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A NOTE ON THE FIRST ORDER COMMUTATOR $\mathcal{C}_2$

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Abstract. This paper gives a counterexample to show that the first order commutator $\mathcal{C}_2$ is not bounded from $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ into $L^{1/2}(\mathbb{R})$. Then we introduce the atomic definition of abstract weighted Hardy spaces $H^1_{ato,\omega}(\mathbb{R})$ and study its properties. At last, we prove that $\mathcal{C}_2$ maps $H^1_{ato,w}(\mathbb{R}) \times H^1_{ato,w}(\mathbb{R})$ into $L^{1/2}_\omega(\mathbb{R})$.

1. Introduction

Calderón commutators first appeared in the study of the Cauchy integral along Lipschitz curves and led to the first proof of the L^2-boundedness of the latter. Let A be a Lipschitz function on $\mathbb{R}$ (i.e., $A' = a \in L^\infty$) and let $\Gamma = (t, A(t))$ be a plane curve. With this parameterization we can regard any function f defined on Γ as a function of t and conversely. Given $f \in \mathcal{S}(\mathbb{R})$, the Cauchy integral

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_\infty^\infty \frac{f(t)(1+ia(t))}{t + iA(t) - z} dt$$

defines an analytic function in the open set $\Omega_+ = \{z = x + iy \in \mathbb{C} : y > A(x)\}$. It can be shown that the limit of $C_\Gamma f(z)$, as z approaches Γ from above and nontangentially, is given by

$$\frac{1}{2}\left[f(x) + \frac{i}{\pi} \lim_{\epsilon\to 0} \int_{|x-t|>\epsilon} \frac{f(t)(1+ia(t))}{x - t + i(A(x) - A(t))} dt\right].$$

This lead to consider the operator

$$Tf(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x - y + i(A(x) - A(y))} dy,$$

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whose kernel

$$K(x,y)=\frac{1}{x-y+i(A(x)-A(y))}.$$

If $\|a\|_\infty<1$, then we can expand this kernel as

$$K(x,y)=\sum_{m=0}^{\infty} i^m K_m(x,y),$$

where

$$K_m(x,y)=\frac{(A(x)-A(y))^m}{(x-y)^{m+1}}.$$

Therefore, it is natural to consider the boundedness of the following Calderón commutators:

$$\mathcal{C}_{m+1}(a,f)(x):=\int_{\mathbb{R}}\frac{(A(x)-A(y))^m}{(x-y)^{m+1}}f(y)dy,\ \text{where } A'=a. \tag{1.1}$$

In this article, we mainly study the first order commutator as follows:

$$\mathcal{C}_2(a,f)(x):=\int_{\mathbb{R}}\frac{A(x)-A(y)}{(x-y)^2}f(y)dy. \tag{1.2}$$

It is known that $\mathcal{C}_2$ is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$, when $1<p,\ q\le\infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r},\ r\neq\infty$. Moreover, if either p or q equals to 1, then $\mathcal{C}_2(a,f_2)$ maps $L^p(\mathbb{R})\times L^q(\mathbb{R})$ to $L^{r,\infty}(\mathbb{R})$, in particular, it maps $L^1(\mathbb{R})\times L^1(\mathbb{R})$ to $L^{1/2,\infty}(\mathbb{R})$. For these results, we refer the articles [3], [4] and [5].

Define

$$e(x)=\begin{cases}1, & x>0,\\ -1, & x<0.\end{cases} \tag{1.3}$$

The first order commutator can be written as

$$\mathcal{C}_2(a,f_2)(x)=\int_{\mathbb{R}}\int_{\mathbb{R}}K(x,y_1,y_2)a(y_1)f_2(y_2)dy_1dy_2, \tag{1.4}$$

where the kernel K is

$$K(x,y_1,y_2)=\frac{e(x-y_1)-e(y_2-y_1)}{(x-y_2)^2}. \tag{1.5}$$

It is natural for us to consider whether the first order commutator $\mathcal{C}_2$ is bounded from $H^{p_1}(\mathbb{R})\times H^{p_2}(\mathbb{R})$ into $L^p(\mathbb{R})$, when $0<p_1,p_2\le 1$, and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. In this paper, we first give a counterexample to illustrate that $\mathcal{C}_2$ is not bounded from $H^1(\mathbb{R})\times H^1(\mathbb{R})$ into $L^{1/2}(\mathbb{R})$. Therefore, we want to seek proper Hardy space to make sure the truth of the boundedness of $\mathcal{C}_2$. Then we introduce the atomic definition of abstract weighted Hardy spaces and obtain the boundedness of $\mathcal{C}_2$ from the abstract weighted Hardy spaces $H^1_{ato,w}(\mathbb{R})\times H^1_{ato,w}(\mathbb{R})$ into $L^{1/2}_w(\mathbb{R})$.

