Bull. Korean Math. Soc. **56** (2019), No. 4, pp. 829–839
https://doi.org/10.4134/BKMS.b180635
pISSN: 1015-8634 / eISSN: 2234-3016

WEAKLY PRIME IDEALS IN COMMUTATIVE SEMIGROUPS

D. D. Anderson, Sangmin Chun, and Jason R. Juett

Abstract. Let S be a commutative semigroup with 0 and 1. A proper ideal P of S is *weakly prime* if for $a, b \in S$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. We investigate weakly prime ideals and related ideals of S. We also relate weakly prime principal ideals to unique factorization in commutative semigroups.

The purpose of this note is to investigate weakly prime ideals and related ideals in commutative semigroups. We will also discuss the relationship between weakly prime principal ideals and unique factorization. Here all commutative semigroups will be nontrivial, multiplicative, and have both 0 and 1. Similarly, all rings will be commutative with identity. Let S be a commutative semigroup. Recall that an *ideal* of S is a nonempty subset I of S with $si \in I$ for $s \in S$ and $i \in I$. Hence 0 is an ideal (we use 0 to denote both the element 0 and the ideal $\{0\}$) and the set of nonunits of S forms the unique maximal ideal of S. A union or intersection of a nonempty family of ideals is again an ideal. A proper ideal P of S is *prime* if $ab \in P$, $a, b, \in S$, implies $a \in P$ or $b \in P$. This is equivalent to $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, where A and B are ideals of S. (When we write AB we mean the ideal $\{ab \mid a \in A, b \in B\}$.) An arbitrary union of prime ideals is again a prime ideal. For ideals A and B of S, we define $(A : B)$ to be the ideal $\{x \in S \mid xB \subseteq A\}$ and for $x \in S$ we abbreviate $(A : x) = (A : (x))$. For a survey of the ideal theory of commutative semigroups the reader is referred to [1].

Let S be a commutative semigroup. A proper ideal P of S is *weakly prime* if $0 \neq ab \in P$, $a, b \in S$, implies $a \in P$ or $b \in P$. Thus 0 and prime ideals are weakly prime. This generalizes the notion of a weakly prime ideal in a commutative ring which was studied in [2]. We are particularly interested in which properties of weakly prime ideals in commutative rings extend to weakly prime ideals in commutative semigroups.

We first state some results from [2] about weakly prime ideals in commutative rings.

Received July 1, 2018; Revised September 17, 2018; Accepted October 11, 2018.
2010 *Mathematics Subject Classification.* 20M12, 20M14.
Key words and phrases. weakly prime ideal, prime ideal, commutative semigroup.

Theorem 1 ([2, Theorem 3]). *For a proper ideal P of a commutative ring R, the following statements are equivalent.*

(1) *P is weakly prime, that is, $0 \neq xy \in P$, $x, y \in R$, implies $x \in P$ or $y \in P$.*
(2) *For $x \in R \setminus P$, $(P : x) = P \cup (0 : x)$.*
(3) *For $x \in R \setminus P$, $(P : x) = P$ or $(P : x) = (0 : x)$.*
(4) *For ideals A and B of R with $0 \neq AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$.*

We note that in Theorem 1(4), like all other times that we will refer to a product of ring ideals, it does not make a difference whether we interpret the product as defined above or as the usual product of ring ideals.

Theorem 2. *Let R be a commutative ring and P a weakly prime ideal of R.*

(1) ([2, Theorem 1, Corollary 2, Theorem 4]) *If P is not prime, then $P^2 = 0$, so $P \subseteq \sqrt{0}$. In fact, $P\sqrt{0} = 0$. So either $P \subseteq \sqrt{0}$ or $\sqrt{0} \subseteq P$. If $P \subsetneq \sqrt{0}$, P is not prime while if $\sqrt{0} \subsetneq P$, P is prime.*
(2) ([2, Theorem 7]) *If R is not indecomposable, then either $P = 0$ or P is prime.*

With the four conditions in Theorem 1 in mind we make the following definition.

Definition 3. Let S be a commutative semigroup and P a proper ideal of S. Then P satisfies

(WP1) if P is weakly prime, that is, $0 \neq xy \in P$, $x, y \in S$, implies $x \in P$ or $y \in P$,
(WP2) if for $x \in S \setminus P$, $(P : x) = P \cup (0 : x)$,
(WP3) if for $x \in S \setminus P$, $(P : x) = P$ or $(P : x) = (0 : x)$, and
(WP4) if for ideals A and B of S with $0 \neq AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$.

The following two results illustrate the relationships between the above properties.

