GRADIENT ESTIMATES OF A NONLINEAR ELLIPTIC EQUATION FOR THE $V$-LAPLACIAN

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**Abstract.** In this paper, we consider gradient estimates for positive solutions to the following nonlinear elliptic equation on a complete Riemannian manifold:

$$\Delta_V u + cu^\alpha = 0,$$

where $c, \alpha$ are two real constants and $c \neq 0$. By applying Bochner formula and the maximum principle, we obtain local gradient estimates for positive solutions of the above equation on complete Riemannian manifolds with Bakry-Emery Ricci curvature bounded from below, which generalize some results of [8].

1. Introduction

Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold. The $V$-Laplacian is defined by

$$\Delta_V = \Delta + \langle V, \nabla \cdot \rangle,$$

where $V$ is a smooth vector field on $M$. Here $\nabla$ and $\Delta$ are the Levi-Civita connection and Laplacian with respect to metric $g$, respectively. The $V$-Laplacian is an important generalization of the Laplacian, as well as $V$-harmonic maps introduced in [2]. We define the $\infty$-Bakry-Emery curvature and $N$-Bakry-Emery curvature as follows: [2, 6]

$$\text{Ric}_V = \text{Ric} - \frac{1}{2} \mathcal{L}_V g,$$

$$\text{Ric}_N^V = \text{Ric}_V - \frac{1}{N} V \otimes V,$$

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where $N > 0$ is a natural number, $\text{Ric}$ is the Ricci curvature of $M$ and $\mathcal{L}_V$ denotes the Lie derivative along the direction $V$. In particular, we use the convention that $N = 0$ if and only if $V \equiv 0$.

In this paper, we want to study positive solutions of the nonlinear elliptic equation with the $V$-Laplacian

$$
(1.3) \quad \Delta_V u + cu^\alpha = 0
$$

on an $n$-dimensional complete Riemannian manifold $(M^n, g)$, where $c, \alpha$ are two real constants and $c \neq 0$. When $V = 0$, the above equation (1.3) reduces to

$$
(1.4) \quad \Delta u + cu^\alpha = 0.
$$

For $c$ a function, the equation (1.4) is studied by Gidas and Spruck in [3] with $1 \leq \alpha \leq \frac{n+2}{n-2}$ when $n > 2$ and later it is studied by Li in [5] to achieve gradient estimates and Liouville type results with $1 \leq \alpha \leq \frac{n}{n-2}$ when $n > 2$. If $c < 0$ and $\alpha < 0$, the equation (1.4) on a bounded smooth domain in $\mathbb{R}^n$ is known as the thin film equation, which describes a steady state of the thin film (see [4]). More progress of this and related equations can be found in [7, 9, 10, 12] and the references therein.

Recently, inspired by the methods used by Yau in [11] and Brighton in [1], Ma, Huang and Luo [8] derived local gradient estimates for positive solutions of equations (1.4). We want to generalize their results to equation (1.3) and we obtain the following results.

**Theorem 1.1.** Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold with $\text{Ric} \geq -K$, where $K$ is a non-negative constant. Suppose that $u$ is a positive solution to the equation (1.3) on $B_p(2R)$. Then on $B_p(R)$, we have the following inequalities.

1. If $c < 0$ and $\alpha > 0$, then we have

$$
|\nabla u(x)| \leq M \epsilon \sqrt{C_1} \left( \frac{2K}{R^2} \right) \left( R \sqrt{(n-1)K + n-1} \right) c_1 + C_2 \left( 2 + \frac{C^2}{C_1} \right) c^2
$$

where $M = \sup_{x \in B_p(2R)} u(x)$, the $c_1$ and $c_2$ are positive constants, and the positive constants $C_1$ and $C_2$ are given by

$$
C_1 = \frac{(\epsilon - 1)^2}{(n+2)} - \frac{\epsilon - 1}{\epsilon}, \quad C_2 = \frac{1 - \epsilon}{\epsilon},
$$

respectively. Here $\epsilon \in (0, 1)$ is close enough to 1.