The layout of the paper is as follows. In Section 2, we will give an example to show that it is not true for the boundedness of $\mathcal{C}_2$ from $H^1(\mathbb{R})\times H^1(\mathbb{R})$

into $L^{1/2}(\mathbb{R})$. In Section 3, we firstly introduce some notations and atomic definition of abstract weighted Hardy spaces and study related basic properties. Then we also get its dual spaces BMO_w. In Section 4, in order to obtain the boundedness of $\mathcal{C}_2$ on abstract weighted Hardy spaces, we establish a weighted version of Proposition 4.7 in [1]. Based on the methods of [1] and [6], we prove the boundedness of $\mathcal{C}_2$ from the abstract weighted Hardy spaces $H^1_{ato,w}(\mathbb{R}) \times H^1_{ato,w}(\mathbb{R})$ into $L^{1/2}_w(\mathbb{R})$.

Throughout, the letter "C" will denote (possibly different) constants that are independent of the essential variables.

2. A counterexample

Theorem 2.1. *The first order commutator $\mathcal{C}_2$ is not bounded from $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ into $L^{1/2}(\mathbb{R})$.*

Proof. Considering the following two functions

$$f_1(x) := \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ -1, & x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$f_2(x) := \begin{cases} 1, & x \in [0, 1), \\ -1, & x \in [1, 2], \end{cases}$$

it is not difficult to check that $f_1, f_2 \in H^1(\mathbb{R})$.

Furthermore, by the definition of the kernel K, one can write (1.4) as

$$\begin{aligned} &\mathcal{C}_2(f_1, f_2)(x) \\ &= \int\int K(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\ &= \int\int \left(\frac{-1}{(x-y_2)^2} \chi_{\{x<y_1<y_2\}} + \frac{1}{(x-y_2)^2} \chi_{\{y_2<y_1<x\}} \right) f_1(y_1) f_2(y_2) dy_1 dy_2. \end{aligned}$$

In what follows, we first consider the case of $x < -4$, then consider the case of $x > 4$ for the first order commutator $\mathcal{C}_2$.

Case 1. $x < -4$.

By the definition of f_1 and f_2, we write

$$\begin{aligned} \mathcal{C}_2(f_1, f_2)(x) &= \int\int \frac{-1}{(x-y_2)^2} \chi_{(\{x<y_1<y_2\}} f_1(y_1) f_2(y_2) dy_1 dy_2 \\ &= \int_0^1 \int_0^1 \frac{-1}{(x-y_2)^2} \chi_{\{x<y_1<y_2\}} f_1(y_1) f_2(y_2) dy_1 dy_2 \\ &\quad + \int_1^2 \int_0^1 \frac{-1}{(x-y_2)^2} f_1(y_1) f_2(y_2) dy_1 dy_2 \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned}$$

For the term II, using the vanishing property of f_1, together with the Mean Value Theorem, one obtains

$$|\mathrm{II}| = \Big|\int_1^2\int_0^1 \Big(\frac{1}{(x-1)^2} - \frac{-1}{(x-y_2)^2}\Big) f_1(y_1)dy_1dy_2\Big| \tag{2.1}$$

$$\leqslant C\int_1^2\int_0^1 \frac{|y_2-1|}{|x-\xi|^3}|f_1(y_1)|dy_1dy_2 \tag{2.2}$$

$$\leqslant C\frac{1}{|x|^3}, \tag{2.3}$$

where $\xi \in (1, y_2)$.

For the term I, we write

$$\begin{aligned}
\mathrm{I} &= \int_0^1\int_0^1 \frac{-1}{(x-y_2)^2}\chi_{\{x<y_1<y_2\}} f_1(y_1)f_2(y_2)dy_1dy_2\\
&= \int_0^{\frac{1}{2}}\int_0^{y_2}\frac{-1}{(x-y_2)^2}dy_1dy_2 + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\frac{-1}{(x-y_2)^2}dy_1dy_2\\
&\quad + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^{y_2}\frac{1}{(x-y_2)^2}dy_1dy_2\\
&= \int_0^{\frac{1}{2}}\frac{-1}{(x-y_2)^2}y_2dy_2 + \frac{1}{2}\int_{\frac{1}{2}}^1\frac{-1}{(x-y_2)^2}dy_2 + \int_{\frac{1}{2}}^1\frac{1}{(x-y_2)^2}(y_2-\frac{1}{2})dy_2\\
&= \int_0^{\frac{1}{2}}\frac{-1}{(x-y_2)^2}y_2dy_2 + \int_{\frac{1}{2}}^1\frac{1}{(x-y_2)^2}y_2dy_2 + \int_{\frac{1}{2}}^1\frac{-1}{(x-y_2)^2}dy_2\\
&= \int_0^{\frac{1}{2}}\frac{-1}{(x-y_2)^2}y_2dy_2 + \int_0^{\frac{1}{2}}\frac{1}{(x-(y_2+\frac{1}{2}))^2}(y_2+\frac{1}{2})dy_2\\
&\quad + \int_{\frac{1}{2}}^1\frac{-1}{(x-y_2)^2}dy_2\\
&= \int_0^{\frac{1}{2}}\Big(\frac{-y_2}{(x-y_2)^2} + \frac{y_2}{(x-(y_2+1/2))^2}\Big)dy_2 + \frac{1}{2}\int_{\frac{1}{2}}^1\frac{-1}{(x-y_2)^2}dy_2\\
&:= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