Proposition 4. *Let S be a commutative semigroup and P a proper ideal of S.*

(1) *P satisfies* (WP3) *if and only if for each $x \in S$ and* (2*-generated*) *ideal B, $0 \neq (x)B \subseteq P \Rightarrow (x) \subseteq P$ or $B \subseteq P$.*
(2) *P satisfies* (WP4) *if and only if for each $x, y \in S$ and* (2*-generated*) *ideal B, $0 \neq (x, y)B \subseteq P \Rightarrow (x, y) \subseteq P$ or $B \subseteq P$.*

Proof. (1) ($\Rightarrow$) Suppose $0 \neq (x)B \subseteq P$ and $(x) \nsubseteq P$. Then $(P : x) \neq (0 : x)$, hence $P = (P : x) \supseteq B$. ($\Leftarrow$) Suppose $x \in S \setminus P$ and $(P : x) \neq (0 : x)$. Then $0 \neq xy \in P$ for some $y \in S$. For each $z \in (P : x)$, $0 \neq (x)(y, z) \subseteq P$, hence $(y, z) \subseteq P$. Therefore $(P : x) = P$.

(2) ($\Rightarrow$) Clear. ($\Leftarrow$) Suppose $0 \neq AB \subseteq P$ and $A \nsubseteq P$. Say $x \in A \setminus P$. Pick $y \in A$ and $z \in B$ with $yz \neq 0$. For each $b \in B$, $0 \neq (x, y)(b, z) \subseteq P$, hence $(b, z) \subseteq P$. Thus $B \subseteq P$. $\square$

Note that in Proposition 4, we can equivalently replace (x), (x,y), and B with $\{x\}$, $\{x,y\}$, and a subset of S (with cardinality at most 2), respectively.

Theorem 5. *Let S be a commutative semigroup. Then* (WP4) $\Rightarrow$ (WP3) $\Rightarrow$ (WP2) $\Leftrightarrow$ (WP1).

Proof. (WP4) $\Rightarrow$ (WP3) $\Rightarrow$ (WP1) Follows from Proposition 4. (WP1) $\Rightarrow$ (WP2) Let $x \in S \setminus P$. Suppose $y \in (P:x)$, so $xy \in P$. Then either $0 \neq xy \in P$ in which case $y \in P$ since P is weakly prime, or $0 = xy$ in which case $y \in (0:x)$. So $(P:x) \subseteq P \cup (0:x)$ and the other inclusion always holds. (WP2) $\Rightarrow$ (WP1) Suppose that $0 \neq xy \in P$. If $x \notin P$, then $(P:x) = P \cup (0:x)$ so $0 \neq xy$ forces $y \in (P:x) \setminus (0:x)$ and hence $y \in P$. $\square$

Example 6 ((WP3) $\not\Rightarrow$ (WP4) or contained in $\sqrt{0}$)**.** Let E be the commutative semigroup with generators x and y and relations $xy = y^2 = 0$. For $1 \leq i \leq n$, let E_n (resp., $E_{n,i}$) be the commutative semigroup with generators x and y and relations $xy = y^2 = 0$ and $x^{n+1} = 0$ (resp., $x^{n+1} = x^i$). (Equivalently, we can define E and E_n as Rees quotients $E = F/(xy, y^2)$ and $E_n = F/(xy, y^2, x^{n+1})$, where F is the free commutative semigroup with generators x and y. We will further discuss Rees quotients below.) Then $P = (x)$ is weakly prime but not prime. We have $P^2 \neq 0$ in E, $E_{n,i}$, and E_n $(n \geq 2)$. Now $P \subsetneq (x,y) = M$, the maximal ideal of S, but $M^2 \subseteq P$ with $M^2 \neq 0$ for $E, E_{n,i}$, and E_n $(n \geq 2)$. Clearly $(P:1) = P$ and $(P:y) = (x,y) = (0:y)$. So P of $E, E_{n,i}$, and E_n $(n \geq 2)$ satisfies (WP3) but not (WP4) and $P^2 \neq 0$. In fact, P of E is not even contained in $\sqrt{0}$.

Example 7 ((WP2) $\not\Rightarrow$ (WP3))**.** Let $P = (x,y)$ in the semigroup $S = \{0, 1, x, y, z, xz, yz\}$ where $x^2 = y^2 = xy = z^2 = 0$. So P is weakly prime. Now for $z \in S \setminus P$, $(P:z) = (x,y,z) \neq P$ and $(0:z) = (z) \neq P$. So P satisfies (WP2) but not (WP3).

We note that (WP2) and (WP3) are equivalent for ring ideals for a rather trivial reason: if a union $I \cup J$ of two ring ideals is a ring ideal, then $I \cup J = I$ or J. However, any nonempty union of semigroup ideals is a semigroup ideal.