2. If $c > 0$ and $\frac{n+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2 + 9(n+N) + 6}{2(n+N)(n+N+2)}$ with $n \geq 3$, then we have

$$
|\nabla u(x)|
$$
\[ \frac{M}{\tilde{\varepsilon}\sqrt{C_3}} \sqrt{2K + \frac{1}{R^2} \left[ R\sqrt{(n-1)\tilde{K} + n - 1} \right] c_1 + c_2 + \left( 2 + \frac{C_2^2}{C_3} \right) c_1^2}, \]

where \( M, c_1 \) and \( c_2 \) are the same as (1.5), and the positive constants \( C_3 \) and \( C_4 \) are given by

\[
C_3 = \frac{1}{2} \left[ \left( \frac{(\tilde{\varepsilon} - 1)^2}{(n + N)\tilde{\varepsilon}^2} - \frac{\tilde{\varepsilon} - 1}{\tilde{\varepsilon}} - \frac{n + N}{n + N - 1} (\tilde{\varepsilon} - 1) + \alpha \right)^2 \right],
\]

\[
C_4 = \frac{4(\alpha - 1)(n + N)(n + N + 2) + (n + N)[2(n + N) + 5]}{[5(n + N) + 6] - 4(\alpha - 1)(n + N)(n + N + 2)},
\]

respectively. Here \( \tilde{\varepsilon} = \frac{[5(n + N) + 6 - 2(\alpha - 1)(n + N)]^2 + 2(n + N)}{2[(n + N)^2 + 3(n + N) + 3]} \).

Letting \( R \to \infty \) in (1.5) and (1.6), we obtain the following gradient estimates on complete noncompact Riemannian manifolds:

**Corollary 1.2.** Let \((M^n, g)\) be an \( n \)-dimensional complete noncompact Riemannian manifold with \( \text{Ric}_V \geq -K \), where \( K \) is a non-negative constant. Let \( u \) be a positive solution to the equation (1.3). Then, we have the following inequalities.

1. If \( c < 0 \) and \( \alpha > 0 \), then we have

\[ |\nabla u|(x) \leq \frac{M}{\varepsilon\sqrt{C_1}} \sqrt{2\tilde{K}}; \]

2. If \( c > 0 \) and \( \frac{n + N + 2}{2(n + N - 1)} < \alpha < \frac{2(n + N)^2 + 9(n + N) + 6}{2[(n + N)^2 + 3(n + N) + 3]} \) with \( n \geq 3 \), then we have

\[ |\nabla u|(x) \leq \frac{M}{\varepsilon\sqrt{C_1}} \sqrt{2\tilde{K}}, \]

where \( M = \sup_{x \in M} u(x) \).

We can also obtain similar results under the assumption that \( \text{Ric}_V \) is bounded by below.

**Theorem 1.3.** Let \((M^n, g)\) be an \( n \)-dimensional Riemannian manifold with \( \text{Ric}_V(B_p(2R)) \geq -\tilde{K} \), and \( |V| \leq \tilde{L} \), where \( \tilde{K} \) and \( \tilde{L} \) are non-negative constants. Suppose that \( u \) is a positive solution to the equation (1.3) on \( B_p(2R) \). Then on \( B_p(R) \), we have the following inequalities.

1. If \( c < 0 \) and \( \alpha > 0 \), then we have

\[ |\nabla u|(x) \leq \frac{M}{\varepsilon\sqrt{C_1}} \sqrt{2\tilde{K}} \]

where \( M = \sup_{x \in M} u(x) \).
where $M = \sup_{x \in B_p(2R)} u(x)$, the $c_1$ and $c_2$ are positive constants, and the positive constants $\tilde{C}_1$ and $\tilde{C}_2$ are given by

$$\tilde{C}_1 = \frac{(\epsilon - 1)^2}{n\epsilon^2} - \frac{1}{\epsilon} - \frac{1}{\epsilon^2}, \quad \tilde{C}_2 = \frac{1}{\epsilon} - \epsilon,$$

respectively. Here $\epsilon \in (0, 1)$ is close enough to 1.

(2) If $c > 0$ and $\frac{n+2}{2(n+1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ with $n \geq 3$, then we have

$$|\nabla u|(x) \leq \frac{M}{\tilde{\epsilon}\sqrt{\tilde{C}_3}} \sqrt{2\tilde{K}},$$

where $M$, $c_1$ and $c_2$ are the same as (1.9), and the positive constants $\tilde{C}_3$ and $\tilde{C}_4$ are given by

$$\tilde{C}_3 = \frac{1}{2} \left[ \frac{(\tilde{\epsilon} - 1)^2}{n\tilde{\epsilon}^2} - \frac{1}{\tilde{\epsilon}} - \frac{1}{\tilde{\epsilon}^2} \left( \frac{n+2}{n} (\tilde{\epsilon} - 1) + \alpha \right)^2 \right],$$

$$\tilde{C}_4 = \frac{4(\alpha - 1)n(n+2) + n(2n+5)}{(5n+6) - 4(\alpha - 1)n(n+2)},$$

respectively. Here $\tilde{\epsilon} = \frac{(5n+6)-2(\alpha-1)(n^2+2n)}{2(n^2+5n+4)}$.

