quad \left(\frac{p}{2} \le \alpha < p,\ \omega(z) \neq -1;\ z \in \mathbb{U}\right).$$

Then $\omega(0) = 0$. Logarithmic differentiation of (7) yields that

$$(8) \qquad \frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} = \frac{(2\alpha - p)\,z\omega'(z)}{p + (2\alpha - p)\,\omega(z)} - \frac{z\omega'(z)}{1 + \omega(z)}; \quad \left(z \in \mathbb{U},\ \frac{p}{2} \le \alpha < p\right).$$

Now, suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\omega(z_0)| = 1 \quad \text{and} \quad |\omega(z)| < 1; \quad \text{when} \quad |z| < |z_0|.$$

Then, by applying Lemma 2.1, we have

$$z_0\omega'(z_0) = m\omega(z_0); \qquad \left(m \geq n \geq 1,\ \omega(z_0) = \mathrm{e}^{\mathrm{i}\theta},\ \theta \neq -\pi\right). \tag{9}$$

Form (8) and (9), we obtain

$$\begin{aligned}\mathsf{Re}\left\{\frac{z_0\mathfrak{p}'(z_0)}{\mathfrak{p}(z_0)}\right\} &= \mathsf{Re}\left\{\frac{m(2\alpha-p)\,\mathrm{e}^{\mathrm{i}\theta}}{p+(2\alpha-p)\,\mathrm{e}^{\mathrm{i}\theta}}\right\} - \mathsf{Re}\left\{\frac{m\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}}\right\} \\ &= \frac{m(2\alpha-p)\,(2\alpha-p+p\cos\theta)}{p^2+(2\alpha-p)^2+2p(2\alpha-p)\cos\theta} - \frac{m}{2} \\ &\leq \frac{n(\alpha-p)}{2\alpha}; \qquad \left(z\in\mathbb{U},\ \frac{p}{2}\leq\alpha<p\right),\end{aligned}$$

which contradicts the hypothesis (6). Thus, we conclude that $|\omega(z)| < 1$ for all $\mathbb{U}$; and equation (7) yields the inequalities

$$\left|\frac{1-\mathfrak{p}(z)}{\mathfrak{p}(z)-\left(\frac{2\alpha}{p}-1\right)}\right| < 1; \qquad \left(z\in\mathbb{U},\ \frac{p}{2}\leq\alpha<p\right),$$

which implies that $\mathsf{Re}\,\{\mathfrak{p}(z)\} > \frac{\alpha}{p}$. □

Putting $\mathfrak{p}_1(z) \coloneqq \mathscr{F}_{p,n}^{\eta;\mu}[f](z);\ (z\in\mathbb{U})$ and $\mathfrak{p}_2(z) \coloneqq \mathscr{G}_{p,n}^{\eta;\mu}[h](z);\ (z\in\mathbb{U})$ in Theorem 2.2, we get the following result:

Corollary 2.3. *If the functions $f \in \mathcal{A}(p,n)$ and $h \in \Sigma(p,n)$ satisfy the following conditions*:

$$\mathsf{Re}\left\{\eta\left(1+\frac{zf''(z)}{f'(z)}-p\right)+\mu\left(p-\frac{zf'(z)}{f(z)}\right)\right\} > \frac{n(\alpha-p)}{2\alpha}; \quad (z\in\mathbb{U}),$$

$$\mathsf{Re}\left\{\eta\left(1+\frac{zh''(z)}{h'(z)}+p\right)-\mu\left(p+\frac{zh'(z)}{h(z)}\right)\right\} > \frac{n(\alpha-p)}{2\alpha}; \quad (z\in\mathbb{U}),$$

for some $\frac{p}{2} \leq \alpha < p$ and η, μ be real numbers not both zero, then

$$\mathsf{Re}\left\{\mathscr{F}_{p,n}^{\eta;\mu}[f](z)\right\} > \frac{\alpha}{p}; \quad (z\in\mathbb{U}),$$

$$\mathsf{Re}\left\{\mathscr{G}_{p,n}^{\eta;\mu}[h](z)\right\} > \frac{\alpha}{p}; \quad (z\in\mathbb{U}).$$