Theorem 8. *Let S be a commutative semigroup and P a proper ideal of S. Then the following are equivalent.*

(1) *P satisfies* (WP4).

(2) *P satisfies* (WP3) *and $P^2 = 0$ if P is not prime.*

Proof. (1) $\Rightarrow$ (2) By Theorem 5, (WP4) $\Rightarrow$ (WP3). Suppose that $P^2 \neq 0$, so there are $p, q \in P$ with $pq \neq 0$. Let $a, b \in S$ with $ab \in P$. Put $A = (a, p)$ and $B = (b, q)$; so A and B are ideals of S with $0 \neq (pq) \subseteq AB \subseteq P$. Hence $a \in A \subseteq P$ or $b \in B \subseteq P$, so P is prime. (2) $\Rightarrow$ (1) Suppose that A and B are ideals of S with $0 \neq AB \subseteq P$. If P is prime, $A \subseteq P$ or $B \subseteq P$. So suppose that P is not prime. Then $P^2 = 0$. Now $0 \neq AB$ gives $a \in A$ and $b \in B$ with $0 \neq ab$. Since $P^2 = 0$ we must have at least one of a or b not in P, say $b \notin P$.

Then $(P:b) = P$ or $(P:b) = (0:b)$. But $a \in (P:b) \setminus (0:b)$, so $(P:b) = P$. Thus $Ab \subseteq AB \subseteq P$ gives $A \subseteq (P:b) = P$. $\square$

However, note that Example 7 shows that (WP2) $+P^2 = 0 \not\Rightarrow$ (WP3).

Theorem 9. *Let S_1 and S_2 be commutative semigroups. The nonzero* (*weakly*) *prime ideals of $S_1 \times S_2$ are $P_1 \times S_2$, $S_1 \times P_2$, and $(P_1 \times S_2) \cup (S_1 \times P_2)$ where P_1 and P_2 are prime ideals of S_1 and S_2, respectively. Hence a nonzero weakly prime ideal of a decomposable commutative semigroup is prime, so* (WP1)-(WP4) *are equivalent in a decomposable commutative semigroup.*

Proof. Let P be a nonzero weakly prime ideal of $S_1 \times S_2$. Let $(0,0) \neq (a,b) \in P$, so $(a,1)(1,b) = (a,b) \in P$. Then $(a,1) \in P$ or $(1,b) \in P$ and hence $(a) \times S_2 \subseteq P$ or $S_1 \times (b) \subseteq P$. So $P = \left(\bigcup_{(a,1)\in P}(a) \times S_2\right) \cup \left(\bigcup_{(1,b)\in P} S_1 \times (b)\right) = (P_1 \times S_2) \cup (S_1 \times P_2)$ where $P_1 = \{a \in S_1 \mid (a,1) \in P\}$ and $P_2 = \{b \in S_2 \mid (1,b) \in P\}$. Now P_1 and P_2 are either empty or ideals of S_1 and S_2, respectively. Suppose $P_1 \neq \emptyset$. Let $xy \in P_1$, $x, y \in S_1$. Then $(0,0) \neq (x,1)(y,1) = (xy,1) \in P \Rightarrow (x,1) \in P$ or $(y,1) \in P \Rightarrow x \in P_1$ or $y \in P_1$. So P_1 is a prime ideal of S_1. Likewise P_2 is empty or a prime ideal of S_2. But it is easily checked that if P_1 (resp., P_2) is a prime ideal of S_1 (resp., S_2), then $P_1 \times S_2$ (resp., $S_1 \times P_2$) is a prime ideal of $S_1 \times S_2$. Since a union of prime ideals is prime, the result follows. $\square$

Theorem 10. *Let S be a commutative semigroup and P a proper ideal of S satisfying* (WP4). *If P is not prime, then $P\sqrt{0} = 0$. Thus if P and Q are non-prime ideals of S satisfying* (WP4), *then $PQ = 0$.*

Proof. Let $0 \neq x \in \sqrt{0}$. If $x \in P$, $xP \subseteq P^2 = 0$ by Theorem 8. So suppose $x \notin P$. Since (WP4) $\Rightarrow$ (WP3), $(P:x) = P$ or $(P:x) = (0:x)$. As $P \subseteq (P:x)$, the second case gives $xP = 0$. So suppose that $(P:x) = P$. Let $x^n = 0$, but $x^{n-1} \neq 0$. Then $0 \neq x^{n-1} \in (P:x) = P$, so $x \in P$, a contradiction. Thus $P\sqrt{0} = 0$. $\square$

Example 11 (P satisfies (WP3) and not prime $\not\Rightarrow P\sqrt{0} = 0$)**.** Let $n \geq 2$ and E_n be as in Example 6. We saw that $P = (x)$ satisfies (WP3). But $\sqrt{0} = (x,y) \supsetneq P$, so P is not prime and $P\sqrt{0} = (x)(x,y) = (x^2) \neq 0$. In fact, $P^2 \neq 0$.

