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REFINED ENUMERATION OF VERTICES AMONG ALL ROOTED ORDERED d-TREES

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Abstract. In this paper, we enumerate the cardinalities for the set of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered d-trees with n edges. Our results unite and generalize several previous works in the literature.

1. Introduction

For a positive integer d, the nth d-Fuss-Catalan number is given by

$$\mathrm{Cat}_n^{(d)} = \frac{1}{dn+1}\binom{(d+1)n}{n} \quad \text{for } n \geq 0.$$

It is a generalization of the well-known nth Catalan number. Like Catalan numbers, there are several combinatorial objects which are enumerated by Fuss-Catalan numbers. The most well-known object is the Fuss-Catalan path. A d-Fuss-Catalan path of length $(d+1)n$ is a lattice path from $(0,0)$ to $((d+1)n, 0)$ using up steps $(1, d)$ and down steps $(1, -1)$ such that it stays weakly above the x-axis. Denote by $\mathcal{FC}_n^{(d)}$ the set of d-Fuss-Catalan paths of length $(d+1)n$. Another example is dissections of a $(dn+2)$-gon into $(d+2)$-gons by diagonals. There are three more combinatorial objects which are enumerated by d-Fuss-Catalan numbers.

Rooted ordered d-trees

A rooted tree can be considered as a process of successively gluing an edge (1-simplex) to a vertex (0-simplex) from the root in a half-plane, where the root is fixed in the line (1-dimensional hyperplane) as the boundary of the given half-plane. In the same way, we can define a *rooted d-tree* in $(d+1)$-dimensional lower Euclidean half-space $\mathbb{R}_-^{d+1}$ as follows: The root $\mathbf{r}$ is a $(d-1)$-simplex fixed in the boundary of $\mathbb{R}_-^{d+1}$. From the root $(d-1)$-simplex $\mathbf{r}$, we glue d-simplices (as edges) successively to one of previous $(d-1)$-simplices (as vertices) in $\mathbb{R}_-^{d+1}$.

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(See [1, d-dimensional trees].) By definition, if $d = 1$, a rooted d-tree is a rooted tree.

In a rooted tree, we can consider a *linear order* among all edges having one common vertex by their positions and such a tree is called a *rooted ordered tree*. Similarly, in higher dimensional cases, we can also give a linear order among d-simplices having one common $(d-1)$-simplices *naturally* by their positions and such a tree is also called a *rooted ordered d-tree*. Jani, Rieper, and Zeleke [5] enumerated ordered K-trees, which was obtained in a similar way using d-simplices with $d \in K$.

Rooted d-ary cacti

A *cactus* is a connected simple graph in which each edge is contained in exactly one *elementary cycle* which is just a polygon. These graphs are also known as 'Husimi trees'. They are introduced by Harary and Uhlenbeck [4]. If each elementary cycle has exactly d edges, a cactus is called a d-ary cactus. Bóna et al. [2] provided enumerations of various combinatorial objects of d-ary cacti.

Rooted d-tuplet trees

Instead of d-simplices used in rooted ordered d-trees, we may use $(d+1)$-gons. A root is a vertex fixed in the bounding hyperplane of a half-plane. One can glue $(d+1)$-gons to a vertex from the root. A tree obtained in this way is called a *rooted d-tuplet tree*, and the $(d+1)$-gons are called *d-tuplets*. As there is a *linear order* on the vertices in a tuplet, one can show that there is a one-to-one correspondence between rooted ordered d-trees with n edges and rooted d-tuplet trees with n tuplets. Thus rooted ordered d-trees and rooted d-tuplet trees are essentially the same. Note that the underlying graph of a d-tuplet tree is a $(d+1)$-ary cactus.

