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GENERALIZED MYERS THEOREM FOR FINSLER MANIFOLDS WITH INTEGRAL RICCI CURVATURE BOUND

Bing-Ye Wu

Abstract. We establish the generalized Myers theorem for Finsler manifolds under integral Ricci curvature bound. More precisely, we show that the forward complete Finsler n-manifold whose part of Ricci curvature less than a positive constant is small in L^p-norm (for $p > n/2$) have bounded diameter and finite fundamental group.

1. Introduction

The celebrated Myers theorem in global Riemannian geometry says that if a Riemannian manifold M satisfies $\mathbf{Ric}(v) \geqslant n-1$ for all unit vectors v, then M is compact with $\mathrm{diam}(M) \leqslant \pi$, and the fundamental group $\pi_1(M)$ is finite. There has been many generalizations of Myers theorem where the point-wise Ricci curvature is replaced by the integral or line integral of Ricci curvature (see e.g., [1, 3, 4, 10]).

Myers theorem has also been generalized to Finsler manifolds [2], and recently we establish a generalized Myers theorem under the line integral curvature bound for Finsler manifolds [8]. The main purpose of the present paper is to establish the generalized Myers theorem for Finsler manifolds under integral Ricci curvature bound. More precisely, we want to show that the forward complete Finsler n-manifold whose part of Ricci curvature less than a positive constant is small in L^p-norm (for $p > n/2$) have bounded diameter and finite fundamental group. To state the main result let us first recall some notations. On a Finsler manifold (M, F) let $dV_{\max}$ and $dV_{\min}$ be the maximal volume form and minimal volume form, respectively, and we shall denote by $\mathrm{vol}_{\min}$ (resp. $\mathrm{vol}_{\max}$) the volume with respect to $dV_{\min}$ (resp. $dV_{\max}$). Let $\underline{\mathbf{Ric}} : M \to \mathbb{R}$ be the function of smallest Ricci curvature at given point. More precisely,

$$\underline{\mathbf{Ric}}(x) = \min_{y \in T_xM\setminus\{0\}} \mathbf{Ric}(y), \quad \forall x \in M.$$

The main purpose of the present paper is to prove the following.

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Main Result. *Let (M,F) be a forward complete Finsler n-manifold with finite uniformity constant $\mu_F =: \Lambda^2$. If $\int_M (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}$ is finite for some $p > n/2$, then the minimal volume $\mathrm{vol}_{\min}(M)$ of M is finite. In this situation, there exist two positive constants $A(n,p,\Lambda)$ and $B(n,p,\Lambda)$ such that when*

$$\epsilon = \left(\frac{\int_M (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(M)}\right)^{\frac{1}{p}} \leqslant A(n,p,\Lambda),$$

then M is compact with $\mathrm{diam}(M) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$, and the fundamental group $\pi_1(M)$ of M is finite.

Remark. When F is Riemannian, the main result is essentially reduced to Aubry's result [1]. It is also clear that $\epsilon = 0$ when $\underline{\mathbf{Ric}} \geqslant n-1$, and the main result is reduced to the classical Myers theorem.

2. Preliminaries

In this section we recall some basic notations and formulas that are needed to prove the main results, for details one is referred to see [6, 7, 9]. Let (M,F) be a Finsler n-manifold with Finsler metric $F : TM \to [0,\infty)$, and $(x,y) = (x^i, y^i)$ be local coordinates on TM. The *fundamental tensor* $\mathbf{g}_y$ on $T_xM\backslash\{0\}$ is defined by $\mathbf{g}_y(u,v) = g_{ij}(x,y)u^iv^j$ for any $u = u^i\frac{\partial}{\partial x^i}, v = v^i\frac{\partial}{\partial x^i}$, here $g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$. A *volume form* on Finsler manifold (M,F) is nothing but a global non-degenerate n-form on M. The frequently used volume forms in Finsler geometry are so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form. In [6] we introduce the maximal and minimal volume forms for Finsler manifolds which play the important role in comparison technique in Finsler geometry. They are defined as follows. Let

$$dV_{\max} = \sigma_{\max}(x)dx^1 \wedge \cdots \wedge dx^n$$

and

$$dV_{\min} = \sigma_{\min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{\max}(x) := \max_{y\in T_xM\backslash 0}\sqrt{\det(g_{ij}(x,y))}, \quad \sigma_{\min}(x) := \min_{y\in T_xM\backslash 0}\sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the n-forms $dV_{\max}$ and $dV_{\min}$ are well-defined on M, and they are called the *maximal volume form* and the *minimal volume form* of (M,F), respectively. Both maximal volume form and minimal volume form are called *extreme volume form*.