Proposition 12. *Let S be a commutative semigroup and $\{P_\alpha\}_{\alpha\in\Lambda}$ be a non-empty family of ideals of S satisfying* (WPn). *Then $P = \bigcup_{\alpha\in\Lambda} P_\alpha$ satisfies* (WPn). *If every P_α is a non-prime ideal satisfying* (WP4), *then $P^2 = 0$.*

Proof. Let $P = \bigcup_{\alpha\in\Lambda} P_\alpha$. The (WP1) case is clear. Now assume each P_α satisfies (WP3). By Proposition 4, we need to show that $0 \neq (x)(y,z) \subseteq P \Rightarrow (x) \subseteq P$ or $(y,z) \subseteq P$. Suppose $(y,z) \nsubseteq P$, say $y \notin P$. Then $xz \in P_\alpha$ for some α. If $xy = 0$, then $0 \neq (x)(y,z) = (xz) \subseteq P_\alpha$, hence $(x) \subseteq P_\alpha \subseteq P$ since P_α satisfies (WP3). But if $xy \neq 0$, then $0 \neq (x)(y) \subseteq P$ implies $(x) \subseteq P$ since P is weakly prime. Now assume each P_α satisfies (WP4). By Theorem 8, each non-prime $P_\alpha \subseteq \sqrt{0}$. So, if at least one P_α is prime, then P is the union of the prime P_α's, hence prime. Now suppose no P_α is prime. Then each $P_\alpha P_\beta = 0$ by Theorem 10. So P satisfies (WP3) and $P^2 = 0$, hence P satisfies (WP4) by Theorem 8. $\square$

Corollary 13. *Let S be a commutative semigroup with $\sqrt{0}$ not a square-zero prime ideal. Then S has a unique largest non-prime ideal P that satisfies* (WP4).

Proof. Let P be the union of the non-prime ideals of S that satisfy (WP4). It suffices to show that P satisfies (WP4) but is not prime. Since 0 is not prime, $P \neq \emptyset$, hence by Proposition 12, P satisfies (WP4) and $P^2 = 0$. So $P \subseteq \sqrt{0}$ and the inclusion is proper if $\sqrt{0}$ is prime. Therefore P is not prime. $\square$

Example 14. (1) (For $1 \leq n \leq 3$, there is no result analogous to Corollary 13 for (WPn).) Let S be the commutative semigroup with generators x, y, z and relations $y^2 = y$, $z^2 = z$, and $x^3 = xy = xz = yz = 0$, so $S = \{0, 1, x, x^2, y, z\}$ and $\sqrt{0} = (x)$ is not prime or square zero. One can easily check that (x), (y), and (z) are non-prime ideals satisfying (WP3). So for $1 \leq n \leq 3$, the union of the non-prime ideals satisfying (WPn) is the maximal ideal (x, y, z), hence there is no largest non-prime ideal satisfying (WPn).

(2) (A commutative semigroup with $\sqrt{0}$ a square-zero prime ideal may or may not have a largest non-prime ideal satisfying (WP4).) Let $S_1 = \{0, 1, x, y\}$ and $S_2 = \{0, 1, x\}$, where $x^2 = y^2 = xy = 0$. In each S_i, $\sqrt{0}$ is a square-zero prime (in fact maximal) ideal. The non-prime ideals of S_1 satisfying (WP4) are 0, (x), and (y), so S_1 has no largest such ideal. However, the largest (in fact only) non-prime ideal of S_2 satisfying (WP4) is 0.

Proposition 15. *Let S be a commutative semigroup and $\{P_\alpha\}_{\alpha \in \Lambda}$ be a non-empty family of ideals of S satisfying* (WP4)*, at least one of which is not prime. Then $\bigcap_{\alpha \in \Lambda} P_\alpha$ satisfies* (WP4).

