GEOMETRIC CHARACTERIZATIONS OF CANAL SURFACES IN MINKOWSKI 3-SPACE $\mathbb{I}$

Xueshan Fu, Seoung Dal Jung, Jinhua Qian, and Mengfei Su

ABSTRACT. The canal surfaces foliated by pseudo spheres $\mathbb{S}_1^2$ along a space curve in Minkowski 3-space are studied. The geometric properties of such surfaces are shown by classifying the linear Weingarten canal surfaces, the developable, minimal and umbilical canal surfaces are discussed at the same time.

1. Introduction

In 1850, Monge first investigated canal surfaces in Euclidean 3-space which is the envelope of a moving sphere whose centers lie on a space curve. The characters of such surfaces have been studied by many experts and geometers [2], [9]. The authors of [2] presented the relationships between the Gaussian curvature, mean curvature and the second Gaussian curvature of canal surfaces. Furthermore, the canal surfaces were classified from different viewpoints, such as the Weingarten canal surfaces, the linear Weingarten canal surfaces and the canal surfaces whose Gauss map satisfies some conditions [1], [6]. With the development of theory of relativity, physicians and geometers extended the topics in classical differential geometry of Riemannian manifolds to that of Lorentzian manifolds. As a nature idea, the canal surfaces can be extended into the spaces with indefinite metric, especially Mikowski 3-space.

Let $\mathbb{E}_1^3$ be Minkowski 3-space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

in terms of the natural coordinate system (x_1, x_2, x_3). A vector $v \in \mathbb{E}_1^3$ is said to be spacelike if $\langle v, v \rangle > 0$ or $v = 0$; timelike if $\langle v, v \rangle < 0$; null (lightlike) if $\langle v, v \rangle = 0$, respectively. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. The timelike or lightlike vector is said to be causal [3]. Due to the causal

character of the tangent vector of a space curve, the curves in Minkowski space can be divided into spacelike curve, timelike curve or null curve. The surfaces are called timelike surface, spacelike surface or lightlike surface if its normal vector is spacelike, timelike or lightlike.

It is well known that $\mathbb{S}_q^n$, $\mathbb{H}_q^n$ are complete semi-Riemannian manifolds with index q of constant sectional curvature r^{-2} and $-r^{-2}$, and $\mathbb{Q}_q^n$ is degenerate hypersurface in $\mathbb{E}_q^m$ respectively. Especially, the semi-Riemannian manifolds $\mathbb{E}_1^n$, $\mathbb{S}_1^n$ and $\mathbb{H}_1^n$ are known as the Minkowski space, the de Sitter space and the anti-de Sitter space. These spaces with index 1 are called Lorentzian space forms (or non-degenerated space forms) and $\mathbb{Q}_1^n$ degenerated space form.

Let p be a point fixed in $\mathbb{E}_q^m$ and $r > 0$ be a constant. Then the pseudo-Riemannian hypersphere is defined by

$$\mathbb{S}_q^n(p, r) = \{x \in \mathbb{E}_q^{n+1} : \langle x - p, x - p \rangle = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$\mathbb{H}_q^n(p, r) = \{x \in \mathbb{E}_{q+1}^{n+1} : \langle x - p, x - p \rangle = -r^2\};$$

the pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$\mathbb{Q}_q^n = \{x \in \mathbb{E}_q^{n+1} : \langle x - p, x - p \rangle = 0\}.$$

Similar to the generating process of canal surfaces in $\mathbb{E}^3$, a canal surface in $\mathbb{E}_1^3$ can be obtained as the envelope of a family of pseudo spheres $\mathbb{S}_1^2$, pseudo hyperbolic spheres $\mathbb{H}_0^2$ or lightlike cones $\mathbb{Q}^2$ whose centers lie on a space curve. In 2016, A. Ucum, K. Ilarslan presented the explicit parametrizations of canal surfaces in $\mathbb{E}_1^3$ and some types of the linear Weingarten tubular surfaces are discussed in [8].

In order to do a complete geometric study for canal surfaces in Minkowski 3-space, we concern on the canal surfaces foliated by three space forms along a space curve (resp. spacelike curve, timelike curve or null curve). First, we denote some notation for different kinds of canal surfaces and review some basic facts, then the geometric properties of canal surfaces are given by discussing the linear Weingarten canal surfaces. The main purposes of the present work is to classify the canal surfaces foliated by pseudo spheres $\mathbb{S}_1^2$ along a space curve in $\mathbb{E}_1^3$, such as the linear Weingarten canal surfaces, the developable canal surfaces, the minimal canal surfaces and the umbilical canal surfaces.

All the surfaces we are dealing with are smooth, regular and topologically connected unless otherwise stated.

2. Preliminaries

In this section, we review some basic facts for curves and canal surfaces in Minkowski 3-space.

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be vectors in $\mathbb{E}_1^3$. Then their scalar product is given by

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3$$

and the exterior product by

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right),$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{E}_1^3$. One can have

$$e_1 \times e_2 = -e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.$$

Let $c(s) : \mathbf{I} \to \mathbb{E}_1^3$ be a space curve with the moving Frenet frame $\{\alpha, \beta, \gamma\}$ consisting of the tangent vector α, the principal normal vector β and the binormal vector γ, respectively.

Case 1. Let $c = c(s)$ be a spacelike curve parameterized by arc length s. Due to the causal character of the principal normal vector, it can be divided into the following two cases:

Case 1.1. If $\langle c''(s), c''(s) \rangle \neq 0$, the following Frenet equations are satisfied

$$\alpha'(s) = \kappa(s)\beta(s), \ \beta'(s) = -\varepsilon\kappa(s)\alpha(s) + \tau(s)\gamma(s), \ \gamma'(s) = \tau(s)\beta(s),$$

where $\langle \alpha, \alpha \rangle = 1$, $\langle \beta, \beta \rangle = \varepsilon = \pm 1$, $\langle \gamma, \gamma \rangle = -\varepsilon$, $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = \langle \gamma, \beta \rangle = 0$. The functions $\kappa(s)$ and $\tau(s)$ are called the curvature and the torsion of $c(s)$, respectively. When $\varepsilon = 1$, $c(s)$ is called the first kind spacelike curve and the second kind spacelike curve when $\varepsilon = -1$.