The special cases $\mathfrak{p}_1(z) \coloneqq \frac{f(z)}{z};\ (z\in\mathbb{U})$, $\mathfrak{p}_2(z) \coloneqq \frac{1}{zh(z)};\ (z\in\mathbb{U})$ and $p = n = 1$ in Theorem 2.2, lead us to the next corollary:

Corollary 2.4. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\mathsf{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{3\alpha - 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

$$-\mathsf{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \frac{3\alpha - 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

for some $\frac{1}{2} \leq \alpha < 1$, then $\mathsf{Re}\left\{\frac{f(z)}{z}\right\} > \alpha$; $(z \in \mathbb{U})$ and $\mathsf{Re}\left\{\frac{1}{zh(z)}\right\} > \alpha$; $(z \in \mathbb{U})$.

Putting $p = n = 1$, $\mathfrak{p}_1(z) := \frac{zf'(z)}{f(z)}$; $(z \in \mathbb{U})$, and $\mathfrak{p}_2(z) := \frac{zh'(z)}{-h(z)}$; $(z \in \mathbb{U})$ in Theorem 2.2, we get the following result:

Corollary 2.5. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\mathsf{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < \frac{\alpha + 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

$$\mathsf{Re}\left\{\frac{zh'(z)}{h(z)} - \frac{zh''(z)}{h'(z)}\right\} < \frac{\alpha + 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

for some $\frac{1}{2} \leq \alpha < 1$, then $f \in \mathcal{S}^*(\alpha)$ and $h \in \mathcal{MS}^*(\alpha)$.

Remark 2.6. A special case of Corollary 2.3 with $h \in \Sigma$ can be found in [3, Corollary 2.2].

Letting $\mathfrak{p}_1(z) := f'(z)$; $(z \in \mathbb{U})$, $\mathfrak{p}_2(z) := -z^2h'(z)$; $(z \in \mathbb{U})$ and $p = n = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.7. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\mathsf{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3\alpha - 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

$$\mathsf{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{\alpha + 1}{2\alpha}; \qquad (z \in \mathbb{U}),$$

for some $\frac{1}{2} \leq \alpha < 1$, then $f \in \mathcal{R}(\alpha)$ and $h \in \mathcal{MR}(\alpha)$.

Putting $\mathfrak{p}_1(z) := \frac{f'(z)}{pz^{p-1}}$; $z \in \mathbb{U}$, and $\mathfrak{p}_2(z) := \frac{h'(z)}{-pz^{-p-1}}$; $(z \in \mathbb{U})$ in Theorem 2.2, we get the following result:

Corollary 2.8. *If the functions $f \in \mathcal{A}(p, n)$ and $h \in \Sigma(p, n)$ satisfy the following conditions*:

$$\mathsf{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > p + \frac{n}{2}\left(\frac{\alpha - p}{\alpha + p}\right); \qquad (z \in \mathbb{U}),$$

$$\mathsf{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -p + \frac{n}{2}\left(\frac{\alpha - p}{\alpha + p}\right); \qquad (z \in \mathbb{U}),$$

for some $0 \leq \alpha < p$*, then*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \frac{p+\alpha}{2}; \qquad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,,$$
$$\operatorname{Re}\left\{-\frac{h'(z)}{z^{-p-1}}\right\} > \frac{p+\alpha}{2}; \qquad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,,$$

or equivalently,

$$f \in \mathcal{R}_p\left(\frac{p+\alpha}{2}\right), \quad h \in \mathcal{MR}_p\left(\frac{p+\alpha}{2}\right) \qquad (0 \leq \alpha < p)\,.$$

Remark 2.9. A special case of Corollary 2.8 with $f \in \mathcal{A}$ and $h \in \Sigma$ can be found in [21, Theorem 1] and [3, Corollary 2.3], respectively.

Putting $\mathfrak{p}_1(z) := \frac{f'(z)}{pz^{p-1}}$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) := \frac{h'(z)}{-pz^{-p-1}}$; $(z \in \mathbb{U})$ in Theorem 2.12, we get the following result.