Proof. By Theorem 8, each non-prime $P_\alpha \subseteq \sqrt{0}$, so we may assume every P_α is non-prime. Suppose that $P = \bigcap_{\alpha \in \Lambda} P_\alpha$ fails (WP4). Then $0 \neq AB \subseteq P$ for some ideals A and B of S not contained in P. So there are $\alpha, \beta \in \Lambda$ with $A \not\subseteq P_\alpha$ and $B \not\subseteq P_\beta$, hence $B \subseteq P_\alpha$ and $A \subseteq P_\beta$ by (WP4). But then $AB \subseteq P_\beta P_\alpha = 0$ by Theorem 10, a contradiction. $\square$

Example 16 (An intersection of two non-prime ideals satisfying (WP3) need not satisfy (WP1))**.** Let S be the commutative semigroup with generators x, y, z, w and relations $x^2 = x$, $y^2 = y$, $xz = z$, $yw = w$, and $xw = yz = z^2 = w^2 = zw = 0$, so $S = \{0, 1, x, y, z, w, xy\}$. The proper ideal $P = (x, z) = \{0, x, xy, z\}$ is not prime since $w^2 \in P$ but $w \notin P$. But P satisfies (WP3) since $(P : 1) = (P : y) = P$ and $(P : w) = \{0, x, z, w, xy\} = (0 : w)$. Similarly, $Q = (y, w) = \{0, y, xy, w\}$ is a non-prime ideal satisfying (WP3). But $P \cap Q = \{0, xy\}$ fails (WP1).

Theorem 17 (c.f. [4, Proposition 3])**.** *Let S be a commutative semigroup. Then*

(1) *Every proper ideal of S satisfies* (WP4) *if and only if for all ideals A and B of S, $AB = 0$, A, or B.*
(2) *Every proper* (*principal*) *ideal of S satisfies* (WP1) *if and only if for principal ideals A and B of S, $AB = 0$, A, or B.*

Proof. (1) ($\Rightarrow$) Let A and B be ideals of S. If $A = B = S$, $AB = A = B$. So suppose that at least one of A and B is proper, so AB is proper. If $AB \neq 0$, then $0 \neq AB \subseteq AB \Rightarrow A \subseteq AB$ or $B \subseteq AB$ since AB satisfies (WP4), so $AB = A$ or $AB = B$. ($\Leftarrow$) Let P be a proper ideal of S and suppose A and B are ideals with $0 \neq AB \subseteq P$. Then $AB = A$ or $AB = B$ so $A \subseteq P$ or $B \subseteq P$.

(2) ($\Rightarrow$) Suppose every proper principal ideal of S satisfies (WP1). Suppose $(a)(b) \neq 0$. Then $0 \neq ab \in (ab) \Rightarrow a \in (ab)$ or $b \in (ab)$ since (ab) is weakly prime, so $(a) = (ab)$ or $(b) = (ab)$. ($\Leftarrow$) Let P be a proper ideal of S. Let $0 \neq ab \in P$. Then $(ab) = (a)$ or $(ab) = (b)$ so $a \in (ab) \subseteq P$ or $b \in (ab) \subseteq P$. $\square$

There is a much stronger version of Theorem 17 for commutative rings: every proper (principal) ideal of a commutative ring R is weakly prime if and only if (R, M) is quasilocal with $M^2 = 0$ or R is a direct product of two fields. (The statement without the "principal" is [2, Theorem 8]. But Theorem 17(2) shows that every proper ideal is weakly prime if every proper principal ideal is.) The paper [4] investigates rings not necessarily commutative or with an identity where every proper ideal is weakly prime.

We now take a moment to consider two concepts related to weakly prime ideals, namely weakly multiplicatively closed subsets and weakly radical ideals. We give some examples to illustrate how many of the familiar facts about multiplicatively closed subsets and radical ideals do not carry over to their "weak" counterparts, even in the commutative ring case. (All of our previous examples have been of phenomena that could appear in commutative semigroups but not commutative rings.)

Let S be a commutative semigroup. Call $T \subseteq S$ *weakly multiplicatively closed* if $s, t \in T \Rightarrow st \in T \cup \{0\}$, or equivalently $T \cup \{0\}$ is multiplicatively closed. Note that a proper ideal P of S is (weakly) prime if and only if $S \setminus P$ is (weakly) multiplicatively closed. The analog of Krull's Lemma does not hold for weakly multiplicatively closed sets (in either commutative semigroups or commutative rings). For let $R = \mathbb{Q}[X]/(X^4)$. Since R is a principal ideal ring, its ring and semigroup ideals coincide. The ideal $(\overline{X}^3)$ is maximal with respect to disjointness from the weakly multiplicatively closed subset $\{\overline{X}^2\}$, but $(\overline{X}^3)$ is not weakly prime.