Let $\mathcal{T}_n^{(d)}$ be the set of rooted d-tuplet trees with n tuplets. It is easy to see that the cardinality of $\mathcal{T}_n^{(d)}$ is the nth d-Fuss-Catalan number $\mathrm{Cat}_n^{(d)}$. For example, there are 22 rooted 3-tuplet trees with 3 tuplets, see Figure 1. Clearly the number of vertices among rooted d-tuplet tree with n tuplets in $\mathcal{T}_n^{(d)}$ is

$$(dn+1)\,\mathrm{Cat}_n^{(d)} = \binom{(d+1)n}{n}. \tag{1}$$

In a rooted d-tuplet tree, the *degree* of a vertex is the number of tuplets it connects. We can have the notion of the *outdegree* of a vertex v, which is the number of tuplets starting at v and pointing away from the root. The *level* of a vertex v in a rooted d-tuplet tree is the distance (number of tuplets) from the root to v. Table 1 shows the number of all vertices of outdegree k at level ℓ among all rooted 3-tuplet trees in $\mathcal{T}_3^{(3)}$. For example, there are 9 vertices of outdegree 1 at level 2 in $\mathcal{T}_3^{(3)}$, see Figure 1.

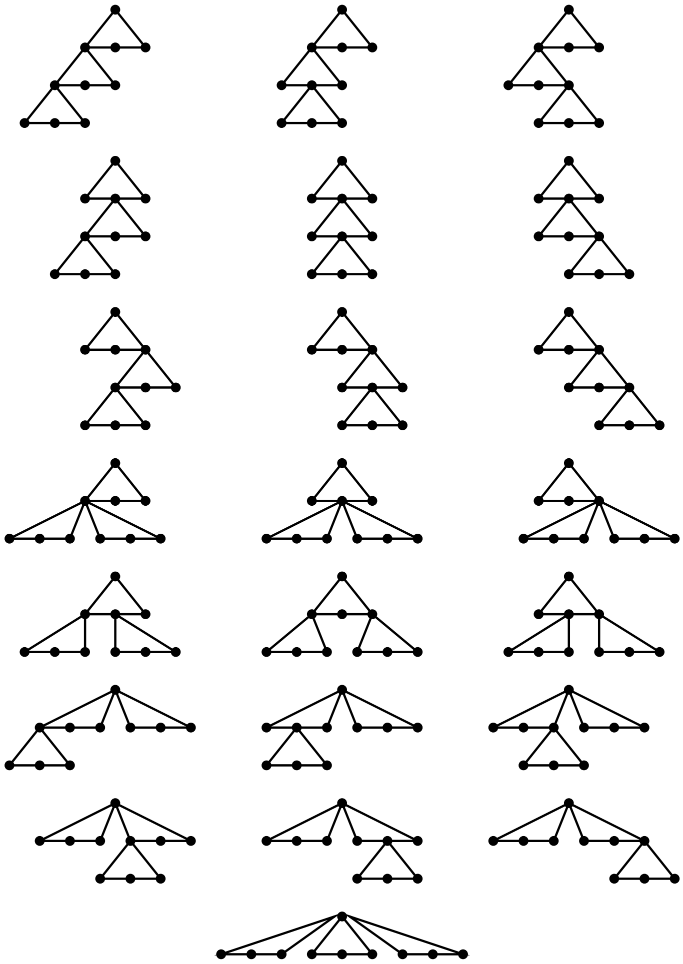

FIGURE 1. All rooted 3-tuplet trees with 3 tuplets in $\mathcal{T}_3^{(3)}$

In a rooted d-tuplet tree, there exists the unique vertex u in each tuplet such that its level is less than levels of the other vertices $v_1, \ldots, v_d$. Here, u is called the *parent* of v_i's and each v_i is called a *child* of u. For each vertex v (except the root), there exists the unique tuplet containing v toward the root, called the *tuplet of* v. Vertices with the same parent are called *siblings*. For two siblings v and w, if v is on the left of w, v is called an *elder* sibling of w; meanwhile, w is called a *younger* sibling of v.

In 2002, using an involution, Seo and Shin [6] gave a formula for the number of leaves among all trees. Recently Eu, Seo, and Shin [3] gave a formula for

TABLE 1. The number of vertices of outdegree k at level ℓ among all rooted 3-tuplet trees in $\mathcal{T}_3^{(3)}$

$\ell\backslash k$	0	1	2	3	$\sum$
0	0	15	6	1	22
1	66	21	3	0	90
2	72	9	0	0	81
3	27	0	0	0	27
$\sum$	165	45	9	1	220

the number of vertices among all trees in the set of rooted ordered trees under some conditions.

Theorem 1 (Eu, Seo, and Shin, 2017)**.** *Given $n \geq 1$, for any nonnegative integers k and ℓ, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered trees with n edges is*

$$\binom{2n-k}{n+\ell}. \tag{2}$$

We give a generalization of the formula (2) for $\mathcal{T}_n^{(d)}$ by generalizing their bijection.