By the Mean Value Theorem, we have

$$|\mathrm{I}_1| \leqslant \int_0^{\frac{1}{2}} \Big|\frac{y_2}{(x-\zeta)^3}\Big|dy_2 \leqslant C\frac{1}{|x|^3},\ \zeta\in(y_2, y_2+\frac{1}{2})$$

and

$$|\mathrm{I}_2| = \Big|\frac{1}{4(x-1)(x-\frac{1}{2})}\Big| \sim \frac{1}{|x|^2}, \quad x\to-\infty.$$

Case 2. $x > 4$.

In this case, proceeding with an argument similar to **Case 1**, we give the following details.

$$\begin{aligned}\mathcal{C}_2(f_1,f_2)(x) &= \int_0^2\int_0^1 \frac{1}{(x-y_2)^2}\chi_{\{y_2<y_1<x\}}f_1(y_1)f_2(y_2)dy_1dy_2\\ &= \int_0^1\int_0^1 \frac{1}{(x-y_2)^2}\chi_{\{y_2<y_1<x\}}f_1(y_1)f_2(y_2)dy_1dy_2\\ &\quad+\int_1^2\int_0^1 \frac{1}{(x-y_2)^2}f_1(y_1)f_2(y_2)dy_1dy_2\\ &:= \mathrm{III}+\mathrm{IV}.\end{aligned}$$

Similarly to II, we have

$$\begin{aligned}|\mathrm{IV}| &= |\int_1^2\int_0^1 (\frac{1}{(x-y_2)^2}-\frac{1}{(x-1)^2})f_1(y_1)dy_1dy_2|\\ &\leqslant C\int_1^2\int_0^1 \frac{|y_2-1|}{|x-\eta|^3}|f_1(y_1)|dy_1dy_2\\ &\leqslant C\frac{1}{|x|^3},\end{aligned}$$

where $\eta\in(1,y_2)$.

Next, we estimate the term III.

$$\begin{aligned}\mathrm{III} = &\int_0^1\int_0^1 \frac{1}{(x-y_2)^2}\chi_{(y_2<y_1<x)}f_1(y_1)f_2(y_2)dy_1dy_2\\ = &\int_0^{y_1}\int_0^{\frac12}\frac{1}{(x-y_2)^2}dy_1dy_2+\int_0^{\frac12}\int_{\frac12}^1\frac{-1}{(x-y_2)^2}dy_1dy_2\\ &+\int_{\frac12}^{y_1}\int_{\frac12}^1\frac{-1}{(x-y_2)^2}dy_1dy_2\\ = &\int_0^{\frac12}(\frac{1}{x}-\frac{1}{x-y_1})dy_1+\frac12(\frac{1}{x-1/2}-\frac{1}{x})+\int_{\frac12}^1(\frac{1}{x-y_1}-\frac{1}{x-1/2})dy_1\\ = &\int_0^{\frac12}\frac{-1}{x-y_1}dy_1+\int_{\frac12}^1\frac{1}{x-y_1}dy_1\\ = &\ \ln\big(\frac{x^2-x+\frac14}{x(x-1)}\big)=\ln\big(1+\frac{1}{4(x^2-x)}\big).\end{aligned}$$

It is obvious that when $x\to\infty$, $\ln\big(1+\frac{1}{4(x^2-x)}\big)\sim\frac{1}{4(x^2-x)}$. According to the estimates of I and III, it is obvious that

$$\int_{|x|>4}|\mathcal{C}_2(f_1,f_2)|^{1/2}dx=\infty,$$

which implies $\int_{\mathbb{R}}|\mathcal{C}_2(f_1,f_2)|^{1/2}dx=\infty$.

The proof of the theorem is completed. □

3. Abstract weighted Hardy space

3.1. Definition of abstract weighted Hardy spaces

In this section, based on the definition and theory of abstract Hardy spaces in [2], we introduce abstract weighted Hardy spaces.