An ideal I of a commutative semigroup is (*weakly*) *radical* if $(0 \neq)\ x^n \in I \Rightarrow x \in I$. It is easily checked that I is radical if and only if $x^2 \in I \Rightarrow x \in I$. However, let $R = \mathbb{Q}[X]/(X^4)$. Then $\overline{0} \neq \overline{f}^2 \in (\overline{X}^3) \Rightarrow \overline{f} \in (\overline{X}^3)$, but $(\overline{X}^3)$ is not weakly radical. Clearly an ideal I is radical if and only if $A^n \subseteq I$, A

an ideal of S, implies $A \subseteq I$. However, strictly stronger than being weakly radical is the condition $(*)$ $0 \neq A^n \subseteq I$, A an ideal of S, implies $A \subseteq I$. For let $R = \mathbb{Q}[X,Y,Z]/J$, where $J = (X^3, X^2Y, XY^2, X^2Z, XZ^2, Y^3, Y^2Z, YZ^2, Z^3)$. Then $0 \neq (\overline{X}, \overline{Y}, \overline{Z})^3 = (\overline{XYZ}) \subsetneq (\overline{X}, \overline{Y}, \overline{Z})$, so $(\overline{XYZ})$ fails $(*)$. However, it is weakly radical. In fact, if $(\overline{f}, \overline{g})^2 \subseteq (\overline{XYZ})$, then $(\overline{f}, \overline{g})^2 = 0$. Indeed, $(\overline{f}, \overline{g})^2 \subseteq (\overline{XYZ}) \Leftrightarrow (f,g)^2 \subseteq (X,Y,Z)^3 \Leftrightarrow (f,g) \subseteq (X,Y,Z)^2 \Rightarrow (f,g)^2 \subseteq J \Leftrightarrow (\overline{f}, \overline{g})^2 = 0$. This example also shows that, unlike (WP4), it is not enough to check $(*)$ for 2-generated ideals. Finally, it is well known that an ideal is radical if and only if it is an intersection of prime ideals. But a weakly radical ideal need not be an intersection of weakly prime ideals. For let $R = \mathbb{Q}[X]/(X^2) \times \mathbb{Q}[X]/(X^2)$. Then $I = (\overline{X}) \times 0$ is weakly radical. By Theorem 9, every nonzero weakly prime (semigroup) ideal of R is actually prime, so I, being a nonzero non-radical ideal, is not weakly prime.

We conclude with some notes on the connection between weakly prime elements and unique factorization. We make the following definition by analogy with [3], which studies factorization in commutative rings.

Definition 18. Let S be a commutative semigroup.

(1) S is *présimplifiable* if for $x, y \in S$, $x = xy \neq 0 \Rightarrow y$ is a unit.
(2) S is *restricted cancellative* or an *r-semigroup* if for $x, y, z \in S$, $xy = xz \neq 0 \Rightarrow (y) = (z)$.
(3) A nonunit $s \in S$ is (*weakly*) *prime* if (s) is (weakly) prime.
(4) A nonunit $s \in S$ is *irreducible* if $s = ab$, $a, b \in S$, implies $(s) = (a)$ or (b).
(5) S is *atomic* if every (nonzero) nonunit is a product of irreducible elements.
(6) S is a *unique factorization semigroup* (UFS) if
   (a) S is atomic and
   (b) if $0 \neq x_1 \cdots x_n = y_1 \cdots y_m$ where x_i, y_j are irreducible, then $n = m$ and each $(x_i) = (y_i)$ after a suitable reordering.

For nonzero elements, prime $\Rightarrow$ weakly prime $\Rightarrow$ irreducible, but none of the implications reverse. On the other hand, the zero element is always weakly prime, but is irreducible if and only if it is prime (c.f. [3, Theorem 2.13]). Using uniqueness, one routinely shows that UFS $\Rightarrow$ restricted cancellative $\Rightarrow$ présimplifiable. It is also easy to see that in a présimplifiable commutative semigroup (i) $(x) = (y) \Leftrightarrow x = \lambda y$ for some unit λ and (ii) a nonunit s is irreducible $\Leftrightarrow$ it is not a product of two nonzero nonunits $\Leftrightarrow s = 0$ is prime or (s) is maximal among the proper principal ideals of S.