Case 1.2. If $\langle c''(s), c''(s) \rangle = 0$, the Frenet equations are given by

$$\alpha'(s) = \beta(s), \quad \beta'(s) = \kappa(s)\beta(s), \quad \gamma'(s) = -\alpha(s) - \kappa(s)\gamma(s),$$

where $\langle \alpha, \alpha \rangle = \langle \beta, \gamma \rangle = 1$, $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = \langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$. The function $\kappa(s)$ is called the curvature of $c(s)$. Such kind spacelike curve is said to be null type spacelike curve.

Case 2. Let $c = c(s)$ be a timelike curve parameterized by arc length s. Then the following Frenet equations are satisfied

$$\alpha'(s) = \kappa(s)\beta(s), \ \beta'(s) = \kappa(s)\alpha(s) - \tau(s)\gamma(s), \ \gamma'(s) = \tau(s)\beta(s),$$

where $\langle \alpha, \alpha \rangle = -1$, $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 1$, $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = \langle \gamma, \beta \rangle = 0$. Similar to spacelike curves, the functions $\kappa(s)$ and $\tau(s)$ are called the curvature and the torsion of $c(s)$, respectively.

Case 3. Let $c = c(s)$ be a null curve parameterized by null arc length s (i.e., $\|c''(s)\| = 1$). Then the following Frenet equations are given by

$$\alpha'(s) = \beta(s), \quad \beta'(s) = \kappa(s)\alpha(s) - \gamma(s), \quad \gamma'(s) = -\kappa(s)\beta(s),$$

where $\langle \alpha, \alpha \rangle = \langle \gamma, \gamma \rangle = \langle \alpha, \beta \rangle = \langle \gamma, \beta \rangle = 0$, $\langle \alpha, \gamma \rangle = \langle \beta, \beta \rangle = 1$. And the function $\kappa(s)$ is called the null curvature of $c(s)$.

Definition. The surface $\mathbb{M}$ in $\mathbb{E}_1^3$ is called a canal surface which is formed as the envelope of a family of pseudo spheres $\mathbb{S}_1^2$ (resp. pseudo hyperbolic spheres $\mathbb{H}_0^2$ or lightlike cones $\mathbb{Q}^2$) whose centers lie on a space curve $c(s)$ framed by $\{\alpha, \beta, \gamma\}$. Then $\mathbb{M}$ can be parametrized by

$$(2.1) \qquad x(s, \theta) = c(s) + \lambda(s, \theta)\alpha(s) + \mu(s, \theta)\beta(s) + \omega(s, \theta)\gamma(s),$$

where λ, μ and ω are differential functions of s and θ, $\|x(s,\theta)-c(s)\|^2 = \epsilon r^2(s)$, ($\epsilon = \pm 1$ or 0). The curve $c(s)$ is called the center curve and $r(s)$ is called the radial function of $\mathbb{M}$.

Explicitly, if $\mathbb{M}$ is foliated by pseudo spheres $\mathbb{S}_1^2$ (resp. pseudo hyperbolic spheres $\mathbb{H}_0^2$ or lightlike cones $\mathbb{Q}^2$), then $\epsilon = 1$ (resp. -1 or 0) and $\mathbb{M}$ is said to be of type $\mathbb{M}_+$ (resp. $\mathbb{M}_-$ or $\mathbb{M}_0$). Also the canal surface of type $\mathbb{M}_+$ can be divided into three types. In the case that $c(s)$ is spacelike (resp. timelike or null), it is said to be of type $\mathbb{M}_+^1$ (resp. $\mathbb{M}_+^2$ or $\mathbb{M}_+^3$). Furthermore, $\mathbb{M}_+^1$ can be divided into $\mathbb{M}_+^{11}$, $\mathbb{M}_+^{12}$ and $\mathbb{M}_+^{13}$ when $c(s)$ is the first kind spacelike curve, the second kind spacelike curve and the null type spacelike curve, respectively. Similar to $\mathbb{M}_+$, the canal surfaces $\mathbb{M}_-$ (resp. $\mathbb{M}_0$) can be divided into $\mathbb{M}_-^1$, $\mathbb{M}_-^2$ and $\mathbb{M}_-^3$ (resp. $\mathbb{M}_0^1$, $\mathbb{M}_0^2$ and $\mathbb{M}_0^3$). Naturally, $\mathbb{M}_-^1$ (resp. $\mathbb{M}_0^1$) can be divided into $\mathbb{M}_-^{11}$, $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^{13}$ (resp. $\mathbb{M}_0^{11}$, $\mathbb{M}_0^{12}$ and $\mathbb{M}_0^{13}$).

Remark. In particular, if the center curve $c(s)$ is a straight line, then the Frenet frame $\{\alpha, \beta, \gamma\}$ of $c(s)$ can be regarded as a trivial orthogonal frame, then the canal surface is nothing but a surface of revolution. If the radius function is constant, then $\mathbb{M}$ is a tube (pipe) surface.

The linear Weingarten surfaces in the general case is almost completely open today. Several geometers are studying linear Weingarten surfaces in the ambient spaces and many interesting results can be found such as [4] and [5].

Definition ([4])**.** For the Gaussian curvature K and the mean curvature H of a surface $\mathbb{M}$ in $\mathbb{E}_1^3$, if $\mathbb{M}$ satisfies

$$(2.2) \qquad 2aH + bK = c \quad (a, b, c \in \mathbb{R} \text{ and } (a, b, c) \neq (0, 0, 0)),$$

then it is said to be a linear Weingarten surface.

Remark. The linear Weingarten surfaces can be considered as a natural generalization of surfaces with constant Gaussian curvature or constant mean curvature, when $a = 0$ or $b = 0$ in (2.2), respectively. Without loss of generality, we always assume $c = 1$ in (2.2).

3. Main results

In this part, we focus on the properties of different types of canal surfaces formed by the movement of the pseudo spheres $\mathbb{S}_1^2$ along a space curve in $\mathbb{E}_1^3$.