We first recall the definition of weights. A weight w is a non-negative locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \le C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(y)\,dy \le Cw(x) \quad \text{for a.e. } x \in B.$$

From [11], we know that whenever $\omega \in A_p$, $1 < p < \infty$, then $\omega(x)dx$ is a doubling measure, which means that for $\forall x \in \mathbb{R}^n$, $r > 0$ and $t \ge 1$, there exists a constant $C > 0$ such that

$$\frac{\omega(B(x,tr))}{\omega(B(x,r))} \le Ct^{np}, \tag{3.4}$$

where $B(x,r)$ is the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. Furthermore, $\omega \in A_p$ if and only if

$$\left(\frac{1}{|Q|}\int_Q f(x)dx\right)^p \le \frac{C}{\omega(Q)}\int_Q f^p(x)\omega(x)dx \tag{3.5}$$

holds for all nonnegative f and all balls B.

Denote by $\mathcal{Q}$ the collection of balls $\mathcal{Q} = \{B(x,r), x \in \mathbb{R}^n, r > 0\}$. Let $\beta \in (1,\infty]$ be a fixed exponent and $\mathbb{B} := (B_Q)_{Q\in\mathcal{Q}}$ be a collection of L^β_w-bounded linear operator. We suppose that these operators B_Q are uniformly bounded on L^β_w: for $\forall f \in L^\beta_w$, and $\forall$ ball Q, there exists a constant C such that $\|B_Q(f)\|_{L^\beta_w} \le C\|f\|_{L^\beta_w}$.

For arbitrary ball Q, we write $S_i(Q)$:

$$S_i(Q) := \left\{x, 2^i \le 1 + \frac{|x - c(Q)|}{r} < 2^{i+1}\right\}, \forall i \ge 0,$$

with the radius r and center $c(Q)$ of the ball Q. Note that $S_0(Q)$ corresponds to the ball Q and $S_i(Q) \subset 2^{i+1}Q$ for $i \ge 1$.

Now we characterize abstract weighted Hardy spaces in terms of atoms in the following way.

Definition 3.1. Suppose that ϵ is a fixed parameter and $w \in A_p$, $1 \le p < \infty$. A function $m \in L^1_{loc}$ is called ϵ-atom with respect to ω associated to a ball Q (or ω-ϵ-atom associated to a ball Q) if there exists a real function f_Q with supp$f_Q \subset Q$ such that

$$m = B_Q(f_Q),$$

with

$$\forall i \ge 0, \quad \|f_Q\|_{L^\beta_\omega(S_i(Q))} \le 2^{-\varepsilon i} w(2^i Q)^{-1/\beta'}.$$

It is easy to show that

$$\|f_Q\|_{L^\beta_\omega} \lesssim \omega(Q)^{-1/\beta'}. \tag{3.6}$$

Then we give the definition of abstract weighted Hardy spaces.

Definition 3.2. A measurable function h belongs to the abstract weighted Hardy space $H^1_{ato,w}(\mathbb{R}^n)$ if there exists a decomposition

$$h = \sum_{i\in\mathbb{N}} \lambda_i b_i \ \mu\text{-a.e.},$$

where for all i, m_i is a ω-ϵ-atom and λ_i are real numbers satisfying

$$\sum_{i\in\mathbb{N}} |\lambda_i| < \infty.$$

We equip $H^1_{ato,w}$ with the norm:

$$\|h\|_{H^1_{ato,w}} := \inf_{h=\sum\limits_{i\in\mathbb{N}} \lambda_i m_i} \sum_i |\lambda_i|.$$

We will see that the "finite abstract weighted Hardy space" are more practicable.

Definition 3.3. A measurable function $f \in H^1_{F,ato,w}(\mathbb{R}^n)$ if f admits a finite atomic decomposition. We equip this space with the norm

$$\|f\|_{H^1_{F,ato,w}} := \inf_{f=\sum\limits_{i=1}^N \lambda_i m_i} \sum_{i=1}^N |\lambda_i|,$$

where we take the infimum over all the finite atomic decomposition.

Remark 3.4. Similar argument as in [2] implies that

(a) $\forall \epsilon > 0$, $H^1_{F,ato,w} \hookrightarrow H^1_{ato,w}$.

(b) $H^1_{F,ato,w} \subset L^\beta \cap H^1_{ato,w}$ is dense in $H^1_{ato,w}$.

(c) Abstract weighted Hardy spaces $H^1_{ato,w}$ are Banach spaces.

(d) Assume that $\mathbb{B}$ satisfies some decay estimates: for a large enough exponent M'', there exists a constant C such that for $\forall i \ge 0$, $\forall k \ge 0$ and $f \in L^2_\omega$ with supp$(f) \subset 2^i Q$,

$$\|B_Q(f)\|_{L^2_\omega, S_k(2^i Q)} \le C 2^{-M'' k} \|f\|_{L^2_\omega, 2^i Q}.$$