The *uniformity function* $\mu : M \to \mathbb{R}$ is defined by

$$\mu(x) = \max_{y,z,u\in T_xM\backslash 0}\frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

$\mu_F = \max_{x\in M} \mu(x)$ is called the *uniformity constant*. Similarly, the *reversibility function* $\lambda : M \to \mathbb{R}$ is defined by

$$\lambda(x) = \max_{y\in T_xM\setminus 0} \frac{F(y)}{F(-y)}.$$

$\lambda_F = \max_{x\in M} \lambda(x)$ is called the *reversibility* of (M, F). It is clear that $\lambda(x)^2 \leqslant \mu(x)$, and $d_F(x, y) \leqslant \lambda_F d_F(y, x)$ for $x, y \in M$, here d_F is the distance induced by F. The forward geodesic ball $B_x(R)$ with radius R centered at $x \in M$ is defined by $B_x(R) = \{y \in M : d_F(x, y) < R\}$. Let $T \subset M$ be a star-shaped subset at $x \in T$, that means that for all $y \in T$ there exists a minimal geodesic from x to y contained in T. Write $T(r) = T \cap B_x(r)$ for $r > 0$. We shall need the following relative volume comparison theorem with integral Ricci curvature bound which is the special case of Theorem 1.1 in [7] (see also [9]).

Theorem 2.1. *Let (M, F) be a forward complete Finsler n-manifold with finite uniformity constant $\mu_F := \Lambda^2$, and $T \subset B_x(R_T)$ be a star-shaped subset at x. For any $p > n/2$ there exists a constant $C(n, p, R_T) > 0$ such that when*

$$\epsilon := \left(\frac{\int_T(\max\{-\underline{\mathbf{Ric}}, 0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)}\right)^{\frac{1}{p}} < \left(\frac{1}{2C(n,p,R_T)}\right)^{\frac{2p-1}{p}},$$

then one has for all $0 < r \leqslant R \leqslant R_T$,

$$\frac{\mathrm{vol}_{\min}(T(R))}{R^n} \leqslant \left(\frac{1 - C(n,p,R_T)\epsilon^{\frac{p}{2p-1}}}{1 - 2C(n,p,R_T)\epsilon^{\frac{p}{2p-1}}}\right)^{2p-1} \cdot \Lambda^n \cdot \frac{\mathrm{vol}_{\min}(T(r))}{r^n}.$$

For given compact Finsler manifold (M, F), let $f : (\widetilde{M}, \widetilde{F}) \to (M, F)$ be the universal covering with pulled-back metric, then it is known that the fundamental group $\pi_1(M)$ is isomorphic to the deck transformation group Γ and each deck transformation is an isometry of $(\widetilde{M}, \widetilde{F})$ (see [5, 9] for details). The following lemma is crucial to prove the finiteness of $\pi_1(M)$.

Lemma 2.2 ([7, 9])**.** *Let $f : (\widetilde{M}, \widetilde{F}) \to (M, F)$ be the universal covering space of (M, F). Then for any forward geodesic ball $B_{\widetilde{x}}(r) \subset \widetilde{M}$ with $r > \mathrm{diam}(M)$ there exists a star-shaped subset T at $\widetilde{x}$ satisfying $B_{\widetilde{x}}(r) \subset T \subset B_{\widetilde{x}}((2 + \lambda_F)r)$ and*

$$\frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}}, 0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)} = \frac{\int_M(\max\{n-1-\underline{\mathbf{Ric}}, 0\})^p dV_{\max}}{\mathrm{vol}_{\min}(M)}.$$

Fix $x \in M$, let $S_x = \{v \in T_xM : F(v) = 1\}$ be the indicatrix at x. For $v \in S_x$, the *cut-value* $c(v)$ is defined by

$$c(v) := \sup\{t > 0 : d_F(x, \exp_x(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(x)$ of x by $\mathbf{C}(x) := \{c(v)v : c(v) < \infty, v \in S_x\}$ and the *cut locus* $C(x)$ of x by $C(x) = \exp_x \mathbf{C}(x)$, respectively. It is known that $C(x)$ has zero Hausdorff measure in M. Also, we set $\mathbf{D}_x = \{tv : 0 \leqslant t < c(v), v \in S_x\}$ and $D(x) = \exp_x \mathbf{D}_x$. It is known that $\mathbf{D}_x$ is the largest domain, which is star-shaped with respect to the origin of T_xM for which $\exp_x$ restricted to that domain is a diffeomorphism, and $D(x) = M\backslash C(x)$.