3.1. The canal surfaces of type $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$

At first, we assume $\mathbb{M}$ be a canal surface formed by the movement of the pseudo spheres $\mathbb{S}_1^2$ along a first kind spacelike curve $c(s)$, i.e., $\mathbb{M}_+^{11}$. According to the definition of $\mathbb{M}_+^{11}$, through detailed calculation, we get

$$\begin{cases} \lambda(s) = -r(s)r'(s), \\ \mu(s,\theta) = r(s)\sqrt{1 - r'^2(s)} \cosh\theta, \\ \omega(s,\theta) = r(s)\sqrt{1 - r'^2(s)} \sinh\theta \end{cases}$$

in (2.1). Then $\mathbb{M}_+^{11}$ can be parametrized by

$$x(s,\theta) = c(s) + r(s)\{-r'(s)\alpha + \sqrt{1 - r'^2(s)}\cosh\theta\beta + \sqrt{1 - r'^2(s)}\sinh\theta\gamma\},$$

where $c(s)$ is parametrized by arc length s. For convenience, we may assume $-r'(s) = \cos\varphi$ for some smooth function $\varphi = \varphi(s)$ in above equation. Then $\mathbb{M}_+^{11}$ can be rewritten by

$$(3.1) \quad x(s,\theta) = c(s) + r(s)(\cos\varphi(s)\alpha + \sin\varphi(s)\cosh\theta\beta + \sin\varphi(s)\sinh\theta\gamma),$$

where $\varphi \in [0, \pi)$.

Initially, we have

$$x_s = x_s^1\alpha + x_s^2\beta + x_s^3\gamma, \quad x_\theta = r\sin\varphi\sinh\theta\beta + r\sin\varphi\cosh\theta\gamma,$$

where

$$x_s^1 = \sin^2\varphi - rr'' - r\kappa\sin\varphi\cosh\theta;$$
$$x_s^2 = r'\sin\varphi\cosh\theta - rr'\kappa - rr'\varphi'\cosh\theta + r\tau\sin\varphi\sinh\theta;$$
$$x_s^3 = r'\sin\varphi\sinh\theta + r\tau\sin\varphi\cosh\theta - rr'\varphi'\sinh\theta.$$

Then, the component functions of the first fundamental form are given by

$$E = \langle x_s, x_s \rangle$$
$$= r^2(\kappa^2\sin^2\varphi\cosh^2\theta + r'^2\kappa^2 + \varphi'^2 - \tau^2\sin^2\varphi + 2\kappa\varphi'\cosh\theta$$
$$(3.2) \qquad - 2r'\kappa\tau\sin\varphi\sinh\theta) + \sin^2\varphi - 2(rr'' + r\kappa\sin\varphi\cosh\theta);$$
$$F = \langle x_s, x_\theta \rangle = -r^2r'\kappa\sin\varphi\sinh\theta - r^2\tau\sin^2\varphi;$$
$$G = \langle x_\theta, x_\theta \rangle = -r^2\sin^2\varphi.$$

And $EG - F^2 = -r^2(rr'' + r\kappa\sin\varphi\cosh\theta - \sin^2\varphi)^2$. The unit normal vector field n to $\mathbb{M}_+^{11}$ is given by

$$(3.3) \qquad n = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = \cos\varphi\alpha + \sin\varphi\cosh\theta\beta + \sin\varphi\sinh\theta\gamma,$$

which point towards $\mathbb{M}_+^{11}$ inside and $\langle n, n \rangle = 1$.

Furthermore, by (3.3) we have

$$n_s = (-r'' - \kappa\sin\varphi\cosh\theta)\alpha + (-r'\kappa - r'\varphi'\cosh\theta + \tau\sin\varphi\sinh\theta)\beta$$
$$+ (\tau\sin\varphi\cosh\theta - r'\varphi'\sinh\theta)\gamma;$$
$$n_\theta = \sin\varphi\sinh\theta\beta + \sin\varphi\cosh\theta\gamma.$$

Then, the component functions of the second fundamental form are given by

$$L = -\langle x_s, n_s \rangle$$
$$= -r(\kappa^2\sin^2\varphi\cosh^2\theta + r'^2\kappa^2 + \varphi'^2 - \tau^2\sin^2\varphi + 2\kappa\varphi'\cosh\theta$$
$$(3.4) \qquad - 2r'\kappa\tau\sin\varphi\sinh\theta) + (r'' + \kappa\sin\varphi\cosh\theta);$$
$$M = -\langle x_\theta, n_s \rangle = rr'\kappa\sin\varphi\sinh\theta + r\tau\sin^2\varphi;$$
$$N = -\langle x_\theta, n_\theta \rangle = r\sin^2\varphi.$$

From (3.2) and (3.4), we have:

Lemma 3.1. *The component functions of the first and second fundamental forms of canal surface* $\mathbb{M}_+^{11}$ *satisfy*

$$L = \frac{E + P_1}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r}$$

and

$$(3.5) \qquad EG - F^2 = -r^2 P_1^2, \quad LN - M^2 = -r P_1 Q_1,$$

where $P_1 = rr'' + r\kappa \sin\varphi\cos\theta - \sin^2\varphi = rQ_1 - \sin^2\varphi$, $Q_1 = r'' + \kappa\sin\varphi\cos\theta$.

Remark. Due to regularity, we see that $P_1 \neq 0$ everywhere by (3.5).