Corollary 2.10. *If the functions* $f \in \mathcal{A}(p,n)$ *and* $h \in \Sigma(p,n)$ *satisfy the conditions*:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p + n\left(\frac{p+\alpha}{2p+\alpha}\right); \quad (z \in \mathbb{U})\,,$$
$$\operatorname{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\} < -p + n\left(\frac{p+\alpha}{2p+\alpha}\right); \quad (z \in \mathbb{U})\,,$$

for some $0 \leq \alpha < p$*, then*

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p+\alpha; \quad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,,$$
$$\left|\frac{h'(z)}{z^{-p-1}} + p\right| < p+\alpha; \quad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,.$$

Remark 2.11. As a special case we obtain [21, Theorem 2] that f is an element of the class $\mathcal{A}$.

Theorem 2.12. *Let* $\mathfrak{p} \in \mathcal{H}(n)$*, and suppose that*

$$\operatorname{Re}\left\{\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}\right\} < n\left(\frac{p+\alpha}{2p+\alpha}\right); \quad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,. \tag{10a}$$

Then,

$$|\mathfrak{p}(z) - 1| < 1 + \frac{\alpha}{p}; \quad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,. \tag{10b}$$

Proof. The function $\omega(z)$ is defined by

$$\mathfrak{p}(z) = \left(1+\frac{\alpha}{p}\right)\omega(z) + 1; \quad (z \in \mathbb{U},\ 0 \leq \alpha < p)\,. \tag{11}$$

Then $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$. Logarithmic differentiation of (11) yields that

$$\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}=\frac{(p+\alpha)\,z\omega'(z)}{(p+\alpha)\,\omega(z)+p};\quad (z\in\mathbb{U},\ 0\le\alpha<p)\,. \tag{12}$$

Now, suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\omega(z_0)|=1 \quad \text{and} \quad |\omega(z)|<1, \quad \text{when} \quad |z|<|z_0|.$$

Then, by applying Lemma 2.1, we have

$$z_0\omega'(z_0)=m\omega(z_0);\qquad \left(m\ge n\ge 1,\ \omega(z_0)=\mathrm{e}^{\mathrm{i}\theta};\ \theta\ne-\pi\right). \tag{13}$$

Form (12) and (13), we obtain

$$\begin{aligned}\mathsf{Re}\left\{\frac{z_0\mathfrak{p}'(z_0)}{\mathfrak{p}(z_0)}\right\}&=\mathsf{Re}\left\{\frac{m(p+\alpha)\,\mathrm{e}^{\mathrm{i}\theta}}{(p+\alpha)\,\mathrm{e}^{\mathrm{i}\theta}+p}\right\}\\&=\frac{m(p+\alpha)\,(p+\alpha+p\cos\theta)}{p^2+2p(p+\alpha)\cos\theta+(p+\alpha)^2}\\&\ge\frac{n(p+\alpha)}{2p+\alpha},\end{aligned}$$

which contradicts the hypothesis (10a). Thus, we conclude that $|\omega(z)|<1$ for all $\mathbb{U}$; and equation (11) yields the inequality (10b). □

Putting $\mathfrak{p}_1(z):=\mathscr{F}^{\eta;\mu}_{p,n}[f](z);\ (z\in\mathbb{U}),\ \mathfrak{p}_2(z):=\mathscr{G}^{\eta;\mu}_{p,n}[h](z);\ (z\in\mathbb{U})$, and $\alpha=0$ in Theorem 2.12, we get the following result.

Corollary 2.13. *If the functions $f\in\mathcal{A}(p,n)$ and $h\in\Sigma(p,n)$ satisfy the conditions*:

$$\mathsf{Re}\left\{\eta\left(1+\frac{zf''(z)}{f'(z)}-p\right)+\mu\left(p-\frac{zf'(z)}{f(z)}\right)\right\}<\frac{n}{2};\quad (z\in\mathbb{U})\,,$$
$$\mathsf{Re}\left\{\eta\left(1+\frac{zh''(z)}{h'(z)}+p\right)-\mu\left(p+\frac{zh'(z)}{h(z)}\right)\right\}<\frac{n}{2};\quad (z\in\mathbb{U})\,,$$

for all η,μ be real numbers not both zero, then $f\in\mathscr{B}^{\eta;\mu}_{p,n}(0)$ and $h\in\mathscr{M}^{\eta;\mu}_{p,n}(0)$.