**Corollary 1.4.** Let $(M^n, g)$ be an $n$-dimensional complete noncompact Riemannian manifold with $\text{Ric}_V \geq -\tilde{K}$, and $|V| \leq L$, where $\tilde{K}$ and $L$ are non-negative constants. Let $u$ be a positive solution to the equation (1.3). Then, we have the following inequalities.

1. If $c < 0$ and $\alpha > 0$, then we have

(1.11) \[ |\nabla u|(x) \leq \frac{M}{\epsilon\sqrt{\tilde{C}_1}} \sqrt{2\tilde{K}}; \]

2. If $c > 0$ and $\frac{n+2}{2(n+1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ with $n \geq 3$, then we have

(1.12) \[ |\nabla u|(x) \leq \frac{M}{\tilde{\epsilon}\sqrt{\tilde{C}_3}} \sqrt{2\tilde{K}}, \]

where $M = \sup_{x \in M} u(x)$.

**Remark 1.1.** Clearly, our results generalize some results of [8] with respect to the nonlinear elliptic equation (1.3) with $V = 0$. 
2. The proof of theorems

We firstly give the following lemma.

**Lemma 2.1.** Let \((M^n, g)\) be an \(n\)-dimensional complete Riemannian manifold with \(\text{Ric}^N_{B_p}(B_p(2R)) \geq -K\), where \(K\) is a nonnegative constant. Assuming that \(u\) is a positive solution to nonlinear elliptic equation \((1.3)\) on \(B_p(R)\). Denote \(h = u^\epsilon\) with \(\epsilon \neq 0\). Then on \(B_p(R)\), the following inequalities hold.

(a) If \(c < 0\) and \(\alpha > 0\), then there exists \(\epsilon \in (0, 1)\) such that

\[
\frac{1}{2} \Delta_V |\nabla h|^2 \geq \left(\frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon}\right) \frac{|\nabla h|^4}{h^2} + \frac{\epsilon - 1}{\epsilon} \frac{\nabla (|\nabla h|^2)}{h} - K |\nabla h|^2.
\]

(b) If \(c > 0\) and for a fixed \(\alpha\), there exist two positive constants \(\epsilon, \delta\) such that

\[
cia\frac{c^2 \epsilon^2}{n + N} - \frac{c}{\delta} \left(\frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha\right) > 0,\n\]

then we have

\[
\frac{1}{2} \Delta_V |\nabla h|^2 \geq \left[\frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha\right] \frac{|\nabla h|^4}{h^2} + \frac{\epsilon - 1}{\epsilon} \frac{\nabla (|\nabla h|^2)}{h} - K |\nabla h|^2.
\]

**Proof.** Let \(h = u^\epsilon\), where \(\epsilon \neq 0\) is a constant to be determined. Then we have

\[\log h = \log u^\epsilon = \epsilon \log u.\]

A simple calculation implies

\[
\Delta_V h = \Delta (u^\epsilon) + \langle V, \nabla (u^\epsilon) \rangle = \epsilon (\epsilon - 1) u^{\epsilon - 2} |\nabla u|^2 + \epsilon u^{\epsilon - 1} \Delta u u = \epsilon (\epsilon - 1) u^{\epsilon - 2} |\nabla u|^2 - cu^{\alpha + \epsilon - 1} = \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ceh^{\alpha + \epsilon - 1}.
\]

Therefore we get

\[
\nabla h \nabla \Delta_V h = \nabla h \nabla \left(\frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ceh^{\alpha + \epsilon - 1}\right) = \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - c(\alpha + \epsilon - 1)h^{\alpha + \epsilon - 1} \frac{|\nabla h|^2}{h}.
\]
Applying (2.5) and (2.6) into the famous Bochner formula to \( h \), we have

\[
\frac{1}{2} \Delta_V |\nabla h|^2
\]

\[
= |\nabla^2 h|^2 + \nabla h \nabla \Delta_V h + \text{Ric}_V(\nabla h, \nabla h)
\]

\[
\geq \frac{1}{n + N} (\Delta_V h)^2 + \nabla h \nabla \Delta_V h + \text{Ric}_V^N(\nabla h, \nabla h)
\]

\[
\geq \frac{1}{n + N} \left( \frac{\epsilon - 1}{\epsilon} |\nabla h|^2 - c\epsilon h^{\alpha}(\epsilon - 1) \right)^2 + \nabla h \nabla \Delta_V h - K|h|^2
\]

\[
= \left( \frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{|\nabla h|^4}{h^2}
\]

\[
- c \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) h^{\alpha}(\epsilon - 1) \frac{|\nabla h|^2}{h}
\]

\[
+ \frac{c^2\epsilon^2}{n + N} h^{2(\alpha + 1)} + \frac{\epsilon - 1}{\epsilon} \nabla h(\nabla h^2) - K|h|^2.
\]

First, we prove (a).