Theorem 2 (Main Result)**.** *Given $n \geq 1$, for any nonnegative integers k and ℓ, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted d-tuplet trees with n tuplets is*

$$d^\ell \binom{(d+1)n-k}{dn+\ell}. \tag{3}$$

We also find a refinement of the formula (3).

Theorem 3. *Given $n \geq 1$, for any two nonnegative integers i, j, one nonnegative integer k which is a multiple of d, and one positive integer ℓ, the number of all vertices among all rooted d-tuplet trees with n tuplets such that*

- *having at least i elder siblings,*
- *having at least j younger siblings,*
- *having at least k children,*
- *at level $\geq \ell$*

is

$$d^\ell \left(1 - \frac{\beta}{d}\frac{dn+\ell}{(d+1)n-\alpha}\right)\binom{(d+1)n-\alpha}{dn+\ell}, \tag{4}$$

where α and β are nonnegative integers satisfying $i+j+k = \alpha d + \beta$ and $0 \leq \beta < d$.

The rest of the paper is organized as follows. In Section 2, we show the Theorem 2 bijectively. In Section 3, we give a combinatorial proof of the

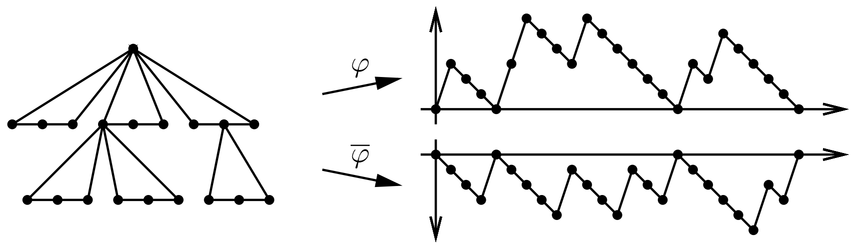

FIGURE 2. Two bijections φ and $\overline{\varphi}$

Theorem 3. In Section 4, we present corollaries induced from Theorems 2 and 3.

2. A bijective proof of Theorem 2

Henceforth, a *tree* is assumed to be a rooted d-tuplet tree. Let $\mathcal{V}$ be the set of pairs (T, v) such that v is a vertex of outdegree $\geq k$ at level $\geq \ell$ in $T \in \mathcal{T}_n^{(d)}$. Let $\mathcal{P}$ be the set of sequences in $\{0, \dots, d-1\}$ of length ℓ. Let $\mathcal{L}$ be the set of *lattice paths* of length $((d+1)n-k)$ from (k, dk) to $((d+1)n, -(d+1)\ell)$, consisting of $(n-k-\ell)$ up-steps along the vector $(1, d)$ and $(dn+\ell)$ down-steps along the vector $(1, -1)$. To show Theorem 2, it is enough to construct a bijection Φ between $\mathcal{V}$ and $\mathcal{P} \times \mathcal{L}$, due to

$$\#\mathcal{P} = d^\ell, \quad \#\mathcal{L} = \binom{(d+1)n-k}{n-k-\ell,\ dn+\ell} = \binom{(d+1)n-k}{dn+\ell}.$$

Three bijections φ, $\overline{\varphi}$, and ψ

Let a *reverse d-Fuss-Catalan path* of length $(d+1)n$ be a lattice path from $(0,0)$ to $((d+1)n, 0)$ using up steps $(1, d)$ and down steps $(1, -1)$ such that it stays weakly below the x-axis. Denote by $\overline{\mathcal{FC}}_n^{(d)}$ the set of reverse d-Fuss-Catalan paths of length $(d+1)n$.

Before constructing the bijection Φ, we introduce three bijections

$$\varphi : \mathcal{T}_n^{(d)} \to \mathcal{FC}_n^{(d)}, \qquad \overline{\varphi} : \mathcal{T}_n^{(d)} \to \overline{\mathcal{FC}}_n^{(d)}, \qquad \psi : \mathcal{T}_n^{(d)} \to \mathcal{FC}_n^{(d)}.$$

The bijection φ corresponds a tree to a lattice path weakly above the x-axis by recording the steps when the tree is traversed in preorder: whenever we go down a side of a tuplet, record an up-step along the vector $(1, d)$ and whenever we go right or up a side of a tuplet, record a down-step along the vector $(1, -1)$.