The cases $p=n=1$, $\mathfrak{p}_1(z):=f'(z)$; $(z\in\mathbb{U})$, and $\mathfrak{p}_2(z):=-z^2h'(z)$; $(z\in\mathbb{U})$ in Theorem 2.12, lead to the following:

Corollary 2.14. *If the functions $f\in\mathcal{A}$ and $h\in\Sigma$ satisfy the following conditions*:

$$\mathsf{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\frac{2\alpha+3}{\alpha+2};\quad (z\in\mathbb{U})\,,$$
$$-\,\mathsf{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\}>\frac{1}{\alpha+2};\quad (z\in\mathbb{U})\,,$$

for some $0\le\alpha<1$, then $|f'(z)-1|<1+\alpha$ and $\left|z^2h'(z)+1\right|<1+\alpha$; $(z\in\mathbb{U})$.

Letting $p = n = 1$, $\alpha = 0$, $\mathfrak{p}_1(z) \coloneqq \frac{z}{f(z)}$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) \coloneqq \frac{1}{zh(z)}$; $(z \in \mathbb{U})$ in Theorem 2.12, we get the following result.

Corollary 2.15. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\mathsf{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{\alpha+2}; \quad (z \in \mathbb{U}),$$

$$-\mathsf{Re}\left\{\frac{zh'(z)}{h(z)}\right\} < \frac{3\alpha+2}{\alpha+2}; \quad (z \in \mathbb{U}),$$

for some $0 \le \alpha < 1$, then $\left|\frac{z}{f(z)} - 1\right| < 1+\alpha$ and $\left|\frac{1}{zh(z)} - 1\right| < 1+\alpha$. Especially for $\alpha = 0$ we have: $f \in \mathscr{B}_{1,1}^{0,1}(0)$ *and* $h \in \mathscr{M}_{1,1}^{0,1}(0)$.

By taking $\alpha = 0$, $\mathfrak{p}_1(z) \coloneqq \frac{zf'(z)}{pf(z)}$; $(z \in \mathbb{U})$, $\mathfrak{p}_2(z) \coloneqq \frac{zh'(z)}{-ph(z)}$; $(z \in \mathbb{U})$, and in Theorem 2.12, we get the following result.

Corollary 2.16. *If the functions $f \in \mathcal{A}(p,n)$ and $h \in \Sigma(p,n)$ satisfy the following conditions*:

$$\mathsf{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} < \frac{n}{2}; \qquad (z \in \mathbb{U}),$$
$$\mathsf{Re}\left\{1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)}\right\} < \frac{n}{2}; \qquad (z \in \mathbb{U}),$$

then $f \in \mathscr{B}_{p,n}^{1,1}(0)$, or equivalently $\left|\frac{zf'(z)}{f(z)} - p\right| < p$; $(z \in \mathbb{U})$ and $h \in \mathscr{M}_{p,n}^{1,1}(0)$, or equivalently $\left|\frac{zh'(z)}{h(z)} + p\right| < p$; $(z \in \mathbb{U})$.

Remark 2.17. A special case of Corollary (2.16) with $p = 1$ was given by Irmak and Çetin [13, Corollary 2], and Ponnusamy and Rajasekaran [23, Example 1].

Applying Corollary 2.16, we get the following sufficient conditions for order of convexity of integral operator $\mathscr{I}^{\eta_i,1-\eta_i}$, where $0 \le \eta_i \le 1$.

Corollary 2.18. *Let $0 \le \eta_i \le 1$ for all $i = 1, \ldots, m$. If the function f satisfy the condition:*

$$\mathsf{Re}\left\{1 + \frac{zf_i''(z)}{f_i'(z)} - \frac{zf_i'(z)}{f_i(z)}\right\} < \frac{n}{2}; \qquad (z \in \mathbb{U})$$

for all $i = 1, \ldots, m$. Then the integral operator $\mathscr{I}^{\eta_i,1-\eta_i}$ define by (5)*, belongs to the class $\mathscr{N}_p(\lambda)$, where $\lambda = 1 + \frac{n}{2}\sum_{i=1}^{m}\eta_i + pm$.*

Proof. Define

$$G(z) \coloneqq \int_0^z \prod_{i=1}^{m}\left[\frac{f_i'(\tau)}{p\tau^{p-1}}\right]^{\eta_i}\left[\frac{f_i(\tau)}{\tau^p}\right]^{1-\eta_i} \mathrm{d}\tau, \qquad (z \in \mathbb{U}). \tag{14}$$