In (2.7), if \( c < 0 \) and \( \alpha > 0 \), we can choose \( \epsilon \in (0, 1) \) close enough to 1 such that

\[-c \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \geq 0,\]

and then (2.1) follows directly.

Next, we prove (b).

For a fixed point \( p \), if there exists a positive constant \( \delta \) such that \( h^{\alpha + 1} \leq \delta |\nabla h|^2 \), according to (2.2), then (2.7) becomes

\[
\frac{1}{2} \Delta_V |\nabla h|^2
\]

\[
\geq \left[ \frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} - c\delta \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \right] \frac{|\nabla h|^4}{h^2}
\]

\[
+ \frac{c^2\epsilon^2}{n + N} h^{2(\alpha + 1)} + \frac{\epsilon - 1}{\epsilon} \nabla h(\nabla h^2) - K|h|^2
\]

\[
\geq \left[ \frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} - c\delta \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \right] \frac{|\nabla h|^4}{h^2}
\]

\[
+ \frac{\epsilon - 1}{\epsilon} \nabla h(\nabla h^2) - K|h|^2.
\]

On the contrary, at the point \( p \), if \( h^{\alpha + 1} \geq \delta |\nabla h|^2 \), then (2.7) becomes

\[
\frac{1}{2} \Delta_V |\nabla h|^2
\]
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\[ \geq \left( \frac{(\epsilon - 1)^2}{(n + N)c^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{|\nabla h|^4}{h^2} + \frac{c^2\epsilon^2}{n + N} - \frac{\epsilon}{\delta} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \right] h^{2(\alpha + \epsilon - 1)} \]

\[ + \frac{\epsilon - 1}{\epsilon} \nabla (|\nabla h|^2) - K|\nabla h|^2 \]

\[ \geq \left\{ \left( \frac{(\epsilon - 1)^2}{(n + N)c^2} - \frac{\epsilon - 1}{\epsilon} \right) + \delta^2 \left[ \frac{c^2\epsilon^2}{n + N} - \frac{\epsilon}{\delta} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \right] \frac{|\nabla h|^4}{h^2} \right\} \]

\[ + \frac{\epsilon - 1}{\epsilon} \nabla (|\nabla h|^2) - K|\nabla h|^2 \]

\[ \geq \left[ \frac{(\epsilon - 1)^2}{(n + N)c^2} - \frac{\epsilon - 1}{\epsilon} - c\delta \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \right] \frac{|\nabla h|^4}{h^2} \]

\[ + \frac{\epsilon - 1}{\epsilon} \nabla (|\nabla h|^2) - K|\nabla h|^2 \]

as long as

\[ (2.10) \quad \frac{c^2\epsilon^2}{n + N} - \frac{\epsilon}{\delta} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) > 0. \]

In both cases, (2.4) holds always. We complete the proof of Lemma 2.1. \qed

In order to obtain the upper bound of $|\nabla h|$ by using the maximum principle, it is sufficient to choose the coefficient of $|\nabla h|^4$ in (2.1) and (2.4) such that it is positive. In (2.4) of Lemma 2.1, we need to choose appropriate $\epsilon$, $\delta$ such that

\[ (2.11) \quad \frac{\epsilon - 1}{(n + N)c^2} - \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) > 0. \]

Under the assumption of (2.2), the inequality (2.3) becomes

\[ (2.12) \quad \delta > \frac{(n + N)c}{c^2\epsilon^2} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) \]

and (2.11) becomes

\[ (2.13) \quad \delta < \frac{\epsilon (\epsilon - 1)^2}{c^2\epsilon^2} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right). \]

In order to ensure we can choose a positive $\delta$, from (2.12) and (2.13), we need choose an $\epsilon$ satisfying

\[ (2.14) \quad \frac{(n + N)c}{c^2\epsilon^2} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right) < \frac{\epsilon (\epsilon - 1)^2}{c^2\epsilon^2} \left( \frac{n + N + 2}{n + N} (\epsilon - 1) + \alpha \right). \]
which is equivalent to

\[
(n + N)^2 + 5(n + N) + 3|\epsilon|^2 + \{2(\alpha - 1)[(n + N)^2 + 2(n + N)]
- [5(n + N) + 6]\} \epsilon + (\alpha - 1)^2(n + N)^2 - 4(\alpha - 1)(n + N) + 3 < 0.
\]