Similarly, the bijection $\overline{\varphi}$ corresponds a tree to a lattice path weakly below the x-axis by recording the steps when the tree is traversed in preorder: whenever we go down or right a side, record a down-step along the vector $(1, -1)$ and whenever we go up a side, record an up-step along the vector $(1, d)$. An example of two bijections φ and $\overline{\varphi}$ is shown in Figure 2.

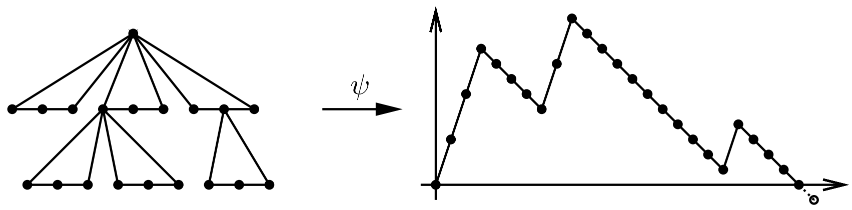

FIGURE 3. The bijection ψ

The bijection ψ corresponds a tree to a lattice path weakly above the x-axis by recording the steps when the tree is traversed in preorder: whenever we meet a vertex of outdegree m, except the last leaf, record m up-steps followed by one down-step. An example of the bijection ψ is shown in Figure 3.

Step 1

Given $(T, v) \in \mathcal{V}$, let D_v be the subtree consisting of v and all its descendants in T, say the *descendant subtree of* v. Letting $\ell'(\geq \ell)$ be the level of v, consider the path from v to the root r of T

$$v(= v_0) \to v_1 \to \cdots \to v_\ell \to \cdots \to v_{\ell'-1} \to r(= v_{\ell'}).$$

Record the number p_i of elder siblings of v_i in the tuplet of v_i for all $0 \leq i \leq \ell - 1$. For all $0 \leq i \leq \ell - 1$, let w_i be the youngest sibling of v_i in the tuplet of v_i. By exchanging two subtrees D_{v_i} and D_{w_i}, we obtain the tree T'.

Step 2

For all $1 \leq i \leq \ell - 1$ and $i = \ell'$, let R_i be the subtree consisting v_i and all its descendants on the right of the tuplet of v_{i-1} in T'. We obtain the tree L by cutting the $\ell + 1$ subtrees $D_v, R_1, \ldots, R_{\ell-1}, R_{\ell'}$ from the tree T', see Figure 4.

A construction of the bijection Φ

We will construct the bijection Φ between $\mathcal{V}$ to $\mathcal{P} \times \mathcal{L}$. Given $(T, v) \in \mathcal{V}$, let $k'(\geq k)$ be the outdegree of v in T and let $\ell'(\geq \ell)$ be the level of v in T. We separate two cases:

Case I. If v is not the root of T, i.e., $\ell' > 0$. We obtain the sequence $p = (p_0, \ldots, p_{\ell-1}) \in \mathcal{P}$ in Step 1 and $(\ell + 2)$ trees $D_v, R_1, R_2, \ldots, R_{\ell-1}, R_{\ell'}, L$ after Step 2 as Figure 4.

Let ρ be the mapping on the set of lattice paths defined by

$$\rho(s_1 s_2 \cdots s_n) = s_2 \cdots s_n s_1,$$

where each s_i is a step. Note that ρ^m means to apply ρ recursively m times.

Clearly, the outdegree of the root of D_v is k'. In the tree L, there are no younger siblings of v in the tuplet of v and the outdegree of vertex v is 0.