By logarithmically differentiating and then taking the real part of both side (14), and applying Corollary 2.16, what obtained is:

$$\begin{aligned}
&\mathsf{Re}\left\{1+\frac{zG''(z)}{G'(z)}\right\} \\
&= 1+\sum_{i=1}^{m}\eta_i\,\mathsf{Re}\left\{1+\frac{zf_i''(z)}{f_i'(z)}-\frac{zf_i'(z)}{f_i(z)}\right\}+\sum_{i=1}^{m}\mathsf{Re}\left\{\frac{zf_i'(z)}{f_i(z)}-p\right\} \\
&\leq 1+\sum_{i=1}^{m}\eta_i\,\mathsf{Re}\left\{1+\frac{zf_i''(z)}{f_i'(z)}-\frac{zf_i'(z)}{f_i(z)}\right\}+\sum_{i=1}^{m}\left|\frac{zf_i'(z)}{f_i(z)}-p\right| \\
&< 1+\frac{n}{2}\sum_{i=1}^{m}\eta_i+pm.
\end{aligned}$$

□

Remark 2.19. A special case of Corollary 2.18 when $p = n = 1$ was given by Frasin [5, Theorem 2.5].

Theorem 2.20. *Let* $\mathfrak{p} \in \mathcal{H}(n)$, *and suppose that*

$$\left|\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}\right| < n\left(\frac{p-\alpha}{2\alpha}\right); \quad \left(z \in \mathbb{U},\ \frac{p}{2} \leq \alpha < p\right). \tag{15a}$$

Then

$$|\mathfrak{p}(z)-1| < 1-\frac{\alpha}{p}; \quad \left(z \in \mathbb{U},\ \frac{p}{2} \leq \alpha < p\right). \tag{15b}$$

Proof. We define $\omega(z)$ by

$$\mathfrak{p}(z) = \frac{p+(p-2\alpha)\,\omega(z)}{p(1-\omega(z))}; \quad \left(\frac{p}{2} \leq \alpha < p,\ \omega(z) \neq 1;\ z \in \mathbb{U}\right). \tag{16}$$

Then $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0) = 0$. Logarithmic differentiation of (16) yields that

$$\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} = \frac{2(p-\alpha)\,z\omega'(z)}{[1-\omega(z)]\,[p+(p-2\alpha)\,\omega(z)]}; \quad (z \in \mathbb{U}). \tag{17}$$

Now, suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\omega(z_0)| = 1 \quad \text{and} \quad |\omega(z)| < 1, \quad \text{when} \quad |z| < |z_0|.$$

Then, by applying Lemma 2.1, we have

$$z_0\omega'(z_0) = m\omega(z_0); \qquad \left(m \geq n \geq 1,\ \omega(z_0) = e^{i\theta};\ \theta \neq 0\right). \tag{18}$$

Form (17) and (18), we get

$$\left|\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}\right| = 2m(p-\alpha)\sqrt{\frac{1}{2\,(1-\cos\theta)\left(p^2+2p(p-2\alpha)\cos\theta+(p-2\alpha)^2\right)}}$$

$$\geq n\left(\frac{p-\alpha}{2\alpha}\right),$$

which contradicts the hypothesis (15a). Thus, we conclude that $|\omega(z)| < 1$ for all $\mathbb{U}$ and equation (16) yields the inequality (15b). □

Putting $\mathfrak{p}_1(z) := \mathscr{F}^{\eta;\mu}_{p,n}[f](z)$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) := \mathscr{G}^{\eta;\mu}_{p,n}[h](z)$; $(z \in \mathbb{U})$ in Theorem 2.20, we get the following result:

Corollary 2.21. *If the functions $f \in \mathcal{A}(p,n)$ and $h \in \Sigma(p,n)$ satisfy the following conditions*:

$$\left|\eta\left(1+\frac{zf''(z)}{f'(z)}-p\right)+\mu\left(p-\frac{zf'(z)}{f(z)}\right)\right| < n\left(\frac{p-\alpha}{2\alpha}\right); \quad (z \in \mathbb{U}),$$

$$\left|\eta\left(1+\frac{zh''(z)}{h'(z)}+p\right)-\mu\left(p+\frac{zh'(z)}{h(z)}\right)\right| < n\left(\frac{p-\alpha}{2\alpha}\right); \quad (z \in \mathbb{U}),$$

for some $\frac{p}{2} \leq \alpha < p$, then $f \in \mathscr{B}^{\eta;\mu}_{p,n}(\alpha)$ and $h \in \mathscr{M}^{\eta;\mu}_{p,n}(\alpha)$.