In the following we consider the polar coordinates on $D(x)$. For any $q \in D(x)$, the polar coordinates of q are defined by $(r,\theta)=(r(q),\theta^1(q),\dots,\theta^{n-1}(q))$, where $r(q) = F(v)$, $\theta^\alpha(q) = \theta^\alpha(u)$, here $v = \exp_x^{-1}(q)$ and $u = v/F(v)$. It is well-known that the unit radial coordinate vector $\partial r = d(\exp_x)\left(\frac{\partial}{\partial r}\right)$ is $\mathbf{g}_{\partial r}$-orthogonal to coordinate vectors ∂_α. Consider the singular Riemannian metric $\widetilde{g} = \mathbf{g}_{\partial r}$ on $D(x)$, we write the corresponding Riemannian volume form by $dV_{\widetilde{g}} = \widetilde{\sigma}(r,\theta)dr \wedge d\theta$. Then

$$dV_{\min} \leqslant dV_{\widetilde{g}} \leqslant dV_{\max} \leqslant \mu_F^{\frac{n}{2}} dV_{\min}. \tag{2.1}$$

Let $\mathbf{D}_x(r) \subset S_x$ be defined by $\mathbf{D}_x(r) = \{v \in S_x : rv \in \mathbf{D}_x\}$. It is easy to see that $\mathbf{D}_x(r_1) \subset \mathbf{D}_x(r_2)$ for $r_1 > r_2$. Since $C(x)$ has zero Hausdorff measure in M, the volume of $B_x(R)$ with respect to $dV_{\widetilde{g}}$ is given by

$$\begin{aligned} \mathrm{vol}_{\widetilde{g}}(B_x(R)) &= \int_{B_x(R)} dV_{\widetilde{g}} = \int_{B_x(R)\cap D(x)} dV_{\widetilde{g}} \\ &= \int_0^R dr \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(r,\theta)d\theta =: \int_0^R A(r)dr. \end{aligned} \tag{2.2}$$

Put

$$h = h(r,\theta) = \frac{\partial}{\partial r}\log\widetilde{\sigma}(r,\theta). \tag{2.3}$$

For $c > 0$ let

$$\sigma_c(r) = \left[\frac{\sin(\sqrt{c}r)}{\sqrt{c}}\right]^{n-1}, \quad h_c(r) = (\log\sigma_c)' = (n-1)\sqrt{c}\cot(\sqrt{c}r). \tag{2.4}$$

Write $\rho_c = \max\{(n-1)c - \underline{\mathbf{Ric}}, 0\}$, and define $\psi_c = \psi_c(t,\theta) = \max\{0, h(t,\theta) - h_c(t)\}$. The following lemma is the special case of Lemma 4.2 in [7] (see also Lemma 2.22 in [9]).

Lemma 2.3. *Let $c > 0$, $p > n/2$ and $0< r < \frac{\pi}{\sqrt{c}}$. Then*

$$\sin^{4p-n-1}(\sqrt{c}r)\psi_c^{2p-1}(r,\theta)\widetilde{\sigma}(r,\theta) \leqslant (2p-1)^p\left(\frac{n-1}{2p-n}\right)^{p-1}\int_0^r \rho_c^p(t,\theta)\widetilde{\sigma}(t,\theta)dt.$$

3. Auxiliary lemmas

In this section we shall derive some auxiliary lemmas that are needed to prove the main result. We always assume that (M,F) is a forward complete Finsler n-manifold with finite uniformity constant $\mu_F =: \Lambda^2$, and $p > n/2$.

Lemma 3.1. *Let $0 < \epsilon' \leqslant \frac{\pi}{4}$, and $A = A(s)$ be defined by (2.2). There exists a positive constant $C_1(n,p)$ such that for all radius $r \geqslant \pi$ the following inequality* *holds*:

$$
\begin{aligned}
A(r)^{\frac{1}{2p-1}} \leqslant & \left(\frac{12}{\pi}\right)^{\frac{1}{2p-1}} (\sqrt{2}\epsilon')^{\frac{n-1}{2p-1}} \left[\mathrm{vol}_{\widetilde{g}}(B_x(r))\right]^{\frac{1}{2p-1}} \\
& + C_1(n,p)\left[\int_{B_x(r)} \rho_1^p dV_{\widetilde{g}}\right]^{\frac{1}{2p-1}} \epsilon'^{\frac{n-2p}{2p-1}} r.
\end{aligned} \tag{3.1}
$$