Then we have the following imbedding:

$$\forall \epsilon > 0,\ H^1_{ato,w} \hookrightarrow L^1_\omega.$$

(e) In order to well understand our abstract weighted Hardy spaces, we compare it with classical weighted Hardy spaces. By the definition of atomic of weighted Hardy space H^1_w of J. Garcia-Cuerva in [8], if we choose the operator B_Q as follows

$$B_Q(f)(x) = f(x)\chi_Q(x) - \frac{1}{|Q|}\int_Q f(y)dy\chi_Q(x).$$

It is easy to check that B_Q is a ω-$(1, q, 0)$-atom with the following properties:
(i) $\mathrm{supp} B_Q(f) \subset Q$;
(ii) $\int B_Q(f)(x)dx = 0$;
(iii) $\|B_Q(f)\|_{L^q_w} \le C\|f\|_{L^q_w} \le C\omega(Q)^{-1/q'}$.
Therefore, our weighted atom are the same as the ones in [8], that is to say

$$H^1_{ato,w}(\mathbb{R}^n) = H^1_w(\mathbb{R}^n).$$

3.2. The duality of abstract Hardy space

In this section, we want to study the dual spaces of the weighted Hardy spaces. We firstly give the following weighted definition.

Definition 3.5. Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $w \in A_p, 1 \le p < \infty$. For an element f is said to belong to BMO_w if

$$\|f\|_{BMO_w} =: \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{w(Q)}\int_Q |B^*_Q(f)|^2 w^{-1}\, dx\right)^{1/2} < \infty,$$

where the sup is taken over all balls Q in $\mathbb{R}^n$ and denote the adjoint operation by $*$.

We have the following inclusion.

Theorem 3.6. $BMO_w \hookrightarrow (H^1_{ato,w})^*$.

Proof. Let $g \in BMO_w$ and $m = B_Q(f_Q)$, by the property of atoms we have

$$\begin{aligned}
\langle g, m\rangle &= \int g(x)B_Q(f_Q)(x)dx \\
&= \int B^*_Q(g)(x)f_Q(x)dx \\
&\le \Big(\int |B^*_Q(g)(x)|^2 w^{-1}dx\Big)^{1/2}\|f_Q\|_{L^2_w} \\
&\le \Big(\int |B^*_Q(g)(x)|^2 w^{-1}dx\Big)^{1/2} w(Q)^{-1/2} \\
&\le \|g\|_{BMO_w},
\end{aligned}$$

which completes the proof of the first inclusion. □

4. Boundedness of the first order commutator

In this section, we will give the proof of the boundedness of the first order commutator $\mathcal{C}_2$ on abstract weighted Hardy spaces as follows. We firstly give some preliminaries.

We choose two collection $\mathbb{B}^1 := (B_Q^1)_{Q\in\mathcal{Q}}$ and $\mathbb{B}^2 := (B_Q^2)_{Q\in\mathcal{Q}}$ and two exponents $\beta_1, \beta_2 \in (1,\infty]$. We assume that $\mathbb{B}^1$ is a collection of $L_\omega^{\beta_1}$-bounded operators and that $\mathbb{B}^2$ is a collection of $L_\omega^{\beta_2}$-bounded operators. We can also define two kind of Hardy space $H^1_{\mathbb{B}^1,ato,\omega}$ and $H^1_{\mathbb{B}^2,ato,\omega}$. According to Definition 3.3, we can construct the spaces $H^1_{F,\mathbb{B}^1,ato,\omega}$ and $H^1_{F,\mathbb{B}^2,ato,\omega}$. In this context, we have the following weighted bilinear results:

Lemma 4.1. *Let T be a bilinear operator with coefficients $(\gamma_j)_{j\geq 0}$ such that for all ball Q_1, Q_2 and for all functions f, g separately supported in Q_1, Q_2, we have for $\forall\ \omega \in A_p$, $1 < p < \infty$ and all $j_1, j_2 \geq 0$*

$$\frac{1}{\omega(2^{j_l+1}Q_l)}\int_{S_{j_1}(Q_1)\cap S_{j_2}(Q_2)} |T(B^1_{Q_1}(f), B^2_{Q_2}(g))|\omega(x)dx$$
$$\leq C\gamma_{j_1}\gamma_{j_2}\frac{\omega(Q_1)}{\omega(2^{j_1}Q_1)}\frac{\omega(Q_2)}{\omega(2^{j_2}Q_2)}\left(\frac{1}{\omega(Q_1)}\int_{Q_1}|f|^{\beta_1}\omega(x)dx\right)^{1/\beta_1}$$
$$\left(\frac{1}{\omega(Q_2)}\int_{Q_2}|g|^{\beta_2}\omega(x)dx\right)^{1/\beta_2}$$

with coefficients γ_l satisfying $\sum_{l\geq 0}\gamma_l \leq C$. Then the operator T is continuous from $H^1_{F,\mathbb{B}^1,\omega}(\mathbb{R}) \times H^1_{F,\mathbb{B}^2,\omega}(\mathbb{R})$ into $L^{1/2}_\omega(\mathbb{R})$.