Before we can state our results, we need to review the following definitions. Let S be a commutative semigroup. We say $x, y \in S$ are *associates*, written $x \sim y$, if $(x) = (y)$. Then $\sim$ is a congruence on S and $S/\sim$ is isomorphic to the semigroup $P(S)$ of principal ideals of S. Note that $P(S)$ is *reduced* in the sense that it has only one unit. Let I be a proper ideal of S. Then the *Rees quotient* of

S by I is the semigroup $S/I = S\setminus(I\setminus\{0\})$ with product $ab = \begin{cases} ab, & \text{if } ab \notin I \\ 0, & \text{if } ab \in I. \end{cases}$
Note that if P is a weakly prime ideal of S, then $P/I = P\setminus(I\setminus\{0\})$ is a weakly prime ideal of S/I. However, even if P is prime (hence satisfies (WP4)), it need not be the case that P/I satisfies (WP3) – see Examples 6 or 7. But if $I \subseteq P$, then (i) P is prime $\Leftrightarrow P/I$ is prime and (ii) P satisfies (WPn) $\Rightarrow P/I$ satisfies (WPn). (These facts are easily checked from the definitions or Proposition 4.) Of course, the converse in (ii) fails: take $P = I$ to be any non-weakly prime proper ideal. For a less trivial example, if S is the free commutative semigroup on x, y, z, then (x, yz) fails (WP1) but $(x, yz)/(xy)$ satisfies (WP4).

Lemma 19. *Let S be a commutative semigroup.*

(1) *S is présimplifiable (resp., restricted cancellative, a UFS) if and only if $P(S)$ is.*
(2) *If S is présimplifiable (resp., restricted cancellative, atomic, a UFS), then so is S/I.*
(3) *Let S be a UFS and X a complete set of non-associate irreducible elements of S. Then $P(S)$ is isomorphic to a Rees quotient of the free commutative semigroup on $|X|$ generators.*

Proof. (1) The présimplifiable and restricted cancellative cases follow routinely from the definitions. For the UFS case, we may assume S and $P(S)$ are présimplifiable, so their irreducible elements are the nonunits that are not products of two nonzero nonunits. Using this characterization, we see that $s \in S$ is irreducible if and only if (s) is irreducible in $P(S)$. If $s = a_1 \cdots a_n$ is a product of irreducible elements, then so is $(s) = (a_1)\cdots(a_n)$. Conversely, if $(s) = (a_1)\cdots(a_n)$ is a product of irreducible elements, then $s = (\lambda a_1)a_2 \cdots a_n$ for some unit λ (by présimplifiability) and $\lambda a_1, a_2, \ldots, a_n$ are irreducible. From the previous two sentences it easily follows that S is a UFS if and only if $P(S)$ is.

(2) The présimplifiable and restrictive cancellative cases are clear. The atomic and UFS cases easily follow from the observation that $x \in S/I$ is irreducible simultaneously as an element of S or S/I.

(3) The proof of (1) shows that each nonzero nonunit in $P(S)$ has a unique representation (up to order) of the form $(x_1)\cdots(x_n)$ with $x_i \in X$. Thus $P(S) \cong F/I$, where F is the free commutative semigroup on X and $I = \{x_1 \cdots x_n \mid x_i \in X \text{ and } x_1 \cdots x_n = 0 \text{ in } S\}$. □

Theorem 20. *For a commutative semigroup S the following are equivalent.*

(1) *S is a UFS.*
(2) *$P(S)$ is a UFS.*
(3) *$P(S)$ is a Rees quotient of a free commutative semigroup.*
(4) *S is restricted cancellative and every (nonzero) nonunit is a product of weakly prime elements.*

(5) *S is a restricted cancellative atomic semigroup in which every irreducible element is weakly prime.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Lemma 19. (1) $\Rightarrow$ (5) Let S be a UFS. We have already observed that S is restricted cancellative. Now let $s \in S$ be a nonzero irreducible element and suppose $0 \neq xy \in (s)$, say $xy = st$. If x or y is a unit, then $(s) \supseteq (xy) = (x)$ or (y). If t is a unit, then $x(yt^{-1}) = s$ and irreducibility implies $(s) = (x)$ or $(s) = (yt^{-1}) = (y)$. So let us assume x, y, and t are nonunits. Write $x = x_1 \cdots x_m$, $y = y_1 \cdots y_n$, and $t = t_1 \cdots t_k$ with each factor on the right-hand side irreducible. Applying uniqueness to $x_1 \cdots x_m y_1 \cdots y_n = st_1 \cdots t_k$ shows that (s) equals some (x_i) or (y_j), hence $x \in (s)$ or $y \in (s)$. (5) $\Rightarrow$ (4) Clear. (4) $\Rightarrow$ (1) Since weakly prime elements are irreducible, S is atomic. Now write $0 \neq x_1 \cdots x_m = y_1 \cdots y_n$ with each x_i and y_j irreducible. Each x_i or y_j is an associate of a weakly prime element by irreducibility, hence weakly prime. Since $y_1 \cdots y_n \in (x_m)$ and (x_m) is weakly prime, we can reorder so that $(x_m) \supseteq (y_n)$, hence by présimplifiability $y_n = \lambda x_m$ for some unit λ. Restricted cancellation then gives $0 \neq x_1 \cdots x_{m-1} = (\lambda y_1) y_2 \cdots y_{n-1}$. By induction, $m - 1 = n - 1$ (hence $m = n$) and we can reorder so that $(x_1) = (\lambda y_1) = (y_1)$, $(x_2) = (y_2)$, $\ldots$, and $(x_{m-1}) = (y_{m-1})$. $\square$

Example 21. Let E, E_n, and $E_{n,i}$ be as in Example 6.