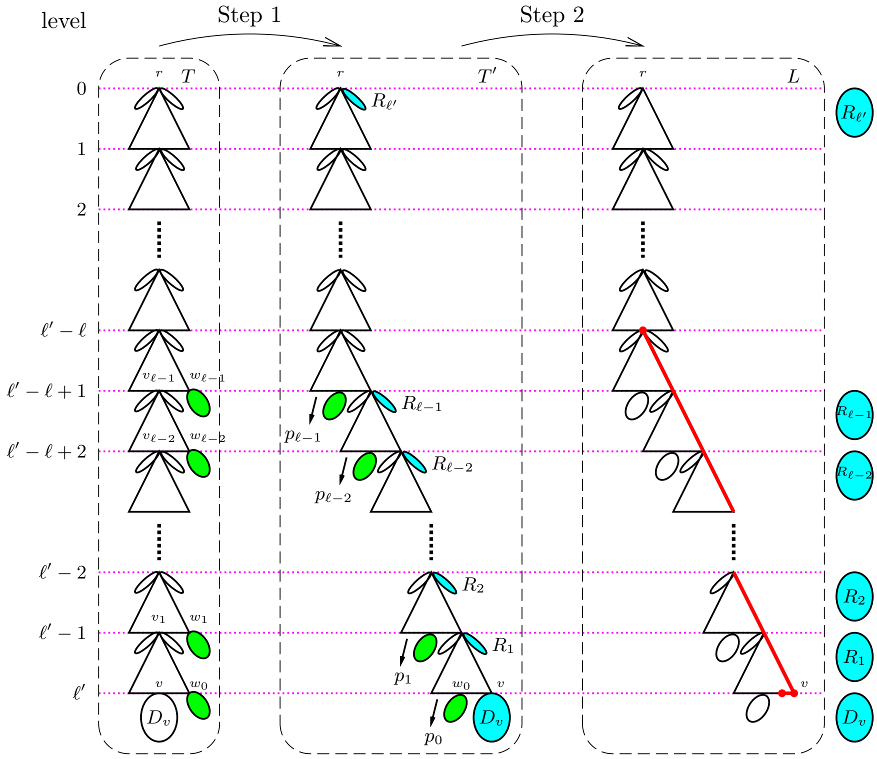

FIGURE 4. Tree decomposition

Thus the lattice path $\rho^{a+\ell}(\overline{\varphi}(L))$ ends with one down-step and ℓ consecutive up-steps, where a is the number of vertices of L which precede v in preorder.

We define a lattice path P from $(0,0)$ to $((d+1)n+(\ell+1), -(\ell+1))$ by

$$P = \psi(D_v) \searrow \varphi(R_1) \searrow \varphi(R_2) \searrow \cdots \searrow \varphi(R_{\ell-1}) \searrow \varphi(R_{\ell'}) \searrow \rho^{a+\ell}(\overline{\varphi}(L)),$$

where $\searrow$ means a down-step.

Case II. If v is the root of T, i.e., $\ell' = 0$. We define a sequence $p = () \in \mathcal{P}$ and a lattice path

$$P = \psi(T) \searrow .$$

In all cases, the lattice path P always starts with at least k (precisely k') consecutive up-steps and ends with one down-step and ℓ consecutive up-steps as red segments in Figure 5.

By removing the first k steps and the last $(\ell+1)$ steps from P, we obtain the lattice path $\hat{P}$ of length $((d+1)n-k)$ from (k, dk) to $((d+1)n, -(d+1)\ell)$,

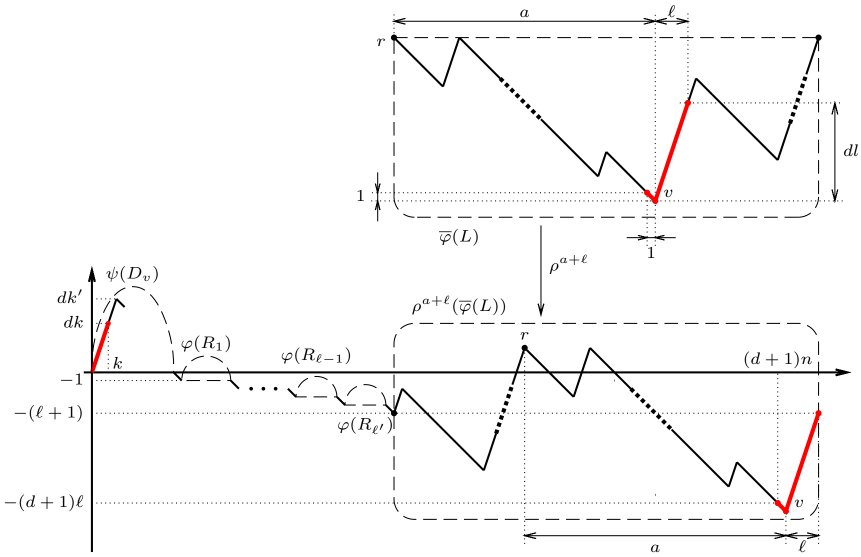

FIGURE 5. Outline of a lattice path P induced from tree decomposition

consisting of $(n-k-\ell)$ up-steps along the vector $(1,d)$ and $(dn+\ell)$ down-steps along the vector $(1,-1)$, so $\hat{P}$ belongs to $\mathcal{L}$.