Proof. For $c > 0$, and $0 < t < r < \pi/\sqrt{c}$, notice that $\mathbf{D}_x(r) \subset \mathbf{D}_x(t)$ for all $t < r$, the function

$$
f(s) = \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(s,\theta) d\theta
$$

is differentiable on $(0, r]$. By (2.3) and (2.4) we easily get

$$
\begin{aligned}
\frac{d}{ds}\left[\frac{f(s)}{\sigma_c(s)}\right]^\alpha &= \alpha\left[\frac{f(s)}{\sigma_c(s)}\right]^{\alpha-1} \int_{\mathbf{D}_x(r)} \frac{\widetilde{\sigma}(s,\theta)}{\sigma_c(s)}(h(s,\theta) - h_c(s)) d\theta \\
&\leqslant \alpha \frac{f(s)^{\alpha-1}}{\sigma_c(s)^\alpha} \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(s,\theta)\psi_c(s,\theta) d\theta,
\end{aligned}
$$

here $\alpha > 0$. Consequently,

$$
\begin{aligned}
\left[\frac{A(r)}{\sigma_c(r)}\right]^\alpha - \left[\frac{A(t)}{\sigma_c(t)}\right]^\alpha &= \left[\int_{\mathbf{D}_x(r)} \frac{\widetilde{\sigma}(r,\theta)}{\sigma_c(r)} d\theta\right]^\alpha - \left[\int_{\mathbf{D}_x(t)} \frac{\widetilde{\sigma}(t,\theta)}{\sigma_c(t)} d\theta\right]^\alpha \\
&\leqslant \left[\int_{\mathbf{D}_x(r)} \frac{\widetilde{\sigma}(r,\theta)}{\sigma_c(r)} d\theta\right]^\alpha - \left[\int_{\mathbf{D}_x(r)} \frac{\widetilde{\sigma}(t,\theta)}{\sigma_c(t)} d\theta\right]^\alpha \\
&= \left[\frac{f(r)}{\sigma_c(r)}\right]^\alpha - \left[\frac{f(t)}{\sigma_c(t)}\right]^\alpha \\
&= \int_t^r \frac{d}{ds}\left[\frac{f(s)}{\sigma_c(s)}\right]^\alpha ds \\
&\leqslant \alpha \int_t^r \frac{f(s)^{\alpha-1}}{\sigma_c(s)^\alpha} \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(s,\theta)\psi_c(s,\theta) d\theta ds.
\end{aligned} \tag{3.2}
$$

By the Hölder inequality we have

$$
\begin{aligned}
& \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(s,\theta)\psi_c(s,\theta) d\theta \\
\leqslant & \left[\int_{\mathbf{D}_x(r)} \psi_c^{2p-1}(s,\theta)\widetilde{\sigma}(s,\theta) d\theta\right]^{\frac{1}{2p-1}} \left[\int_{\mathbf{D}_x(r)} \widetilde{\sigma}(s,\theta) d\theta\right]^{\frac{2p-2}{2p-1}} \\
= & \left[\int_{\mathbf{D}_x(r)} \psi_c^{2p-1}(s,\theta)\widetilde{\sigma}(s,\theta) d\theta\right]^{\frac{1}{2p-1}} [f(s)]^{\frac{2p-2}{2p-1}}.
\end{aligned} \tag{3.3}
$$

Now (3.2) and (3.3) yield

$$\left[\frac{A(r)}{\sigma_c(r)}\right]^{\alpha}-\left[\frac{A(t)}{\sigma_c(t)}\right]^{\alpha} \tag{3.4}$$
$$\leqslant \alpha\int_t^r \frac{f(s)^{\alpha-\frac{1}{2p-1}}}{\sigma_c(s)^{\alpha}}\left[\int_{\mathbf{D}_x(r)}\psi_c^{2p-1}(s,\theta)\widetilde{\sigma}(s,\theta)d\theta\right]^{\frac{1}{2p-1}}ds.$$