By a direct calculation, under the condition

\[
-\frac{(n + N - 4) - \sqrt{(n + N)^2 + 5(n + N) + 3}}{2(n + N - 1)} < \alpha - 1 < \frac{-\frac{(n + N - 4) + \sqrt{(n + N)^2 + 5(n + N) + 3}}{2(n + N - 1)}}{2(n + N - 1)},
\]

we have

\[
2(\alpha - 1)[(n + N)^2 + 2(n + N)] - [5(n + N) + 6]\]
\[\times [(\alpha - 1)^2(n + N)^2 - 4(\alpha - 1)(n + N) + 3]
\[= (n + N)^2\{-4(n + N - 1)(\alpha - 1)^2 - 4(n + N - 4)(\alpha - 1) + 13\} > 0,
\]

which shows the quadratic inequality (2.15) with respect to $\epsilon$ has two real roots.

Now we are ready to prove the following proposition which plays a key role in the proof of main results.

**Proposition 2.2.** Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric}^2_N(B_p(2R)) \geq -K$, where $K$ is a nonnegative constant. Assuming that $u$ is a positive solution to nonlinear elliptic equation (1.3) on $B_p(2R)$. Denote $h = u^\epsilon$ with $\epsilon \neq 0$. Then on $B_p(R)$ the following inequalities hold.

(c) If $c < 0$ and $\alpha > 0$, then we have

\[
\frac{1}{2} \Delta V |\nabla h|^2 \geq C_1 \frac{|\nabla h|^4}{h^2} - C_2 h \nabla h \nabla (|\nabla h|^2) - K |\nabla h|^2,
\]

where positive constants $C_1$ and $C_2$ are given by

\[
C_1 = \frac{(\epsilon - 1)^2}{(n + N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon},
\]
\[
C_2 = 1 - \frac{\epsilon}{\epsilon},
\]

respectively.

(d) If $c > 0$ and \(\frac{n + N + 2}{2(n + N - 1)} < \alpha < \frac{2(n + N)^2 + 9(n + N) + 6}{2(n + N)(n + N + 2)}\) with $n \geq 3$, then we have

\[
\frac{1}{2} \Delta V |\nabla h|^2 \geq C_3 \frac{|\nabla h|^4}{h^2} - C_4 h \nabla h \nabla (|\nabla h|^2) - K |\nabla h|^2,
\]
where positive constants $C_3$ and $C_4$ are given by

\[
C_3 = \frac{1}{2} \left[ \frac{(\bar{\epsilon} - 1)^2}{(n+N)\bar{\epsilon}^2} - \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \right] + \frac{n + N}{\bar{\epsilon}^2} \left( \frac{n + N + 2}{n + N} \right) \left( \bar{\epsilon} - 1 + \alpha \right)^2,
\]

\[
C_4 = \frac{4(\alpha - 1)(n + N)(n + N + 2) + (n + N)[2(n + N) + 5]}{5(n + N) + 6} - \frac{4(\alpha - 1)(n + N)(n + N + 2)}{2(n + N)(n + N + 2)} + \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \left( \frac{n + N + 2}{n + N} \right) \left( \bar{\epsilon} - 1 + \alpha \right)^2.
\]

respectively. Here $\bar{\epsilon} = \frac{[5(n+N)+6-2(\alpha-1)((n+N)^2+2(n+N))]^{1/2}}{2[(n+N)^2+5(n+N)+3]}$.

Proof. We prove this proposition case by case.

(c) The case of $c < 0$ and $\alpha > 0$. In the proof of Lemma 2.1 we see that by choosing an $\epsilon \in (0,1)$ such that $\frac{n+N+2}{n+N} (\epsilon - 1) + \alpha \geq 0$ we get the

\[
\frac{1}{2} \Delta \psi |\nabla h|^2 \geq \left( \frac{(\epsilon - 1)^2}{(n+N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{|\nabla h|^4}{\epsilon^2} + \frac{\epsilon - 1}{\epsilon} \frac{\nabla(|\nabla h|^2)}{h^2} - K |\nabla h|^2.
\]

Then we see that $C_1 = \frac{(\epsilon - 1)^2}{(n+N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} > 0$ and $C_2 = \frac{\epsilon - 1}{\epsilon} > 0$.

(d) The case of $c > 0$ and $\frac{n+N+2}{n+N-1} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ when $n \geq 3$. In this case, (2.2) is equivalent to

\[
\epsilon > 1 - \frac{(n+N)\alpha}{n+N+2}.
\]

We can check

\[
\frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]} < -\frac{(n+N-4)+\sqrt{(n+N)^2+5(n+N)+3}}{2(n+N-1)}.
\]

Hence, when $n \geq 3$, for any $\alpha$ satisfies

\[
-\frac{n+N-4}{2(n+N-1)} < \alpha - 1 < \frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]}
\]

which is equivalent to

\[
-\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}.
\]

then (2.21) is satisfied by choosing

\[
\epsilon := \bar{\epsilon} = \frac{[5(n+N)+6]-2(\alpha-1)[(n+N)^2+2(n+N)]}{2[(n+N)^2+5(n+N)+3]},
\]

and it is easy to check that $\epsilon \in (0,1)$.