Putting $p = n = 1$, $\mathfrak{p}_1(z) := \frac{zf'(z)}{f(z)}$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) := -\frac{zh'(z)}{h(z)}$; $(z \in \mathbb{U})$ in Theorem 2.20, we get the following result:

Corollary 2.22. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2\alpha}; \quad (z \in \mathbb{U}),$$

$$\left|1+\frac{zh''(z)}{h'(z)}-\frac{zh'(z)}{h(z)}\right| < \frac{1-\alpha}{2\alpha}; \quad (z \in \mathbb{U}),$$

for some $\frac{1}{2} \leq \alpha < 1$, then $\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha$ and $\left|\frac{zh'(z)}{h(z)}+1\right| < 1-\alpha$.

Letting $p = n = 1$, $\mathfrak{p}_1(z) := \frac{z}{f(z)}$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) := \frac{1}{zh(z)}$; $(z \in \mathbb{U})$ in Theorem 2.20, we have the following corollary:

Corollary 2.23. *If the functions $f \in \mathcal{A}$ and $h \in \Sigma$ satisfy the following conditions*:

$$\left|1-\frac{zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2\alpha}; \quad (z \in \mathbb{U}),$$

$$\left|1+\frac{zh'(z)}{h(z)}\right| < \frac{1-\alpha}{2\alpha}; \quad (z \in \mathbb{U}),$$

for some $\frac{1}{2} \leq \alpha < 1$, then $f \in \mathscr{B}^{1,0}_{1,1}(\alpha)$ and $h \in \mathscr{M}^{1,0}_{1,1}(\alpha)$.

Finally, taking $\mathfrak{p}_1(z) := \frac{f'(z)}{pz^{p-1}}$; $(z \in \mathbb{U})$ and $\mathfrak{p}_2(z) := \frac{h'(z)}{-pz^{-p-1}}$; $(z \in \mathbb{U})$ in Theorem 2.20, we have the following result.

Corollary 2.24. *If the functions* $f \in \mathcal{A}(p,n)$ *and* $h \in \Sigma(p,n)$ *satisfy the following conditions*:

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < n\left(\frac{p-\alpha}{2\alpha}\right); \quad (z \in \mathbb{U}),$$

$$\left|1+\frac{zh''(z)}{h'(z)}+p\right| < n\left(\frac{p-\alpha}{2\alpha}\right); \quad (z \in \mathbb{U}),$$

for some $\frac{p}{2} \le \alpha < p$, *then* $f \in \mathscr{B}_{p,n}^{1,0}(\alpha)$ *and* $h \in \mathscr{M}_{p,n}^{1,0}(\alpha)$.

3. Strongly starlikeness and strongly convexity p-valent functions of order γ and type β

Theorem 3.1. *If* $f \in \mathcal{A}(p,n)$ *satisfies the following condition*

$$\left|\left[\frac{f(z)}{z^p}\right]^{\frac{1}{p-\beta}}\left(\frac{zf'(z)}{f(z)}-\beta\right)-p+\beta\right| < \frac{(n+1)(p-\beta)\sin\left(\frac{\pi}{2}\alpha\right)}{\sqrt{1+(n+1)^2+2(n+1)\cos\left(\frac{\pi}{2}\alpha\right)}},$$

where $0<\alpha\le 1$ *and* $0 \le \beta < p$, *then* $f \in \widetilde{\mathcal{S}}_p^*\left(\alpha, \frac{\beta}{p}\right)$.