Notice that $\sigma_c(r)=\left[\frac{\sin(\sqrt{c}r)}{\sqrt{c}}\right]^{n-1}$, and put $\alpha=\frac{1}{2p-1}$, from (3.4) and Lemma 2.3 we reach at

$$\left[\frac{A(r)}{\sin^{n-1}(\sqrt{c}r)}\right]^{\frac{1}{2p-1}}-\left[\frac{A(t)}{\sin^{n-1}(\sqrt{c}t)}\right]^{\frac{1}{2p-1}} \tag{3.5}$$
$$\leqslant \frac{1}{2p-1}\int_t^r\frac{1}{\sin^2(\sqrt{c}s)}\left[\int_{\mathbf{D}_x(r)}\sin^{4p-n-1}(\sqrt{c}s)\psi_c^{2p-1}(s,\theta)\widetilde{\sigma}(s,\theta)d\theta\right]^{\frac{1}{2p-1}}ds$$
$$\leqslant \left[\frac{n-1}{(2p-n)(2p-1)}\right]^{\frac{p-1}{2p-1}}\left[\int_{B_x(r)}\rho_c^p dV_{\widetilde{g}}\right]^{\frac{1}{2p-1}}\int_t^r\frac{ds}{\sin^2(\sqrt{c}s)}.$$

For given $0<\epsilon'\leqslant\frac{\pi}{4}$, let $c=\left(\frac{\pi-\epsilon'}{r}\right)^2<1$, $t\in[\frac{2}{3}r,\frac{3}{4}r]$, then $\rho_c\leqslant\rho_1$, $\sqrt{c}t\in[\frac{\pi}{2},\frac{3}{4}\pi]$, and

$$\sin(\sqrt{c}r)=\sin(\pi-\epsilon')<\epsilon',\quad \sin(\sqrt{c}t)\geqslant\sin\frac{3}{4}\pi=\frac{1}{\sqrt{2}},$$

$$\int_t^r\frac{ds}{\sin^2(\sqrt{c}s)}=\frac{1}{\sqrt{c}}[\cot(\sqrt{c}t)-\cot(\sqrt{c}r)]<-\frac{\cot(\pi-\epsilon')}{\sqrt{c}}<\frac{1}{\sqrt{c}\epsilon'}\leqslant\frac{4r}{3\pi\epsilon'}.$$

Thus from (3.5) it follows that

$$A(r)^{\frac{1}{2p-1}}\leqslant(\sqrt{2}\epsilon')^{\frac{n-1}{2p-1}}A(t)^{\frac{1}{2p-1}}+C_1(n,p)\left[\int_{B_x(r)}\rho_1^p dV_{\widetilde{g}}\right]^{\frac{1}{2p-1}}\epsilon'^{\frac{n-2p}{2p-1}}r, \tag{3.6}$$

here

$$C_1(n,p)=\frac{4}{3\pi}\left[\frac{n-1}{(2p-n)(2p-1)}\right]^{\frac{p-1}{2p-1}}. \tag{3.7}$$

Finally, by the mean value property we may choose $t\in[\frac{2}{3}r,\frac{3}{4}r]$ so that

$$A(t)=\frac{12}{r}\int_{\frac{2}{3}r}^{\frac{3}{4}r}A(s)ds\leqslant\frac{12}{\pi}\int_0^r A(s)ds=\frac{12}{\pi}\mathrm{vol}_{\widetilde{g}}(B_x(r)). \tag{3.8}$$

Substituting (3.8) into (3.6) we easily get (3.1). □

Lemma 3.2. *Let*

$$C_2(n,p) := \left[\left(\frac{12}{\pi}\right)^{\frac{1}{2p-1}} (\sqrt{2})^{\frac{n-1}{2p-1}} + C_1(n,p)(\pi+2)\right]^{2p-1}, \tag{3.9}$$

$$B(n,p,\Lambda) := \Lambda^3 R \left[2^{2p} C_2(n,p)\right]^{\frac{1}{n}}, \tag{3.10}$$

(3.11)

$$A_1(n,p,\Lambda) := \min\left\{\left(\frac{\pi}{4}\right)^{\frac{2p-1}{p}}, \left(\frac{1}{3C(n,p,R)}\right)^{\frac{2p-1}{p}}, \left(\frac{0.9}{B(n,p,\Lambda)}\right)^{\frac{n(2p-1)}{p(n-1)}}\right\},$$

here

$$R = R(\Lambda) = (\Lambda+1)(\pi+1) + \Lambda^{-1}. \tag{3.12}$$

If

$$\epsilon = \left(\frac{\int_{B_y(R)} (\max\{n-1-\underline{\mathbf{Ric}}, 0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_y(R))}\right)^{\frac{1}{p}} \leqslant A_1(n,p,\Lambda) \tag{3.13}$$

holds for some $y \in M$*, then* M *is compact with* $\mathrm{diam}(M) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$.