In particular, we let

\[
\delta := \bar{\delta} = \frac{1}{2} \left[ \frac{(n+N)c}{\epsilon^2 \bar{\epsilon}^2} \left( \frac{n+N+2}{n+N} \bar{\epsilon} - 1 + \alpha \right) + \frac{(\bar{\epsilon} - 1)^2}{(n+N)\bar{\epsilon}^2} - \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \right] + \frac{(n+N+2)(\bar{\epsilon} - 1 + \alpha)}{\epsilon} \left( \frac{n+N+2}{n+N} \bar{\epsilon} - 1 + \alpha \right),
\]

where positive constants $C_3$ and $C_4$ are given by

\[
C_3 = \frac{1}{2} \left[ \frac{(\bar{\epsilon} - 1)^2}{(n+N)\bar{\epsilon}^2} - \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \right] + \frac{n + N}{\bar{\epsilon}^2} \left( \frac{n + N + 2}{n + N} \right) \left( \bar{\epsilon} - 1 + \alpha \right)^2,
\]

\[
C_4 = \frac{4(\alpha - 1)(n + N)(n + N + 2) + (n + N)[2(n + N) + 5]}{5(n + N) + 6} - \frac{4(\alpha - 1)(n + N)(n + N + 2)}{2(n + N)(n + N + 2)} + \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \left( \frac{n + N + 2}{n + N} \right) \left( \bar{\epsilon} - 1 + \alpha \right)^2.
\]

respectively. Here $\bar{\epsilon} = \frac{[5(n+N)+6-2(\alpha-1)((n+N)^2+2(n+N))]^{1/2}}{2[(n+N)^2+5(n+N)+3]}$. 

Proof. We prove this proposition case by case.

(c) The case of $c < 0$ and $\alpha > 0$. In the proof of Lemma 2.1 we see that by choosing an $\epsilon \in (0,1)$ such that $\frac{n+N+2}{n+N} (\epsilon - 1) + \alpha \geq 0$ we get the

\[
\frac{1}{2} \Delta \psi |\nabla h|^2 \geq \left( \frac{(\epsilon - 1)^2}{(n+N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{|\nabla h|^4}{\epsilon^2} + \frac{\epsilon - 1}{\epsilon} \frac{\nabla(|\nabla h|^2)}{h^2} - K |\nabla h|^2.
\]

Then we see that $C_1 = \frac{(\epsilon - 1)^2}{(n+N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon} > 0$ and $C_2 = \frac{\epsilon - 1}{\epsilon} > 0$.

(d) The case of $c > 0$ and $\frac{n+N+2}{n+N-1} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ when $n \geq 3$. In this case, (2.2) is equivalent to

\[
\epsilon > 1 - \frac{(n+N)\alpha}{n+N+2}.
\]

We can check

\[
\frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]} < -\frac{(n+N-4)+\sqrt{(n+N)^2+5(n+N)+3}}{2(n+N-1)}.
\]

Hence, when $n \geq 3$, for any $\alpha$ satisfies

\[
-\frac{n+N-4}{2(n+N-1)} < \alpha - 1 < \frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]}
\]

which is equivalent to

\[
-\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}.
\]

then (2.21) is satisfied by choosing

\[
\epsilon := \bar{\epsilon} = \frac{[5(n+N)+6]-2(\alpha-1)[(n+N)^2+2(n+N)]}{2[(n+N)^2+5(n+N)+3]},
\]

and it is easy to check that $\epsilon \in (0,1)$.