Hence the map $\Phi : \mathcal{V} \to \mathcal{P} \times \mathcal{L}$ is defined by

$$\Phi(T,v) = (p, \hat{P}).$$

A description of the bijection Φ^{-1}

In the Case I of the construction of the bijection Φ, given a lattice path P from $(0,0)$ to $((d+1)n+(\ell+1), -(\ell+1))$, we decompose P into $(\ell+2)$ paths $P_D, P_1, \dots, P_{\ell-1}, P_{\ell'}, P_L$ by removing the leftmost down-steps from height $-i$ to height $-(i+1)$ for $0 \le i \le \ell$. Some of those paths may be empty.

Clearly all the paths $P_D, P_1, \dots, P_{\ell-1}, P_{\ell'}$ are d-Fuss-Catalan path. By moving all the steps after the leftmost highest vertex in the lattice path P_L to the beginning, we obtain a reverse d-Fuss-Catalan path $\overline{P}_L$ from P_L. Since φ, $\overline{\varphi}$, and ψ are bijections, we can restore trees $D_v, R_1, \dots, R_{\ell-1}, R_{\ell'}, L$ from $P_D, P_1, \dots, P_{\ell-1}, P_{\ell'}, \overline{P}_L$.

Therefore, Φ is a bijection between $\mathcal{V}$ and $\mathcal{P} \times \mathcal{L}$ since all the remaining processes are also reversible.

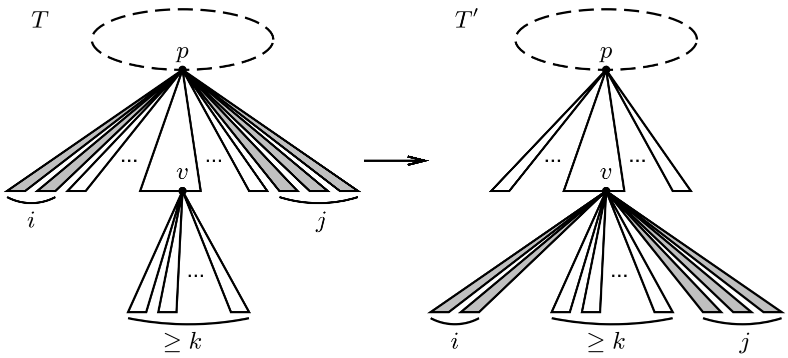

FIGURE 6. Cut-and-paste bijection $\gamma_{i,j}$

3. Proof of Theorem 3

For any three nonnegative integers i, j, k and one positive integer ℓ, denote by $\mathcal{V}_n^{(d)}(i,j,k;\ell)$ the set of pairs (T,v) whose tree T in $\mathcal{T}_n^{(d)}$ and vertex v in T such that

- v has at least i elder siblings in T,
- v has at least j younger siblings in T,
- v has at least k children in T,
- v is at level $\geq \ell$ in T.

We show the following lemma, which is a particular case of Theorem 3, that is, i and j are multiples of d.

Lemma 4. *Given $n \geq 1$, for any three nonnegative integers i, j, k, all of which are multiples of d, and one positive integer ℓ, the cardinality of $\mathcal{V}_n^{(d)}(i,j,k;\ell)$ is*

$$d^\ell \binom{(d+1)n-\alpha}{dn+\ell},$$

where α is the nonnegative integer satisfying $i+j+k=\alpha d$.

Proof. That a vertex v has at least i elder (or younger resp.) siblings means that there exists at least i/d (or j/d resp.) d-tuplets directly connected from the parent of v on its left (or right resp.).