From Lemma 3.1, the Gaussian curvature K and the mean curvature H of $\mathbb{M}_+^{11}$ are given by, respectively

$$(3.6) \qquad K = \frac{LN - M^2}{EG - F^2} = \frac{Q_1}{rP_1},$$

$$(3.7) \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{-2P_1 - \sin^2\varphi}{2rP_1}.$$

Secondly, for the canal surface $\mathbb{M}_+^{12}$, by the definition of $\mathbb{M}_+^{12}$ and Frenet equations of second kind spacelike curve, similar to $\mathbb{M}_+^{11}$, we obtain

$$\begin{cases} \lambda(s) = -r(s)r'(s), \\ \mu(s,\theta) = r(s)\sqrt{1 - r'^2(s)}\sinh\theta, \\ \omega(s,\theta) = r(s)\sqrt{1 - r'^2(s)}\cosh\theta \end{cases}$$

in (2.1). Then $\mathbb{M}_+^{12}$ can be parametrized by

$$x(s,\theta) = c(s) + r(s)\{-r'(s)\alpha + \sqrt{1 - r'^2(s)}\sinh\theta\beta + \sqrt{1 - r'^2(s)}\cosh\theta\gamma\},$$

where $c(s)$ is parametrized by arc length s. We assume $-r'(s) = \cos\varphi$ for some smooth function $\varphi = \varphi(s)$, then the canal surface $\mathbb{M}_+^{12}$ can be rewritten by

$$x(s,\theta) = c(s) + r(s)(\cos\varphi(s)\alpha + \sin\varphi(s)\sinh\theta\beta + \sin\varphi(s)\cosh\theta\gamma),$$

where $\varphi \in [0, \pi)$.

Do similar calculations to those of $\mathbb{M}_+^{11}$, we have the following conclusions.

Lemma 3.2. *The component functions of the first and second fundamental forms of canal surface* $\mathbb{M}_+^{12}$ *satisfy*

$$L = \frac{E + P_2}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r}$$

and

$$(3.8) \qquad EG - F^2 = -r^2 P_2^2, \quad LN - M^2 = -r P_2 Q_2,$$

where $P_2 = rr'' - r\kappa\sin\varphi\sinh\theta - \sin^2\varphi = rQ_2 - \sin^2\varphi$, $Q_2 = r'' - \kappa\sin\varphi\sinh\theta$.

Remark. Due to regularity, we see that $P_2 \neq 0$ everywhere by (3.8).

From Lemma 3.2, the Gaussian curvature K and the mean curvature H of $\mathbb{M}_+^{12}$ are given by, respectively

$$(3.9) \qquad\qquad K = \frac{LN - M^2}{EG - F^2} = \frac{Q_2}{rP_2},$$

$$(3.10) \qquad\qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{-2P_2 - \sin^2 \varphi}{2rP_2}.$$

Based on the Gaussian curvature and mean curvature of $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$, it is not difficult to find the following results.

Proposition 3.3. *The Gaussian curvature K and the mean curvature H of the canal surface $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) can be related by*

$$H = -\frac{1}{2}(Kr + \frac{1}{r}).$$

Proof. For $\mathbb{M}_+^{11}$, from (3.6) and (3.7), we can get the conclusion easily. And for $\mathbb{M}_+^{12}$, we can refer to (3.9) and (3.10). $\qquad\square$

Next, we study the canal surface $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) whose Gaussian curvature and mean curvature satisfy some particular conditions.

Remark. In the following, we just prove the results for $\mathbb{M}_+^{11}$ and omit the proof for $\mathbb{M}_+^{12}$ since it can be similarly done to those of $\mathbb{M}_+^{11}$ and the results are same.

Theorem 3.4. *Let $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) be a linear Weingarten canal surface. Then it is an open part of the following surfaces*:

    (1) *a surface of revolution such as*

$$x(s, \theta) = (s + r(s) \cos \varphi(s), r(s) \sin \varphi(s) \cosh \theta, r(s) \sin \varphi(s) \sinh \theta),$$

        where $r(s)$ is given by (3.11);

    (2) *a tube with radius $r = -a$ ($a < 0$).*

Proof. From (2.2) with $c = 1$ and Proposition 3.3, we obtain

$$(br - ar^2)K = r + a.$$

By (3.6), we get

$$\frac{(br - ar^2)(r'' + \kappa \sin \varphi \cosh \theta)}{r(rr'' + r\kappa \sin \varphi \cosh \theta - \sin^2 \varphi)} = r + a,$$

i.e.,

$$\kappa(-r^2 - 2ar + b) \sin \varphi \cosh \theta + (-r^2 - 2ar + b)r'' + (r + a)(1 - r'^2) = 0.$$

Therefore, we get

$$\kappa(-r^2 - 2ar + b) \sin \varphi = 0 \quad \text{and} \quad (-r^2 - 2ar + b)r'' + (r + a)(1 - r'^2) = 0.$$

Case (1). If $-r^2 - 2ar + b \neq 0$, i.e., $a^2 + b < 0$, then $\kappa = 0$. Thus, $\mathbb{M}_+^{11}$ is a surface of revolution and its radial function satisfies

$$(-r^2 - 2ar + b)r'' + (r + a)(1 - r'^2) = 0.$$

Solving the above equation, we get

$$(3.11) \qquad s = c_2 \pm \int \sqrt{\frac{r^2 + 2ar - b}{r^2 + 2ar - b - c_1}}\, dr,$$

where $0 < c_1 < r^2 + 2ar - b, c_2$ are constants.

Since $\kappa = 0$, without loss of generality, we may assume the center curve $c(s) = (s, 0, 0)$ and $\alpha = (1, 0, 0), \beta = (0, 1, 0), \gamma = (0, 0, 1)$, respectively. Then, by (3.1), $\mathbb{M}_+^{11}$ can be expressed by

$$x(s, \theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cosh\theta, r(s)\sin\varphi(s)\sinh\theta),$$

where $r(s)$ satisfy (3.11) and notice $-r' = \cos\varphi$.

Case (2). If $\kappa \neq 0$, then $-r^2 - 2ar + b = 0$. Hence, $r = -a$ is a non-zero constant. $\mathbb{M}_+^{11}$ is a tube and a, b satisfy $a^2 + b = 0$.

Note that $\mathbb{M}_+^{11}$ is a circular cylinder if $\kappa = -r^2 - 2ar + b \equiv 0$. $\square$

Corollary 3.5. *Let $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) be a canal surface with non-zero constant Gaussian curvature. Then it is a surface of revolution with negative constant Gaussian curvature such as*

$$x(s, \theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cosh\theta, r(s)\sin\varphi(s)\sinh\theta),$$

where $r(s)$ is given by (3.12).