Proof. We mainly follow the idea of proof of Proposition 4.7 in [1]. For the sake of completeness and for the reader's convenience we give the details of the proof of this theorem.

Let $f \in H^1_{F,\mathbb{B}^1,ato,\omega}$ and $f \in H^1_{F,\mathbb{B}^2,ato,\omega}$, we can also write them with a finite atomic decomposition:

$$f = \sum_Q \lambda_Q B^1_Q(f_Q), \qquad g = \sum_R \tau_R B^2_R(g_R) \tag{4.7}$$

with the appropriate properties for f_Q and g_R: f_Q is supported in Q with

$$\|f_Q\|_{L_\omega^{\beta_1}} \leq \omega(Q)^{-1/\beta_1'}, \qquad \sum_Q |\lambda_Q| \leq 2\|f\|_{H^1_{\mathbb{B}^1,\omega}}, \tag{4.8}$$

and similarly for g_R, relatively to the ball R. So it is sufficient to estimate

$$\|T(f,g)\|_{L_\omega^{1/2}} = \left\|\sum_{Q,R}\sum_{i,j\geq 0}\lambda_Q\tau_R T(B^1_Q f_Q, B^2_R f_R)\mathbf{1}_{S_i(Q)\cap S_j(R)}\right\|_{L_\omega^{1/2}}.$$

By symmetry, we just need to study the sum over the extra condition

$$2^i r_Q \leq 2^j r_R, \tag{4.9}$$

where r denotes the radius of the ball. Meanwhile, we recall following lemma in [10].

Lemma 4.2 ([10]). *For $r \leq 1$ and $\omega \in A_p$, $1 \leq p < \infty$, there exist constants C and $\delta > 1$ such that for all collection $(Q_k)_k$ of balls and $(g_k)_k$ collection of nonnegative L^r_ω functions supported in Q_k, we have*

$$\Big\|\sum_k g_k\Big\|_{L^r_\omega} \leq C\Big\|\sum_k \Big(\frac{1}{\omega(Q_k)}\int g_k(x)\omega(x)dx\Big)\mathbf{1}_{\delta Q_k}\Big\|_{L^r_\omega}. \tag{4.10}$$

Next we use this lemma with $r = 1/2$ and the doubling property (3.4) of the weight ω, together with Lemma 4.1 and the estimate (4.9), then finally we get

$$\begin{aligned}
&\|T(f,g)\|_{L^{1/2}_\omega} \\
\leq &\Big\|\sum_{Q,R}\sum_{i,j\geq 0}|\lambda_Q||\tau_R||T(B^1_Q f_Q, B^2_R g_R)|\mathbf{1}_{S_i(Q)\cap S_j(R)}\Big\|_{L^{1/2}_\omega} \\
\leq &\Big\|\sum_{Q,i}|\lambda_Q|\Big(\sum_{R,j}|\tau_R||T(B^1_Q f_Q, B^2_R g_R)|\mathbf{1}_{S_i(Q)\cap S_j(R)}\Big)\mathbf{1}_{2^{i+1}Q}\Big\|_{L^{1/2}_\omega} \\
\leq &\Big\|\sum_{Q,R}\sum_{i,j\geq 0}|\lambda_Q||\tau_R|\Big(\frac{1}{\omega(2^{i+1}Q)}\int_{S_i(Q)\cap S_j(R)}|T(B^1_Q f_Q, B^2_R g_R)(x)|\omega(x)dx\Big)\mathbf{1}_{\delta 2^{i+1}Q}\Big\|_{L^{1/2}_\omega} \\
\leq &C\Big\|\sum_{Q,R}\sum_{i,j\geq 0}|\lambda_Q||\tau_R|\gamma_i\gamma_j\frac{\omega(Q)\omega(R)}{\omega(2^iQ)\omega(2^jR)}\Big(\frac{1}{\omega(Q)}\int_Q|f_Q(x)|^{\beta_1}\omega(x)dx\Big)^{1/\beta_1} \\
&\quad\times\Big(\frac{1}{\omega(R)}|g_R(x)|^{\beta_2}\omega(x)dx\Big)^{1/\beta_2}\mathbf{1}_{\delta 2^{i+1}Q}\mathbf{1}_{2\delta 2^{j+1}R}\Big\|_{L^{1/2}_\omega} \\
\leq &C\Big\|\sum_{Q,R}\sum_{i,j\geq 0}|\lambda_Q||\tau_R|\frac{\gamma_i\gamma_j}{\omega(2^iQ)\omega(2^jR)}\mathbf{1}_{\delta 2^{i+1}Q}\mathbf{1}_{2\delta 2^{j+1}R}\Big\|_{L^{1/2}_\omega} \\
= &C\Big\|\sum_{Q,i}|\lambda_Q|\frac{\gamma_i}{\omega(2^iQ)}\mathbf{1}_{\delta 2^{i+1}Q}\sum_{R,j}|\tau_R|\frac{\gamma_j}{\omega(2^jR)}\mathbf{1}_{2\delta 2^{j+1}R}\Big\|_{L^{1/2}_\omega} \\
\leq &C\Big\|\sum_{Q,i}|\lambda_Q|\frac{\gamma_i}{\omega(2^iQ)}\mathbf{1}_{\delta 2^{i+1}Q}\Big\|_{L^1_\omega}\Big\|\sum_{R,j}|\tau_R|\frac{\gamma_j}{\omega(2^jR)}\mathbf{1}_{2\delta 2^{j+1}R}\Big\|_{L^1_\omega}.
\end{aligned}$$