In particular, we let

\[
\delta := \bar{\delta} = \frac{1}{2} \left[ \frac{(n+N)c}{\epsilon^2 \bar{\epsilon}^2} \left( \frac{n+N+2}{n+N} \bar{\epsilon} - 1 + \alpha \right) + \frac{(\bar{\epsilon} - 1)^2}{(n+N)\bar{\epsilon}^2} - \frac{\bar{\epsilon} - 1}{\bar{\epsilon}} \right] + \frac{(n+N+2)(\bar{\epsilon} - 1 + \alpha)}{\epsilon} \left( \frac{n+N+2}{n+N} \bar{\epsilon} - 1 + \alpha \right),
\]
then (2.10) and (2.11) are satisfied and (2.4) becomes
\[ (2.27) \quad \frac{1}{2} \Delta V |\nabla h|^2 \geq C_3 \frac{|\nabla h|^4}{h^2} - C_4 \frac{\nabla h}{h} \nabla(|\nabla h|^2) - K |\nabla h|^2, \]
where positive constants $C_3$ and $C_4$ are given by
\[ C_3 = \frac{1}{2} \left[ \left( \frac{(\tilde{\epsilon} - 1)^2}{(n+N)\tilde{\epsilon}^2} - \frac{n+N}{\tilde{\epsilon}^2} \left( \frac{n+N+2}{n+N}(\tilde{\epsilon} - 1) + \alpha \right)^2 \right) \right], \]
\[ C_4 = \frac{4(\alpha - 1)(n+N)(n+N+2) + (n+N)[2(n+N)+5]}{[5(n+N)+6] - 4(n-1)(n+N)(n+N+2)}, \]
respectively. We conclude the proof of Proposition 2.2.

Now we begin to prove Theorem 1.1.

**Proof of Theorem 1.1.** We first prove the case of $c < 0$ and $\alpha > 0$. Choose a smooth function $\eta(r)$ such that $0 \leq \eta(r) \leq 1$, $\eta(r) = 1$ if $r \leq 1$, $\eta(r) = 0$ if $r \geq 2$, and $0 \geq \eta(r)^{-\frac{1}{2}} \eta'(r) \geq -c_1$, $\eta''(r) \geq -c_2$ for some $c_1$, $c_2 \geq 0$. For a fixed point $p \in M$, let $\rho(x) = \text{dist}(p,x)$ and $\psi = \eta \left( \frac{\rho(x)}{R} \right)$. Therefore,
\[ (2.28) \quad \frac{\nabla \psi}{\psi} = \frac{\nabla \eta}{\eta} = \frac{1}{\eta(r)} \frac{\eta'(r)^2}{R^2} |\nabla \rho(x)|^2 \leq \frac{c_2^2}{R^2}. \]
Since $\text{Ric}^N \geq -K$, the Laplacian comparison theorem in [6] implies that
\[ (2.29) \quad \Delta V \rho \leq \sqrt{(n-1)K \coth \left( \sqrt{\frac{K}{n-1}} \rho \right)} \leq \sqrt{(n-1)K + \frac{n-1}{\rho}}. \]
Hence,
\[ (2.30) \quad \Delta V \psi = \frac{\eta(r)'' |\nabla \rho|^2}{R^2} + \frac{\eta(r)' \Delta V \rho}{R} \]
\[ \geq -\frac{c_2}{R^2} + \frac{c_1}{R} \left( \sqrt{(n-1)K + \frac{n-1}{\rho}} \right) \]
\[ \geq - \frac{R \left( \sqrt{(n-1)K + \frac{n-1}{R}} \right) c_1 + c_2}{R^2} \]
\[ = - \frac{R(\sqrt{(n-1)K + n-1}) c_1 + c_2}{R^2}. \]

Denote by $B_p(R)$ the geodesic ball centered at $p$ with radius $R$. Let $G = \psi |\nabla h|^2$. Assume $G$ achieves its maximum at the point $x_0 \in B_p(2R)$ and assume $G(x_0) > 0$ (otherwise this is obvious). Then at the point $x_0$, it holds that
\[ \Delta V G \leq 0, \quad \nabla(|\nabla h|^2) = - \frac{|\nabla h|^2}{\psi} \nabla \psi. \]
Using (2.18) in Proposition 2.2, we obtain

\begin{align}
0 & \geq \Delta V G \\
& = \psi \Delta V (|\nabla h|^2) + |\nabla h|^2 \Delta V \psi + 2\nabla \psi \nabla |\nabla h|^2 \\
& = \psi \Delta V (|\nabla h|^2) + \frac{\Delta V \psi}{\psi} G - 2\frac{|\nabla \psi|^2}{\psi^2} G \\
& \geq 2\psi \left[ C_1 \frac{|\nabla h|^4}{h^2} - C_2 \frac{\nabla h}{h} \nabla ((|\nabla h|^2) - K|\nabla h|^2) \right] + \frac{\Delta V \psi}{\psi} G - 2\frac{|\nabla \psi|^2}{\psi^2} G \\
& = 2C_1 \frac{G^2}{\psi h^2} + 2C_2 \frac{G}{\psi} \nabla \psi \nabla h - 2KG + \frac{\Delta V \psi}{\psi} G - 2\frac{|\nabla \psi|^2}{\psi^2} G.
\end{align}