Proof. By Theorem 3.4 with $a = 0$, when $\mathbb{M}_+^{11}$ has non-zero constant Gaussian curvature $K = \frac{1}{b}$, from $a^2 + b < 0$, then it is nothing but a surface of revolution with negative constant Gaussian curvature. And it can be expressed by

$$x(s, \theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cosh\theta, r(s)\sin\varphi(s)\sinh\theta),$$

where $r(s)$ satisfies

$$(3.12) \qquad s = c_2 \pm \int \sqrt{\frac{r^2 - b}{r^2 - b - c_1}}\, dr \quad (0 < c_1 < r^2 - b, c_2 \in \mathbb{R}).$$
$\square$

Corollary 3.6. *The canal surface $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) with non-zero constant mean curvature does not exist.*

Proof. By Theorem 3.4 with $b = 0$, it must be a surface of revolution. However, from $a^2 + b < 0$, then $a^2 < 0$, it is a contradiction. $\square$

Theorem 3.7. *The canal surfaces $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) is developable if and only if it is congruent to a part of a circular cylinder or a circular cone.*

Proof. $\mathbb{M}_+^{11}$ is developable if and only if $K \equiv 0$. By (3.6), we have $Q_1 \equiv 0$. From Lemma 3.1, we get

$$r''(s) + \kappa(s)\sin\varphi(s)\cosh\theta = 0.$$

It follows that $r'' = 0$ and $\kappa = 0$. Then $r(s) = c_1 s + c_2$, where c_1, c_2 are constants and $c_1 \neq \pm 1$ (or else $\sin\varphi = 0$, a contradiction). Therefore, $\mathbb{M}_+^{11}$ is a circular cylinder ($c_1 = 0$) or a circular cone ($c_1 \neq 0$, $c_1 \neq \pm 1$) in $\mathbb{E}_1^3$, respectively. The converse is obvious. $\square$

Theorem 3.8. *The canal surfaces* $\mathbb{M}_+^{11}$ ($\mathbb{M}_+^{12}$) *is minimal if and only if it is a part of a surface of revolution such as*

$$x(s,\theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cosh\theta, r(s)\sin\varphi(s)\sinh\theta),$$

where $r(s)$ satisfies (3.13).

Proof. $\mathbb{M}_+^{11}$ is minimal if and only if $H \equiv 0$. From (3.7), $H \equiv 0$ implies

$$-2P_1 - \sin^2\varphi = 0.$$

By Lemma 3.1, we get

$$2rr'' + 2r\kappa\sin\varphi\cosh\theta - \sin^2\varphi = 0.$$

Therefore, one can obtain $r\kappa\sin\varphi = 0$ and $2rr'' - \sin^2\varphi = 0$. Since $r \neq 0$, $\sin\varphi \neq 0$, then $\kappa = 0$ and $\mathbb{M}_+^{11}$ is a surface of revolution. Solving $2rr'' - \sin^2\varphi = 0$, we get

$$(3.13) \qquad s = c_2 \pm \int \sqrt{\frac{r}{r - c_1}}\, dr, \quad (0 < c_1 < r, c_2 \in \mathbb{R}). \qquad \square$$

3.2. The canal surfaces of type $\mathbb{M}_+^2$

In this section, we concern on the canal surface $\mathbb{M}_+^2$. According to the definition of $\mathbb{M}_+^2$, through calculation, we get

$$\begin{cases} \lambda(s) = r(s)r'(s), \\ \mu(s,\theta) = r(s)\sqrt{1 + r'^2(s)}\cos\theta, \\ \omega(s,\theta) = r(s)\sqrt{1 + r'^2(s)}\sin\theta, \end{cases}$$

in (2.1). Then $\mathbb{M}_+^2$ can be parametrized by

$$(3.14) \quad x(s,\theta) = c(s) + r(s)\{r'(s)\alpha + \sqrt{1 + r'^2(s)}\cos\theta\beta + \sqrt{1 + r'^2(s)}\sin\theta\gamma\},$$

where $c(s)$ is parametrized by arc length s. We may assume $r'(s) = \tan\varphi$ for some smooth function $\varphi = \varphi(s)$. Then $\mathbb{M}_+^2$ can be written as

$$x(s,\theta) = c(s) + r(s)(\tan\varphi(s)\alpha + \sec\varphi(s)\cos\theta\beta + \sec\varphi(s)\sin\theta\gamma),$$

where $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark. Do similar calculations to those of $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$, for $\mathbb{M}_+^2$, we just state the following results and omit their proofs.

Lemma 3.9. *The component functions of the first and second fundamental forms of canal surface* $\mathbb{M}_+^2$ *satisfy*

$$L = \frac{E + P_3}{-r}, \quad M = \frac{F}{-r}, \quad N = \frac{G}{-r}$$

and

(3.15) $$EG - F^2 = -r^2 P_3^2, \quad LN - M^2 = -rP_3 Q_3,$$

where $P_3 = rr'' + r\kappa \sec\varphi \cos\theta + \sec^2\varphi = rQ_3 + \sec^2\varphi$, $Q_3 = r'' + \kappa \sec\varphi \cos\theta$.

Remark. Due to regularity, we see that $P_3 \neq 0$ everywhere by (3.15).

From Lemma 3.9, the Gaussian curvature K and the mean curvature H of $\mathbb{M}_+^2$ are given by, respectively

$$K = \frac{LN - M^2}{EG - F^2} = \frac{Q_3}{rP_3},$$

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\sec^2\varphi - 2P_3}{2rP_3}.$$

Proposition 3.10. *The Gaussian curvature K and the mean curvature H of the canal surface $\mathbb{M}_+^2$ can be related by*

$$H = -\frac{1}{2}\left(\frac{1}{r} + Kr\right).$$

Theorem 3.11. *Let $\mathbb{M}_+^2$ be a linear Weingarten canal surface. Then it is an open part of the following surfaces*:

(1) *a surface of revolution such as*

$$x(s, \theta) = (r(s)\sec\varphi(s)\cos\theta, r(s)\sec\varphi(s)\sin\theta, r(s)\tan\varphi(s) + s),$$

where $r(s)$ is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 + 2ar - b}{c_1 - r^2 - 2ar + b}}\,dr, \quad (c_1 > r^2 + 2ar - b > 0, c_2 \in \mathbb{R});$$