(1) Let F be the free commutative semigroup with generators x and y. Then $E = F/(xy, y^2)$ and $E_n = F/(xy, y^2, x^{n+1})$, so E and E_n are UFS's.
(2) The semigroup $E_{n,i}$ is not présimplifiable, let alone a UFS. But $x, y \in E_{n,i}$ are weakly prime, so every nonunit of $E_{n,i}$ is a product of weakly prime elements.

We note that Theorem 20 can be sharpened for commutative rings. From [2, Theorem 14] it follows that every (nonzero) nonunit element of a commutative ring R is a product of weakly prime elements if and only if (R, M) is quasilocal with $M^2 = 0$ or R is a finite direct product of unique factorization domains and Artinian local principal ideal rings. So a commutative ring R is a *unique factorization ring* (meaning $(R, \cdot)$ is a UFS) if and only if it has no nontrivial idempotents and every (nonzero) nonunit is a product of weakly prime elements.

Example 22 (A présimplifiable commutative semigroup in which every nonunit is a product of prime elements need not be a UFS)**.** Let S be the commutative semigroup with generators x and y and relation $x^2y = xy^2$. It can be checked that x and y are non-associate primes and S is présimplifiable (hence has no nontrivial idempotents). So every nonunit is a product of prime elements but S is not a UFS.

Note however that a présimplifiable commutative semigroup has every proper (principal) ideal weakly prime if and only if it has a square-zero maximal ideal. So such a semigroup is a UFS.

Example 23 (A commutative semigroup in which every proper ideal is prime need not be a UFS)**.** Let S be a nontrivial chain with greatest element 1 and least element 0. Then $(S, \wedge)$ is a commutative semigroup. Note that every proper ideal is prime, so every nonunit is prime, hence irreducible. But S is présimplifiable (resp., a UFS) $\Leftrightarrow S = \{0, 1\}$.

We note one final connection between weakly prime ideals and factorization. Let S be a commutative semigroup and P be a proper ideal of S. If every nonzero element of P is irreducible, then P is weakly prime (c.f. [2, Theorem 9(2)]). But P need not satisfy (WP3). Consider the commutative semigroup with generators x, y, z and relations $x^2 = y^2 = z^2 = xz = yz = 0$ and $xy = y$. Then $(y) = \{0, y\}$ is weakly prime and y is irreducible, but from $0 \neq (x)(y, z) = (y)$ we see that (y) fails (WP3). On the other hand, if S has a unique prime ideal M and P satisfies (WP4), then $P = M$ or every nonzero element of P is irreducible (c.f. [2, Theorem 9(3)]). However, the commutative semigroup E_2 from Example 6 has a unique prime ideal $M = (x, y)$, but $P = (x) = \{0, x, x^2\}$ satisfies (WP3), x^2 is not irreducible, and $P \neq M$.

Acknowledgment. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant no.NRF-2018R1D1A1B07046733).

References

[1] D. D. Anderson and E. W. Johnson, *Ideal theory in commutative semigroups*, Semigroup Forum **30** (1984), no. 2, 127–158. `https://doi.org/10.1007/BF02573445`

[2] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (2003), no. 4, 831–840.

[3] D. D. Anderson and S. Valdes-Leon, *Factorization in commutative rings with zero divisors*, Rocky Mountain J. Math. **26** (1996), no. 2, 439–480. `https://doi.org/10.1216/rmjm/1181072068`

[4] Y. Hirano, E. Poon, and H. Tsutsui, *On rings in which every ideal is weakly prime*, Bull. Korean Math. Soc. **47** (2010), no. 5, 1077–1087. `https://doi.org/10.4134/BKMS.2010.47.5.1077`

D. D. Anderson
Department of Mathematics
The University of Iowa
Iowa City, IA 52242, USA
Email address: `dan-anderson@uiowa.edu`

Sangmin Chun
Da Vinci College of General Education
Chung-Ang University
Seoul 06974, Korea
Email address: `schun@cau.ac.kr`

Jason R. Juett
Department of Mathematics
Texas State University
San Marcos, TX 78666, USA
Email address: jason.juett@txstate.edu