Multiplying both sides of (2.31) by \( \frac{\psi}{G} \) yields

\begin{align}
2C_1 \frac{G}{h^2} \leq -2C_2 \nabla \psi \nabla h + 2\psi K - \Delta V \psi + 2\frac{|\nabla \psi|^2}{\psi}.
\end{align}

Using the Cauchy inequality

\[-2C_2 \nabla \psi \nabla h \leq 2C_2|\nabla \psi| \frac{|\nabla h|}{h} \leq \frac{C_2^2}{C_1} \frac{|\nabla \psi|^2}{\psi} + C_1 \frac{G}{h^2},\]

into (2.32) yields

\begin{align}
C_1 \frac{G}{h^2} \leq 2\psi K - \Delta V \psi + \left(2 + \frac{C_2^2}{C_1}\right) \frac{|\nabla \psi|^2}{\psi}.
\end{align}

Hence, for \( x \in B_p(R) \), we have

\begin{align}
C_1 G(x) & \leq C_1 G(x_0) \\
& \leq h^2(x_0) \left\{2K + \frac{1}{R^2} \left[ (R \sqrt{(n-1)K + n-1}) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1}\right) c_1^2 \right]\right\}.
\end{align}

It shows that

\begin{align}
|\nabla u|^2(x) & \leq M^2 \frac{1}{c^2 C_1} \left\{2K + \frac{1}{R^2} \left[ (R \sqrt{(n-1)K + n-1}) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1}\right) c_1^2 \right]\right\},
\end{align}

and hence,

\begin{align}
|\nabla u|(x) & \leq \frac{M}{\epsilon \sqrt{C_1}} \left\{2K + \frac{1}{R^2} \left[ (R \sqrt{(n-1)K + n-1}) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1}\right) c_1^2 \right]\right\}.
\end{align}

It yields the desired inequality (1.5) of Theorem 1.1.
Next, we prove the case $c > 0$ and $\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ with $n \geq 3$. In a similar way as the case $c < 0$ and $\alpha > 0$, on $\bar{B}_p(R)$, we have

\[
|\nabla u|(x) \leq \frac{M}{\epsilon^2\sqrt{C_3}} \sqrt{2K + \frac{1}{R^2} \left( (R\sqrt{(n-1)K}+n-1) c_1+c_2+\left( 2+\frac{C_2}{C_3} \right) \epsilon^2 \right)}.
\]

This concludes the proof of inequality (1.6) of Theorem 1.1. We complete the proof of Theorem 1.1.

Now we are in the position to give a brief proof of Theorem 1.3.

Skeptic of the proof of Theorem 1.3. Noticing that we have the following Bochner formula to $h$ with Ric$_V$,

\[
\frac{1}{2} \Delta_V |\nabla h|^2 = |\nabla^2 h|^2 + \nabla h \nabla \Delta_V h + \text{Ric}_V(\nabla h, \nabla h),
\]

then (2.7) becomes

\[
\frac{1}{2} \Delta_V |\nabla h|^2 = |\nabla^2 h|^2 + \nabla h \nabla \Delta_V h + \text{Ric}_V(\nabla h, \nabla h)
\geq \frac{1}{n} \left( \frac{n-1}{n} \left| \nabla h \right|^2 - c\epsilon h^{\frac{n+1}{n}} \right)^2 + \nabla h \nabla \Delta_V h - \tilde{K} |\nabla h|^2
\geq \left( \frac{\epsilon - 1}{n} \right) \left( \frac{n-1}{n} \right) \left| \nabla h \right|^2 - c \left( \frac{n+2}{n} (\epsilon - 1) + \alpha \right) h^{\frac{n+1}{n}} \frac{|\nabla h|^2}{h}
\geq \frac{c^2}{n} h^{\frac{n(n+1)}{n}} + \frac{\epsilon - 1}{\epsilon} h \nabla(|\nabla h|^2) - \tilde{K} |\nabla h|^2.
\]

Moreover, the Laplacian comparison theorem in [2] implies: if Ric$_V$ $\geq -\tilde{K}$ and $|V| \leq L$, we have

\[
\Delta_V \rho \leq \sqrt{(n-1)\tilde{K} + \frac{n-1}{\rho} + L}.
\]

So (2.28) and (2.30) also hold true in almost the same forms

\[
\frac{|\nabla \psi|^2}{\psi} \leq \frac{c_1^2}{R^2}
\]

and

\[
\Delta_V \psi \geq - \frac{R \sqrt{(n-1)\tilde{K} + RL + n-1} c_1 + c_2}{R^2}.
\]

Noticing the above facts, the proof of Theorem 1.3 is the same to that of Theorem 1.1, so we omit it here.
References


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