(2) *a tube with radius $r = -a$ $(a < 0)$.*

Corollary 3.12. *Let $\mathbb{M}_+^2$ be a canal surface with non-zero constant Gaussian curvature. Then it is a surface of revolution with negative constant Gaussian curvature such as*

$$x(s, \theta) = (r(s)\sec\varphi(s)\cos\theta, r(s)\sec\varphi(s)\sin\theta, r(s)\tan\varphi(s) + s),$$

where $r(s)$ is given by

$$s = c_2 \pm \int \sqrt{\frac{r^2 - b}{c_1 - r^2 + b}}\,dr, \quad (c_1 > r^2 - b > 0, c_2 \in \mathbb{R}).$$

Corollary 3.13. *The canal surface $\mathbb{M}_+^2$ with non-zero constant mean curvature does not exist.*

Theorem 3.14. *The canal surfaces $\mathbb{M}_+^2$ is developable if and only if it is congruent to a part of a circular cylinder or a circular cone.*

Theorem 3.15. *The canal surfaces $\mathbb{M}_+^2$ is minimal if and only if it is a part of a surface of revolution such as*

$$x(s, \theta) = (r(s) \sec \varphi(s) \cos \theta, r(s) \sec \varphi(s) \sin \theta, r(s) \tan \varphi(s) + s),$$

where $r(s)$ satisfies

$$s = c_2 \pm \int \sqrt{\frac{r}{c_1 - r}} dr, \ (c_1 > r > 0, c_2 \in \mathbb{R}).$$

3.3. The canal surfaces of type $\mathbb{M}_+^{13}$ and $\mathbb{M}_+^3$

At first, let $\mathbb{M}$ be a canal surface formed by the movement of the pseudo spheres $\mathbb{S}_1^2$ along a null type spacelike curve $c(s)$, i.e., $\mathbb{M}_+^{13}$. By the definition of $\mathbb{M}_+^{13}$ and Frenet equations, we obtain

$$(3.16) \qquad \begin{cases} \lambda(s) = -r(s)r'(s), \\ 2\mu(s, \theta)\omega(s, \theta) = r^2(s)(1 - r'^2(s)) \end{cases}$$

in (2.1). Then $\mathbb{M}_+^{13}$ can be parametrized by [8]

$$x(s, \theta) = c(s) - r(s)r'(s)\alpha + \mu(s, \theta)\beta + \omega(s, \theta)\gamma,$$

where $c(s)$ is parametrized by arc length s.

Initially, we have

$$(3.17) \qquad x_s = U_1(s, \theta)\alpha + V_1(s, \theta)\beta + W_1(s, \theta)\gamma, \quad x_\theta = \mu_\theta \beta + \omega_\theta \gamma,$$

where

$$U_1 = 1 - r'^2 - rr'' - \omega, \ V_1 = -rr' + \mu_s + \mu\kappa, \ W_1 = \omega_s - \omega\kappa.$$

By differentiating (3.16) with respect to s and θ, respectively. We can get

$$(3.18) \qquad \mu_\theta = -\frac{\mu\omega_\theta}{\omega}, \quad \mu_s = \frac{rr'(U_1 + \omega) - \mu\omega_s}{\omega}.$$

Then, the component functions of the first fundamental form are given by

$$(3.19) \qquad E = U_1^2 + 2V_1 W_1; \quad F = \omega_\theta V_1 + \mu_\theta W_1; \quad G = 2\mu_\theta \omega_\theta.$$

By (3.18) and (3.19), we get

$$(3.20) \qquad EG - F^2 = -\frac{r^2\omega_\theta^2 U_1^2}{\omega^2}.$$

From (3.17) and (3.20), the unit normal vector field n of $\mathbb{M}_+^{13}$ is given by

$$(3.21) \qquad n = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = -\frac{1}{r}(-rr'\alpha + \mu\beta + \omega\gamma),$$

which point outwards the canal surface $\mathbb{M}_+^{13}$ and $\langle n, n \rangle = 1$.

Furthermore, by (3.21) we have

$$n_s = \frac{1}{r^2}\{(-rr'^2 - rU_1 + r)\alpha + (r'\mu - rV_1)\beta + (r'\omega - rW_1)\gamma\};$$

$$n_\theta = -\frac{1}{r}(\mu_\theta\beta + \omega_\theta\gamma).$$

Then, the component functions of the second fundamental form are given by

$$L = -\frac{1}{r^2}\{(-rr'^2 - rU_1 + r)U_1 + (r'\omega - 2rW_1)V_1 + r'\mu W_1\};$$

(3.22) $$M = \frac{\omega_\theta}{r\omega}(rr'U_1 - 2\mu W_1);$$

$$N = \frac{2\mu_\theta\omega_\theta}{r}.$$

From (3.19) and (3.22), we have:

Lemma 3.16. *The component functions of the first and second fundamental forms of canal surfaces* $\mathbb{M}_+^{13}$ *satisfy*

$$L = \frac{E - U_1}{r}, \qquad M = \frac{F}{r}, \qquad N = \frac{G}{r}$$

and

(3.23) $$EG - F^2 = -\frac{r^2\omega_\theta^2 U_1^2}{\omega^2}, \quad LN - M^2 = \frac{\omega_\theta^2 U_1 P_4}{\omega^2},$$

where $P_4 = -U_1 - r'^2 + 1 = rr'' + \omega$.

Remark. Due to regularity, we see that $U_1 \neq 0$ everywhere by (3.23).

From Lemma 3.16, the Gaussian curvature K and the mean curvature H of $\mathbb{M}_+^{13}$ are given by, respectively

(3.24) $$K = \frac{LN - M^2}{EG - F^2} = -\frac{P_4}{r^2 U_1},$$

(3.25) $$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{U_1 - P_4}{2rU_1}.$$

Secondly, let $\mathbb{M}$ be a canal surface formed by the movement of the pseudo spheres $\mathbb{S}_1^2$ along a null curve $c(s)$, i.e., $\mathbb{M}_+^3$. By the definition of $\mathbb{M}_+^3$ and Frenet equations of null curves, we obtain

$$\begin{cases} \omega(s) = -r(s)r'(s), \\ \mu^2(s,\theta) - 2\lambda(s,\theta)r(s)r'(s) = r^2(s) \end{cases}$$

in (2.1). Then $\mathbb{M}_+^3$ can be parametrized by [8]

(3.26) $$x(s,\theta) = c(s) + \lambda(s,\theta)\alpha + \mu(s,\theta)\beta - r(s)r'(s)\gamma,$$

where the center curve $c(s)$ is parametrized by null arc length s.