Proof. By (3.11) it is clear that $B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}} \leqslant 0.9$ if (3.13) holds. Now we claim that if (3.13) holds, then $d_F(x,y) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$ holds for any $x \in M$. Otherwise, there exists $x \in M$ such that $d_F(x,y) = \pi + \delta$ with $B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}} < \delta \leqslant 1$. In this situation, it is easy to check that

$$B_y(\Lambda^{-1}\delta) \subset B_x(\pi+\delta+\Lambda^{-1}\delta) \backslash B_x(\pi) \subset B_y(R), \tag{3.14}$$

here $R = R(\Lambda)$ is defined by (3.12). By (3.11) $A_1(n,p,\Lambda) \leqslant (\frac{\pi}{4})^{\frac{2p-1}{p}}$, thus $\epsilon' := \epsilon^{\frac{p}{2p-1}} \leqslant \frac{\pi}{4}$. Substituting $\epsilon' = \epsilon^{\frac{p}{2p-1}}$ into (3.1), and notice (2.1) and (3.14) we have, for any $\pi \leqslant r \leqslant \pi + \delta + \Lambda^{-1}\delta (\leqslant \pi+2)$,

$$\begin{aligned} A(r)^{\frac{1}{2p-1}} \leqslant\ & \left(\frac{12}{\pi}\right)^{\frac{1}{2p-1}} (\sqrt{2}\epsilon')^{\frac{n-1}{2p-1}} [\mathrm{vol}_{\widetilde{g}}(B_y(R))]^{\frac{1}{2p-1}} \\ & + C_1(n,p)(\pi+2)\left[\frac{1}{\mathrm{vol}_{\widetilde{g}}(B_y(R))}\int_{B_y(R)} \rho_1^p dV_{\widetilde{g}}\right]^{\frac{1}{2p-1}} \\ & [\mathrm{vol}_{\widetilde{g}}(B_y(R))]^{\frac{1}{2p-1}} \epsilon'^{\frac{n-2p}{2p-1}} \\ \leqslant\ & \left[\left(\frac{12}{\pi}\right)^{\frac{1}{2p-1}} (\sqrt{2})^{\frac{n-1}{2p-1}} + C_1(n,p)(\pi+2)\right] \\ & [\mathrm{vol}_{\widetilde{g}}(B_y(R))]^{\frac{1}{2p-1}} \epsilon^{\frac{p(n-1)}{(2p-1)^2}}, \end{aligned}$$

which together with (3.9) yields

$$A(r) \leqslant C_2(n,p)\mathrm{vol}_{\widetilde{g}}(B_y(R))\epsilon^{\frac{p(n-1)}{2p-1}}, \quad \pi \leqslant r \leqslant \pi+\delta+\Lambda^{-1}\delta. \tag{3.15}$$

By (3.14) and (3.15) we have

$$\begin{aligned}\mathrm{vol}_{\widetilde{g}}(B_y(\Lambda^{-1}\delta)) &\leqslant \mathrm{vol}_{\widetilde{g}}(B_x(\pi+\delta+\Lambda^{-1}\delta)\backslash B_x(\pi))\\ &= \int_{\pi}^{\pi+\delta+\Lambda^{-1}\delta} A(r)dr\\ &\leqslant 2C_2(n,p)\mathrm{vol}_{\widetilde{g}}(B_y(R))\epsilon^{\frac{p(n-1)}{2p-1}},\end{aligned}$$

which together with (2.1) yields

$$\frac{\mathrm{vol}_{\min}(B_y(\Lambda^{-1}\delta))}{\mathrm{vol}_{\min}(B_y(R))} \leqslant \Lambda^n \frac{\mathrm{vol}_{\widetilde{g}}(B_y(\Lambda^{-1}\delta))}{\mathrm{vol}_{\widetilde{g}}(B_y(R))} \leqslant 2\Lambda^n C_2(n,p)\epsilon^{\frac{p(n-1)}{2p-1}}. \tag{3.16}$$

On the other hand, (3.11) and (3.13) implies that

$$\epsilon := \left(\frac{\int_{B_y(R)}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_y(R))}\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{3C(n,p,R)}\right)^{\frac{2p-1}{p}},$$

thus by Theorem 2.1 we have

$$\begin{aligned}\frac{\mathrm{vol}_{\min}(B_y(\Lambda^{-1}\delta))}{\mathrm{vol}_{\min}(B_y(R))} &\geqslant \frac{\Lambda^{-n}\delta^n}{R^n}\cdot\Lambda^{-n}\cdot\left(\frac{1-2C(n,p,R)\epsilon^{\frac{p}{2p-1}}}{1-C(n,p,R)\epsilon^{\frac{p}{2p-1}}}\right)^{2p-1}\\ &\geqslant \frac{\delta^n}{R^n\Lambda^{2n}2^{2p-1}},\end{aligned}$$