Then the proof is finished with the properties (4.8), the assumption $\sum_l \gamma_l \leq C$ in Lemma 4.1 and the doubling property of the wight ω. $\square$

In order to use Lemma 4.1 to obtain the boundedness of the commutator C_2 on abstract weighted Hardy spaces, we still need to introduce a class of integral operators A_t, $t > 0$, which play the role of approximations to the identity as in [7]. We assume that the operators A_t can be represented by kernels $a_t(x,y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x,y)f(y)dy$$

for every function $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, and the kernel $a_t(x,y)$ satisfy the following size conditions

$$|a_t(x,y)| \leqslant h_t(x,y) = t^{-1/s} h\left(\frac{|x-y|^s}{t}\right), \tag{4.11}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{1+\eta} h(r^s) = 0. \tag{4.12}$$

Theorem 4.3 ([7]). *The first order commutator $\mathcal{C}_2$ satisfies: for each $i = 1,2$, there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x,y)$ that satisfy condition* (4.11) *and* (4.12) *with constants s and η and there exist kernels $K_t^{(i)}$ such that*

$$\langle \mathcal{C}_2(f_1, A_t^{(i)} f_i), g\rangle = \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} K_t^{(i)}(x,y_1,y_2) \prod_{i=1}^{2} f_i(y_i) d\vec{y} \right| g(x) dx$$

for f_1, f_2, and g in $\mathcal{S}(\mathbb{R})$ with $\bigcap_{k=1}^2 \operatorname{supp} f_k \cap \operatorname{supp} g = \emptyset$. There exist a function $\varphi \in C(\mathbb{R})$ with $\operatorname{supp}\varphi \in [-1,1]$ and a constant $\epsilon > 0$ so that for every $i = 1,2$, we have

$$|K(x,\vec{y}) - K_t^{(i)}(x,\vec{y})| \leq \frac{A}{(\sum_{j=1}^{2} |x-y_j|)^2} \varphi(\frac{|y_1-y_2|}{t^{1/s}}) + \frac{A t^{\epsilon/s}}{(\sum_{j=1}^{2} |x-y_j|)^{2+\epsilon}}, \tag{4.13}$$

whenever $t^{1/s} \leq |x-y_i|/2$.

In fact, from the proof of inequality (4.13) in [7], we can see that the operators $A_t^{(i)}$ are of form

$$A_t^{(i)}(f)(x) = \int_{\mathbb{R}} a_t^{(i)}(x,y) f(y) dy, \; a_t^{(i)}(x,y) = \Phi_t(x-y)\chi_{(-\infty,x)}(y), \tag{4.14}$$

where $\Phi = \phi'$ and $\phi \in C^\infty(\mathbb{R})$ be even, $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\operatorname{supp}\phi \subset [-1,1]$. It is not difficult to check that $a_t^{(i)}(x,y)$ satisfy conditions (4.11) and (4.12) with constant $s = \eta = 1$, specifically,

$$|a_t^{(i)}(x,y)| \leq C \frac{t}{(t+|x-y|)^2}. \tag{4.15}$$

In the sequel, we will obtain the main theorem:

Theorem 4.4. *Let $1 < p < \min\{2, 1+\epsilon\}$, where ϵ is decided in Theorem 4.3. If $\omega \in A_p$, then the first order commutator $\mathcal{C}_2$ is bounded from $H^1_{F,ato,\omega}(\mathbb{R}) \times H^1_{F,ato,\omega}(\mathbb{R})$ into $L^{1/2}_\omega(\mathbb{R})$.*

Proof. We mainly follow the idea of Chapter 5.2 in [1]. Set $r_i = r_{Q_i}^{1/s}$, where r_{Q_i} is the radius of the ball $Q_i, i = 1,2$. Let $\beta_1 = \beta_2 = \infty$. We define $B_{Q_i} = Id - A_{r_i}$. By the estimate (4.15) we can define our Hardy space $H^1_{ato,\omega}$.

We claim that the first order commutator $\mathcal{C}_2$ satisfies Lemma 4.1, which gives us that the boundedness of the commutator $\mathcal{C}_2$ from $H^1_{F,ato,\omega}(\mathbb{R}) \times H^1_{F,ato,\omega}(\mathbb{R})$ into $L^{1/2}_\omega(\mathbb{R})$.