Do similar calculation to those of $\mathbb{M}_+^{13}$, we have the following conclusions.

Lemma 3.17. *The component functions of the first and second fundamental forms of canal surfaces $\mathbb{M}_+^3$ satisfy*

$$L = \frac{E - W_2}{r}, \qquad M = \frac{F}{r}, \qquad N = \frac{G}{r}$$

and

$$(3.27) \qquad EG - F^2 = -\frac{r^2 \lambda_\theta^2 W_2^2}{\mu^2}, \quad LN - M^2 = \frac{\lambda_\theta^2 W_2 P_5}{\mu^2},$$

where $P_5 = \mu + rr'' = -W_2 - r'^2$.

Remark. Due to regularity, we see that $W_2 \neq 0$ everywhere by (3.27).

From Lemma 3.17, the Gaussian curvature K and the mean curvature H of $\mathbb{M}_+^3$ are given by, respectively

$$(3.28) \qquad K = \frac{LN - M^2}{EG - F^2} = -\frac{P_5}{r^2 W_2},$$

$$(3.29) \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{W_2 - P_5}{2rW_2}.$$

Based on the Gaussian curvature and mean curvature of $\mathbb{M}_+^{13}$ and $\mathbb{M}_+^3$, it is easy to get the following results.

Proposition 3.18. *The Gaussian curvature K and the mean curvature H of the canal surface $\mathbb{M}_+^{13}$ ($\mathbb{M}_+^3$) can be related by*

$$H = \frac{1}{2}(Kr + \frac{1}{r}).$$

Proof. For $\mathbb{M}_+^{13}$, from (3.24) and (3.25), we can get the conclusion easily. And for $\mathbb{M}_+^3$, we can refer to (3.28) and (3.29). $\qquad\square$

Next, we study the canal surface $\mathbb{M}_+^{13}$ ($\mathbb{M}_+^3$) whose Gaussian curvature and mean curvature satisfy some particular conditions.

Remark. In the following, we just prove the results for $\mathbb{M}_+^{13}$ and omit the proof for $\mathbb{M}_+^3$ since it can be similarly done to those of $\mathbb{M}_+^{13}$ and the results are same.

Theorem 3.19. *Let $\mathbb{M}_+^{13}$ ($\mathbb{M}_+^3$) be a linear Weingarten canal surface. Then it is a tube with radius $r = a$ ($a > 0$).*

Proof. From (2.2) with $c = 1$ and Proposition 3.18, we obtain

$$(br + ar^2)K = r - a.$$

By (3.24), we get

$$\frac{-(br + ar^2)(rr'' + \omega)}{r^2(1 - r'^2 - rr'' - \omega)} = r - a,$$

i.e.,

$$(-r^2 + 2ar + b)\omega + (-r^2 + 2ar + b)rr'' - (r^2 - ar)(r'^2 - 1) = 0.$$

Therefore, we get

$$(-r^2 + 2ar + b)\omega = 0 \quad \text{and} \quad (-r^2 + 2ar + b)rr'' - (r^2 - ar)(r'^2 - 1) = 0.$$

Assume $-r^2 + 2ar + b \neq 0$, then $\omega = 0$. By (3.20), $\mathbb{M}_+^{13}$ is degenerate. Thus, $-r^2 - 2ar + b = 0$. Hence, $r = a$ $(a > 0)$ is a non-zero constant. $\mathbb{M}_+^{13}$ is a tube and a, b satisfy $a^2 + b = 0$. □

Corollary 3.20. *The canal surface $\mathbb{M}_+^{13}$ ($\mathbb{M}_+^3$) with non-zero constant Gaussian curvature or non-zero constant mean curvature does not exist.*

Proof. If $\mathbb{M}_+^{13}$ has non-zero constant Gaussian curvature or non-zero constant mean curvature, by (3.24) and (3.25), the functions $\omega = \omega(s)$ and $\mu = \mu(s)$, obviously. It is impossible. The proof is completed. □

Similar to Corollary 3.20, when the Gaussian curvature or mean curvature equal to zero, by (3.24) and (3.25), the functions $\omega = \omega(s)$ and $\mu = \mu(s)$, obviously. Then we have:

Theorem 3.21. *The canal surface $\mathbb{M}_+^{13}$ ($\mathbb{M}_+^3$) is non-developable and non-minimal.*

From the calculations as stated in Subsections 3.1, 3.2 and 3.3, we have some common conclusions as following:

Theorem 3.22. *The umbilical canal surface $\mathbb{M}_+$ does not exist.*

Proof. The canal surfaces $\mathbb{M}_+$ is umbilical means

$$E : F : G = L : M : N,$$

from Lemma 3.1, Lemma 3.2, Lemma 3.9, Lemma 3.16 and Lemma 3.17, we obtain $P_1 = P_2 = P_3 = U_1 = W_2 = 0$. It is impossible by the regularity of those canal surfaces. □

Theorem 3.23. *The canal surfaces $\mathbb{M}_+$ are timelike surfaces in $\mathbb{E}_1^3$.*

Proof. Because the normal vector of canal surface $\mathbb{M}_+$ satisfies $\langle n, n \rangle = 1$, then it is achieved easily. □

Remark. The conclusions obtained in this paper for the canal surfaces of type $\mathbb{M}_+^{11}$, $\mathbb{M}_+^{12}$ and $\mathbb{M}_+^2$ are similar to those of canal surfaces in Euclidean 3-space [2]. However, the results for the canal surfaces of type $\mathbb{M}_+^{13}$ and $\mathbb{M}_+^3$ are quite different due to the causal character of lightlike vector in Minkowski 3-space.