which together with (3.10) and (3.16) yields

$$\delta \leqslant \Lambda^3\left[2^{2p}C_2(n,p)\right]^{\frac{1}{n}} R\epsilon^{\frac{p(n-1)}{n(2p-1)}} = B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}},$$

which contradicts with the assumption $\delta > B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$. In summary, we have proved our claim that

$$d_F(x,y) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}} \leqslant \pi+0.9 < \pi+1$$

for any $x \in M$ whenever (3.13) holds. In this situation, we also have $d_F(y,x) < \Lambda(\pi+1)$, and thus $d_F(x,z) \leqslant d_F(x,y)+d_F(y,z) < (1+\Lambda)(\pi+1)$ for any $x,z \in M$. In other words, if (3.13) holds, then $M \subset \overline{B_x((1+\Lambda)(\pi+1))} \subset B_x((1+\Lambda)(\pi+1)+\Lambda^{-1}) = B_x(R)$ for any $x \in M$. Now by the same argument as above we have $d_F(x,y) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$ for any $x,y \in M$, which imply that $\mathrm{diam}(M) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$. □

Lemma 3.3. *Let $T \subset M$ be a star-shaped subset at x such that $B_x(R) \subset T \subset B_x(R_T)$. If*

$$\left(\frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)}\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{3C(n,p,R_T)}\right)^{\frac{2p-1}{p}},$$

then

$$\left(\frac{\int_{B_x(R)}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_x(R))}\right)^{\frac{1}{p}}$$

$$\leqslant \left(\frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)}\right)^{\frac{1}{p}} \cdot 2^{\frac{2p-1}{p}} \left(\frac{R_T\Lambda}{R}\right)^{\frac{n}{p}}. \tag{3.17}$$

Proof. By Theorem 2.1 it is clear that

$$\begin{aligned}
&\frac{\int_{B_x(R)}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_x(R))}\\
&\leqslant \frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_x(R))}\\
&= \frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)} \cdot \frac{\mathrm{vol}_{\min}(T)}{\mathrm{vol}_{\min}(B_x(R))}\\
&\leqslant \frac{\int_T(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)} \cdot 2^{2p-1} \cdot \Lambda^n \cdot \frac{R_T^n}{R^n},
\end{aligned}$$

which clearly implies (3.17). □

4. The proof of Main Result

In this section we shall complete the proof of the main result of this paper. Let us first prove the following.

Theorem 4.1. *Let (M,F) be a forward complete Finsler n-manifold with finite uniformity constant $\mu_F := \Lambda^2$. If $\int_M(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}$ is finite for some $p > n/2$, then the minimal volume $\mathrm{vol}_{\min}(M)$ of M is finite. In this situation, there exists a positive constant $A_2(n,p,\Lambda)$ such that when*

$$\epsilon = \left(\frac{\int_M(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(M)}\right)^{\frac{1}{p}} \leqslant A_2(n,p,\Lambda), \tag{4.1}$$

then M is compact with $\mathrm{diam}(M) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$, here $B(n,p,\Lambda)$ is given in Lemma 3.2.

Proof. Let $R = R(\Lambda) = (\Lambda+1)(\pi+1)+\Lambda^{-1}$ be given as in Lemma 3.2, and let $\{B_{x_i}(R)\}_{i\in I}$ be a maximal family of disjoint geodesic balls in M. It is not difficult to verify that the Dirichlet domains $T_i = \{y \in M : d_F(x_i,y) < d_F(x_j,y), \forall j \neq i\}$ satisfy the following facts:

(1) $B_{x_i}((1+\Lambda)R) \supset T_i \supset B_{x_i}(R)$;

(2) T_i is star-shaped at x_i;

(3) except for a set of zero measure, M is the disjoint union of the sets T_i.