Proof. Let $f \in \mathcal{A}(p,n)$ given by (1). Define $g(z)$ by

$$g(z) := \left[\frac{f(z)}{z^\beta}\right]^{\frac{1}{p-\beta}} = z+\frac{a_{n+p}}{p-\beta}z^{n+1}+\cdots; \qquad (0 \le \beta < p,\ z \in \mathbb{U}). \tag{19}$$

Differentiating (19) logarithmically,

$$\frac{zg'(z)}{g(z)} = \frac{1}{p-\beta}\left(\frac{zf'(z)}{f(z)}-\beta\right), \tag{20}$$

thus

$$g'(z) = \frac{1}{p-\beta}\left[\frac{f(z)}{z^p}\right]^{\frac{1}{p-\beta}}\left(\frac{zf'(z)}{f(z)}-\beta\right).$$

By applying Lemma 1.1, we conclude that $g \in \widetilde{\mathcal{S}}^*(\alpha)$. From (20):

$$\arg\left\{\frac{zf'(z)}{pf(z)}-\frac{\beta}{p}\right\} = \arg\left\{\left(\frac{p-\beta}{p}\right)\frac{zg'(z)}{g(z)}\right\},$$

therefore $f \in \widetilde{\mathcal{S}}_p^*\left(\alpha, \frac{\beta}{p}\right)$ and this completes the proof of the Theorem. □

By taking $p=n=1$ and $\alpha=1$ in Theorem 3.1, we get the following result.

Corollary 3.2. *If* $f \in \mathcal{A}$ *satisfies the following condition*

$$\left|\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\beta}}\left(\frac{zf'(z)}{f(z)}-\beta\right)-1+\beta\right| < \frac{2(1-\beta)}{\sqrt{5}},$$

where $0 \le \beta < 1$, *then* $f \in \mathcal{S}^*(\beta)$.

Theorem 3.3. *If* $f \in \Sigma(p,n)$ *satisfies the following condition*

$$\left|[z^p f(z)]^{\frac{1}{\beta-p}}\left(\frac{zf'(z)}{f(z)}+\beta\right)+p-\beta\right| < \frac{(n+1)(p-\beta)\sin\left(\frac{\pi}{2}\alpha\right)}{\sqrt{1+(n+1)^2+2(n+1)\cos\left(\frac{\pi}{2}\alpha\right)}},$$

where $0<\alpha\leq 1$ *and* $0\leq\beta<p$*, then* $f \in \mathcal{M}\widetilde{\mathcal{S}}_p^*\left(\alpha,\frac{\beta}{p}\right)$.

Proof. Let $f \in \Sigma(p,n)$ given by (2). The proof is similar to that of Theorem 3.1 with the function g defined by

$$g(z)=\left[z^{\beta} f(z)\right]^{\frac{1}{\beta-p}} = z+\frac{a_{n-p}}{\beta-p}z^{n+1}+\cdots; \qquad (z\in\mathbb{U},\ 0\leq\beta<p). \quad \square$$

Putting $p=n=1$, $\alpha=1$ and $\beta=0$ in Theorem 3.3, we get the following result.

Corollary 3.4. *If* $f \in \Sigma$ *satisfies the following condition*

$$\left|\frac{f'(z)}{f^2(z)}+1\right| < \frac{2}{\sqrt{5}}; \qquad (z\in\mathbb{U}),$$

then $f\in\mathcal{MS}^*$.

Putting $p=n=1$ and $\alpha=1$ in Theorem 3.3, the following result is obtained:

Corollary 3.5. *If* $f \in \Sigma$ *satisfies the following condition*

$$\left|[zf(z)]^{\frac{1}{\beta-1}}\left(\frac{zf'(z)}{f(z)}+\beta\right)+1-\beta\right| < \frac{2(1-\beta)}{\sqrt{5}},$$

where $0\leq\beta<1$*, then* $f\in\mathcal{MS}^*(\beta)$.

Theorem 3.6. *If* $f \in \mathcal{A}(p,n)$ *satisfies the following condition*

$$\left|\left[\frac{f'(z)}{pz^{p-1}}\right]^{\frac{1}{p-\beta}}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right)-p+\beta\right| < \frac{(n+1)(p-\beta)\sin\left(\frac{\pi}{2}\alpha\right)}{\sqrt{1+(n+1)^2+2(n+1)\cos\left(\frac{\pi}{2}\alpha\right)}},$$

where $0<\alpha\leq 1$ *and* $0\leq\beta<p$*, then* $f\in\widetilde{\mathcal{K}}_p\left(\alpha,\frac{\beta}{p}\right)$.