Remark. The canal surfaces obtained by pseudo hyperbolic spheres $\mathbb{H}_0^2$ along a space curve, i.e., $\mathbb{M}_-$ are discussed in [7]. And the canal surfaces foliated by lightcones $\mathbb{Q}^2$ along a space curve, i.e., $\mathbb{M}_0$ are degenerate surfaces by simple calculation. Here, the proof is omitted.

4. Example

In this section, we present two examples of canal surface $\mathbb{M}_+^2$ and $\mathbb{M}_+^3$, the other types of canal surfaces can be characterized similarly.

Example 4.1. Let $c(s) = (\sin \frac{s}{2}, \cos \frac{s}{2}, \frac{\sqrt{5}}{2}s)$ be a timelike curve. Then its Frenet frame is

$$\begin{cases} \alpha(s) = (\frac{1}{2} \cos \frac{s}{2}, -\frac{1}{2} \sin \frac{s}{2}, \frac{\sqrt{5}}{2}), \\ \beta(s) = (-\sin \frac{s}{2}, -\cos \frac{s}{2}, 0), \\ \gamma(s) = (\frac{\sqrt{5}}{2} \cos \frac{s}{2}, -\frac{\sqrt{5}}{2} \sin \frac{s}{2}, \frac{1}{2}). \end{cases}$$

By (3.14), we have

(1) when the radius function $r(s) = s$, the canal surface as

$$x(s,\theta) = (\sin \frac{s}{2} + \frac{s}{2} \cos \frac{s}{2} - \sqrt{2}s \sin \frac{s}{2} \cos \theta + \frac{\sqrt{10}}{2} s \cos \frac{s}{2} \sin \theta,$$

$$\cos \frac{s}{2} - \frac{s}{2} \sin \frac{s}{2} - \sqrt{2}s \cos \frac{s}{2} \cos \theta - \frac{\sqrt{10}}{2} s \sin \frac{s}{2} \sin \theta, \sqrt{5}s + \frac{\sqrt{2}}{2} s \sin \theta);$$

(2) when the radius function $r(s) = 1$, the tube as

$$x(s,\theta) = (\sin \frac{s}{2} - \sin \frac{s}{2} \cos \theta + \frac{\sqrt{5}}{2} \cos \frac{s}{2} \sin \theta,$$

$$\cos \frac{s}{2} - \cos \frac{s}{2} \cos \theta - \frac{\sqrt{5}}{2} \sin \frac{s}{2} \sin \theta, \frac{\sqrt{5}}{2} s + \frac{1}{2} \sin \theta).$$
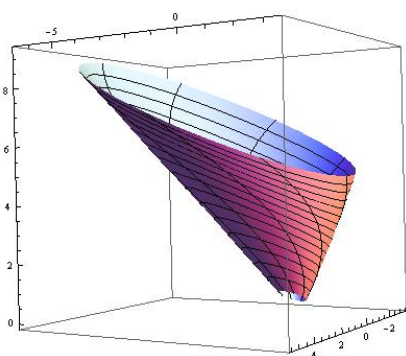

FIGURE 1. Canal surface $\mathbb{M}_+^2$ with $r(s) = s$.

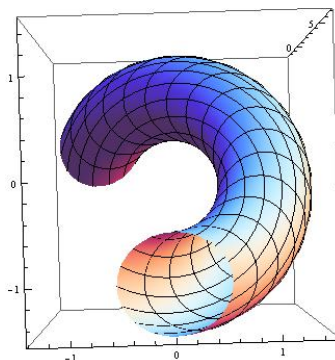FIGURE 2. Tube $\mathbb{M}_+^2$ with $r(s) = 1$.

Example 4.2. Let $c(s) = (\cos s, \sin s, s)$ be a null curve. Then its Frenet frame is

$$\begin{cases} \alpha(s) = (-\sin s, \cos s, 1), \\ \beta(s) = (-\cos s, -\sin s, 0), \\ \gamma(s) = (-\frac{1}{2} \sin s, \frac{1}{2} \cos s, -\frac{1}{2}). \end{cases}$$

By denoting $\lambda(s, \theta) = \cosh\theta$ in (3.26), we have

(1) when the radius function $r(s) = s$, the canal surface as

$$x(s, \theta) = (\cos s - \cosh\theta \sin s - \cos s\sqrt{s^2 + 2s\cosh\theta} + \frac{s}{2}\sin s,$$
$$\sin s + \cosh\theta \cos s - \sin s\sqrt{s^2 + 2s\cosh\theta} - \frac{s}{2}\cos s, s + \cosh\theta + \frac{s}{2});$$

(2) when the radius function $r(s) = 1$, the tube as

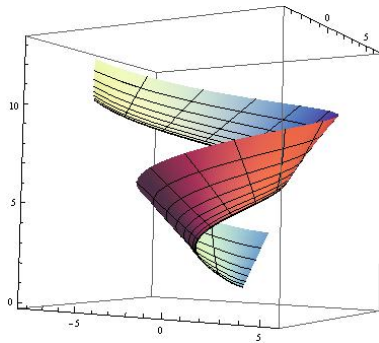$$x(s, \theta) = (\cos s - \cosh\theta \sin s - \cos s, \sin s + \cosh\theta \cos s - \sin s, s + \cosh\theta).$$

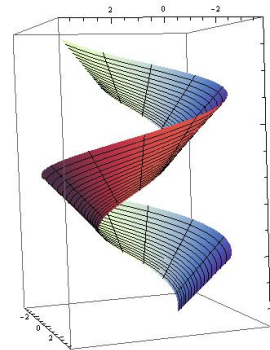FIGURE 3. Canal surface $\mathbb{M}^3_+$ with $r(s) = s$.

FIGURE 4. Tube $\mathbb{M}^3_+$ with $r(s) = 1$.

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Xueshan Fu
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
Email address: xueshanfu000@163.com

Seoung Dal Jung
Department of Mathematics
Jeju National University
Jeju 63243, Korea
Email address: sdjung@jejunu.ac.kr

Jinhua Qian
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
Email address: qianjinhua@mail.neu.edu.cn

Mengfei Su
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
Email address: mengfeisu@126.com