A pair (T,v) in $\mathcal{V}_n^{(d)}(i,j,k,\ell)$ corresponds to a pair (T',v) in $\mathcal{V}_n^{(d)}(0,0,i+j+k,\ell)$ under a *cut-and-paste* bijection $\gamma_{i,j} : (T,v) \mapsto (T',v)$ which cuts the leftmost i/d tuplets connected from the parent p of v and pastes them at v on the left and does again the rightmost j/d tuplets connected from the parent p of v on the right, as Figure 6.

Since that v has at least $i+j+k$ children means that the outdegree of v greater than or equal to $\alpha = \frac{i+j+k}{d}$, this case corresponds to $k \leftarrow \alpha$ of Theorem 2. □

In Theorem 3, what to find is the cardinality of $\mathcal{V}_n^{(d)}(i,j,k;\ell)$ for any two nonnegative integers i, j, one nonnegative integer k which is a multiple of d, and one positive integer ℓ.

Given $(T,v) \in \mathcal{V}_n^{(d)}(i,j,k;\ell)$, let w be the jth younger sibling of v. By exchanging two subtrees D_v and D_w, we obtain (T',v) in $\mathcal{V}_n^{(d)}(i+j,0,k;\ell)$ from (T,v) in $\mathcal{V}_n^{(d)}(i,j,k;\ell)$. Let α and β be the quotient and the remainder when $i+j+k$ is divided by d, that is,

$$i+j+k = \alpha d + \beta.$$

By applying the cut-and-paste bijection $\gamma_{i+j-\beta,0}$, we obtain (T'',v) in $\mathcal{V}_n^{(d)}(\beta,0,\alpha d;\ell)$ from (T',v) in $\mathcal{V}_n^{(d)}(i+j,0,k;\ell)$. One can show that the values

$$\#\mathcal{V}_n^{(d)}(i,0,\alpha d;\ell) - \#\mathcal{V}_n^{(d)}(i+1,0,\alpha d;\ell)$$

are the same for all $0 \le i \le d-1$ under exchanging two descendant subtrees of two sibling in the same tuplet. By telescoping, we get the formula

$$\begin{aligned}&\#\mathcal{V}_n^{(d)}(0,0,\alpha d;\ell) - \#\mathcal{V}_n^{(d)}(\beta,0,\alpha d;\ell)\\ &= \frac{\beta}{d}\left[\#\mathcal{V}_n^{(d)}(0,0,\alpha d;\ell) - \#\mathcal{V}_n^{(d)}(d,0,\alpha d;\ell)\right].\end{aligned}$$

By Lemma 4, we have

$$\begin{aligned}\#\mathcal{V}_n^{(d)}(0,0,\alpha d;\ell) &= d^\ell \binom{(d+1)n-\alpha}{dn+\ell},\\ \#\mathcal{V}_n^{(d)}(d,0,\alpha d;\ell) &= d^\ell \binom{(d+1)n-\alpha-1}{dn+\ell}.\end{aligned}$$

Thus we get the cardinality of $\mathcal{V}_n^{(d)}(\beta,0,\alpha d;\ell)$ and the desired formula (4).

4. Further results

From Theorem 2, we can obtain the following result.

Corollary 5. *Given $n \ge 1$, for any two nonnegative integers k and ℓ, the number of all vertices of outdegree k at level ℓ among d-trees in $\mathcal{T}_n^{(d)}$ is*

$$d^\ell \frac{dk+(d+1)\ell}{(d+1)n-k}\binom{(d+1)n-k}{dn+\ell}. \tag{5}$$

Proof. By the sieve method with (3), we obtain the formula (5) from

$$\begin{aligned}&d^\ell\binom{(d+1)n-k}{dn+\ell} - d^\ell\binom{(d+1)n-k-1}{dn+\ell}\\ &- d^{\ell+1}\binom{(d+1)n-k}{dn+\ell+1} + d^{\ell+1}\binom{(d+1)n-k-1}{dn+\ell+1}.\end{aligned}$$

□

The next result follows from Theorem 3 for $d = 1$.

Corollary 6. *Given $n \geq 1$, for any three nonnegative integers i, j, k, and one positive integer ℓ, the number of all vertices among trees in $\mathcal{T}_n$ such that*

- *having at least i elder siblings,*
- *having at least j younger siblings,*
- *having at least k children,*
- *at level $\geq \ell$*

is

$$\binom{2n-i-j-k}{n+\ell}.$$

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