Let

$$A_2(n,p,\Lambda) = \min\left\{A_1(n,p,\Lambda)\cdot 2^{-\frac{2p-1}{p}}\cdot(\Lambda(1+\Lambda))^{-\frac{n}{p}}, \left(\frac{1}{3C(n,p,(1+\Lambda)R)}\right)^{\frac{2p-1}{p}}\right\},$$

here $A_1(n,p,\Lambda)$ is given by Lemma 3.2. Setting

$$\alpha = \inf_{i\in I}\left(\frac{\int_{T_i}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T_i)}\right)^{\frac{1}{p}},$$

then we have

$$\begin{aligned}
&\int_M (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}\\
&= \sum_i \int_{T_i}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}\\
&\geqslant \alpha^p \sum_{i\in I}\mathrm{vol}_{\min}(T_i) = \alpha^p \mathrm{vol}_{\min}(M). \qquad (4.2)
\end{aligned}$$

If $\alpha > A_2(n,p,\Lambda)$, then by (4.2) it is clear that $\mathrm{vol}_{\min}(M)$ is finite. Elsewhere, there exists a star-shaped set T_i such that

$$\left(\frac{\int_{T_i}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T_i)}\right)^{\frac{1}{p}} \leqslant A_2(n,p,\Lambda), \tag{4.3}$$

which together with (3.17) and the property (1) of T_i yields

$$\begin{aligned}
&\left(\frac{\int_{B_{x_i}(R)}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_{x_i}(R))}\right)^{\frac{1}{p}}\\
&\leqslant A_2(n,p,\Lambda)\cdot 2^{\frac{2p-1}{p}}\cdot(\Lambda(1+\Lambda))^{\frac{n}{p}} \leqslant A_1(n,p,\Lambda).
\end{aligned}$$

Now by Lemma 3.2 we conclude that M is compact, and thus $\mathrm{vol}_{\min}(M)$ is finite. On the other hand, if (4.1) holds, then by (4.2) it follows that $\alpha \leqslant A_2(n,p,\Lambda)$, and we may argue similarly as above to conclude that M is compact with $\mathrm{diam}(M) \leqslant \pi + B(n,p,\Lambda)\epsilon^{\frac{p(n-1)}{n(2p-1)}}$. □

Theorem 4.2. *Let (M,F) be a compact Finsler n-manifold with uniformity constant $\mu_F := \Lambda^2$. For any $p > n/2$ there exists a positive constant $A(n,p,\Lambda)$ such that when*

$$\left(\frac{\int_{M}(\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(M)}\right)^{\frac{1}{p}} \leqslant A(n,p,\Lambda), \tag{4.4}$$

then the fundamental group $\pi_1(M)$ of M is finite.

Proof. Let

$$A(n,p,\Lambda) = \min\left\{A_1(n,p,\Lambda)\cdot 2^{-\frac{2p-1}{p}}\cdot(\Lambda(2+\Lambda))^{-\frac{n}{p}}, \left(\frac{1}{3C(n,p,(2+\Lambda)R)}\right)^{\frac{2p-1}{p}}\right\},$$

here $A_1(n,p,\Lambda)$ and R are given by Lemma 3.2. Notice that $A(n,p,\Lambda) < A_2(n,p,\Lambda)$, Theorem 4.1 and (4.4) implies that $\mathrm{diam}(M) < \pi + 1 < R$. Let $f : (\widetilde{M},\widetilde{F}) \to (M,F)$ be the universal covering space of (M,F). In order to prove the finiteness of $\pi_1(M)$, we need only to prove $\widetilde{M}$ is compact. We note that the uniformity constant of $\widetilde{M}$ is just the uniformity constant of M since f is a local isometry. Fix a base point $\widetilde{x} \in \widetilde{M}$. Since $R >$diam(M), by Lemma 2.2 there is a star-shaped subset T at $\widetilde{x}$ satisfying $B_{\widetilde{x}}(R) \subset T \subset B_{\widetilde{x}}((2+\Lambda)R)$ and

$$\frac{\int_T (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(T)} = \frac{\int_M (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(M)}. \tag{4.5}$$

By (4.4), (4.5) and Lemma 3.3 we see that

$$\begin{aligned} &\left(\frac{\int_{B_{\widetilde{x}}(R)} (\max\{n-1-\underline{\mathbf{Ric}},0\})^p dV_{\max}}{\mathrm{vol}_{\min}(B_{\widetilde{x}}(R))}\right)^{\frac{1}{p}} \\ &\leqslant A(n,p,\Lambda)\cdot 2^{\frac{2p-1}{p}}\cdot(\Lambda(2+\Lambda))^{\frac{n}{p}} \leqslant A_1(n,p,\Lambda). \end{aligned}$$

Now by Lemma 3.2 it is clear that $\widetilde{M}$ is compact, and thus the theorem is proved. □

Proof of Main Result. It is the direct consequence of Theorems 4.1 and 4.2. □

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Bing-Ye Wu
Institution of Mathematics
Minjiang University
Fuzhou 350108
Fujian, P. R. China
Email address: wubingye@mju.edu.cn