In fact, for $j_1 = j_2 = 0$, by Theorem 4.3 and Theorem 1.2 in [9], we know that $\mathcal{C}_2$ is a bounded operator from $L^p_\omega \times L^\infty_\omega \to L^p_\omega$. So using Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(2Q_1)}\int_{Q_1\cap Q_2} |\mathcal{C}_2(B_{Q_1}f_1, B_{Q_2}f_2)(x)|\omega(x)dx \\
&\le C\frac{1}{\omega(2Q_1)^{1/p}}\left(\int_{Q_1\cap Q_2} |\mathcal{C}_2(B_{Q_1}f_1, B_{Q_2}f_2)(x)|^p\omega(x)dx\right)^{1/p} \\
&\le C\frac{1}{\omega(2Q_1)^{1/p}}\|B_{Q_1}f_1\|_{L^p_\omega}\|B_{Q_2}f_2\|_{L^\infty_\omega} \\
&\le C\|f_1\|_{L^\infty_\omega}\|f_2\|_{L^\infty_\omega}.
\end{aligned}$$

For $j_1 > 0$ and $j_2 = 0$, $x \in S_{j_1}(Q_1)\cap Q_2$, and $\text{supp} f_i \subset Q_i$, $i = 1, 2$. Set $g = B_{Q_2}(f_2)$. Hence by (4.13) we have

$$\begin{aligned}
|\mathcal{C}_2(B_{Q_1}(f_1), g)| &= |\int\int (K(x,y_1,y_2) - K^{(1)}_{Q_1}(x,y_1,y_2))f_1(y_1)g(y_2)dy_1dy_2| \\
&\leqslant \int\int \Big(\frac{A}{(|x-y_1|+|x-y_2|)^2}\varphi\left(\frac{|y_1-y_2|}{r_1}\right) \\
&\quad + \frac{A\ r_1^\epsilon}{(|x-y_1|+|x-y_2|)^{2+\epsilon}}\Big)|f_1(y_1)||g(y_2)|dy_1dy_2 \\
&:= \text{I} + \text{II}.
\end{aligned}$$

It is easy to see that $g = B_{Q_2}(f_2) \le Mf_2$, where M is the Hardy-Littlewood maximal operator. Together with the property of φ, $|x-y_1| \gtrsim 2^{j_1}r_1 = 2^{j_1}|Q_1|$ and the estimates (3.5) and (3.4), we have

$$\begin{aligned}
\text{I} &= \int\int \frac{A}{(|x-y_1|+|x-y_2|)^2}\varphi\left(\frac{|y_1-y_2|}{r_1}\right)|f_1(y_1)||g(y_2)|dy_1dy_2 \\
&\le C\|f_2\|_\infty 2^{-2j_1}\frac{1}{|Q_1|}\int_{Q_1}|f_1(y_1)|dy_1 \\
&\le C\|f_2\|_\infty 2^{-2j_1}\left(\frac{1}{\omega(Q_1)}\int_{Q_1}|f_1(y_1)|^p\omega(y_1)dy_1\right)^{1/p} \\
&\le C2^{-j_1(2-p)}\frac{\omega(Q_1)}{\omega(2^jQ_1)}\|f_1\|_\infty\|f_2\|_\infty.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\text{II} &\le CA\|f_2\|_\infty\frac{r^\epsilon}{|2^jQ_1|}\int_{Q_1}|f_1(y_1)|dy_1\int_{\mathbb{R}}(|2^{j_1}Q_1|+|x-y_2|)^{1+\epsilon}dy_2 \\
&\le CA2^{-j(1+\epsilon)}\|f_2\|_\infty\frac{1}{|Q_1|}\int_{Q_1}f_1(y_1)dy_1
\end{aligned}$$

$$\le CA2^{-j(1+\epsilon-p)}\|f_2\|_\infty \frac{\omega(Q_1)}{\omega(2^jQ_1)}\left(\frac{1}{\omega(Q_1)}\int_{Q_1}|f_1(y_1)|^p\omega(y_1)dy_1\right)^{1/p}$$
$$\le CA2^{-j(1+\epsilon-p)}\frac{\omega(Q_1)}{\omega(2^jQ_1)}\|f_1\|_\infty\|f_2\|_\infty.$$

According to the above two inequalities, we choose

$$\gamma_j = C\min\{2^{-j_1(2-p)}, 2^{-j(1+\epsilon-p)}\}.$$

Since $1 < p < \min\{2, 1+\epsilon\}$, $\sum_j \gamma_j < \infty$.

Symmetrically, we can discuss the case of $j_1 = 0$ and $j_2 > 0$. For the case of $j_1 > 0$ and $j_2 > 0$, we can deduce it to the above cases, and just consider the smaller ball.

The proof of the theorem is completed. □

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