Proof. Let $f \in \mathcal{A}(p,n)$ given by (1). Define $g(z)$ by

$$g(z) := z\left[\frac{f'(z)}{pz^{p-1}}\right]^{\frac{1}{p-\beta}} = z+\frac{n+p}{p(p-\beta)}a_{n+p}z^{n+1}+\cdots \tag{21}$$

for $z\in\mathbb{U}$ and $0\leq\beta<p$. Differentiating (21) logarithmically, we obtain

$$\frac{zg'(z)}{g(z)} = \frac{1}{p-\beta}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right), \tag{22}$$

thus

$$g'(z)=\frac{1}{p-\beta}\left[\frac{f'(z)}{pz^{p-1}}\right]^{\frac{1}{p-\beta}}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right).$$

By applying Lemma 1.1, it is concluded that $g \in \widetilde{\mathcal{S}}^*(\alpha)$. From (22) we have

$$\arg\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{\beta}{p}\right\}=\arg\left\{\left(\frac{p-\beta}{p}\right)\frac{zg'(z)}{g(z)}\right\}, \tag{23}$$

therefore $f \in \widetilde{\mathcal{K}}_p\left(\alpha, \frac{\beta}{p}\right)$ and this completes the proof of the Theorem. □

The cases $p = n = 1$, $\alpha = 1$ and $\beta = 0$ in Theorems 3.6, lead to the following:

Corollary 3.7 ([18])**.** *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|f'(z)+zf''(z)-1\right|<\frac{2}{\sqrt{5}}; \qquad (z\in\mathbb{U}),$$

then $f \in \mathcal{K}$.

Putting $p = n = 1$ and $\alpha = 1$ in Theorem 3.6, we get the following result.

Corollary 3.8. *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|[f'(z)]^{\frac{1}{1-\beta}}\left(1+\frac{zf''(z)}{f'(z)}-\beta\right)-1+\beta\right|<\frac{2(1-\beta)}{\sqrt{5}},$$

where $0 \le \beta < 1$, then $f \in \mathcal{K}(\beta)$.

Theorem 3.9. *If $f \in \Sigma(p, n)$ satisfies the following condition*

$$\left|\left[\frac{f'(z)}{-pz^{-p-1}}\right]^{\frac{1}{\beta-p}}\left(1+\frac{zf''(z)}{f'(z)}+\beta\right)+p-\beta\right|$$
$$<\frac{(n+1)(p-\beta)\sin\left(\frac{\pi}{2}\alpha\right)}{\sqrt{1+(n+1)^2+2(n+1)\cos\left(\frac{\pi}{2}\alpha\right)}},$$

where $0 < \alpha \le 1$ and $0 \le \beta < p$, then $f \in \mathcal{M}\widetilde{\mathcal{K}}_p\left(\alpha, \frac{\beta}{p}\right)$.

Proof. Let $f \in \Sigma(p, n)$ given by (2). The proof is similar to that of Theorem 3.6 with the function g defined by

$$g(z)=z\left[\frac{f'(z)}{-pz^{-p-1}}\right]^{\frac{1}{\beta-p}}=z+\frac{n-p}{p(p-\beta)}a_{n-p}z^{n+1}+\cdots$$

for some $z \in \mathbb{U}$ and $0 \le \beta < p$. □

The cases $p = n = 1$, $\alpha = 1$ and $\beta = 0$ in Theorems 3.9, lead to the following:

Corollary 3.10. *If $f \in \Sigma$ satisfies the following condition*

$$\left|1-\frac{1}{z^2f'(z)}-\frac{f''(z)}{z[f'(z)]^2}\right|<\frac{2}{\sqrt{5}}; \qquad (z\in\mathbb{U}),$$

then $f \in \mathcal{MK}$.

The special cases $p = n = 1$ and $\alpha = 1$ in Theorem 3.9 brings us to the next corollary.

Corollary 3.11. *If $f \in \Sigma$ satisfies the following condition*

$$\left| \left[-z^2 f'(z)\right]^{\frac{1}{\beta-1}} \left(1 + \frac{zf''(z)}{f'(z)} + \beta\right) + 1 - \beta \right| < \frac{2(1-\beta)}{\sqrt{5}}; \qquad (z \in \mathbb{U}),$$

where $0 \leq \beta < 1$, then $f \in \mathcal{MK}(\beta)$.

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