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ON THE DENSITY OF VARIOUS SHADOWING PROPERTIES

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Abstract. In this paper we deal with some shadowing properties of discrete dynamical systems on a compact metric space via the density of subdynamical systems. Let $f : X \to X$ be a continuous map of a compact metric space X and A be an f-invariant dense subspace of X. We show that if $f|_A : A \to A$ has the periodic shadowing property, then f has the periodic shadowing property. Also, we show that f has the finite average shadowing property if and only if $f|_A$ has the finite average shadowing property.

1. Introduction

Let X be a compact metric space with a metric d and $f : X \to X$ be a continuous map (or homeomorphism). As usual, we identify a continuous map (or homeomorphism) f with a discrete semidynamical system (or discrete dynamical system) generated by f on X.

The theory of shadowing of pseudo orbits in dynamical systems is now an important and rapidly developing branch of the modern global theory of dynamical systems. The notion of a pseudo orbit goes back to G. Birkhoff [5]. The real development of the shadowing theory started after the classical results of D. V. Anosov [1] and R. Bowen [9]. The main results concerning various shadowing properties obtained in recent years were reflected in the monographs [19–21]. It is known that every Axiom A diffeomorphism to the nonwandering set has the shadowing property (see [9]). P. Kościelniak and M. Mazur introduced the notion of the periodic shadowing property for homeomorphisms on a compact metric space and showed that the periodic shadowing property is C^0 generic in the space of discrete dynamical systems on a closed smooth manifold M (see [14, Theorem 1.1] and [13]).

L. Fernández and C. Good [11] showed that for a continuous map $f : X \to X$ of a compact metric space X and an f-invariant dense subspace Y of X, f has the finite shadowing property if and only if $f|_Y : Y \to Y$ has the finite

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shadowing property. Also, they studied shadowing and h-shadowing for the induced map on a hyperspace by using the density of some subspaces of a hyperspace.

A. Darabi and A.-M. Forouzanfar [10] showed that expansive dynamical systems with the shadowing property have the periodic shadowing property. Also, they showed that transitive dynamical systems with the periodic shadowing property have the shadowing property. C. Pugh [22] showed that C^1-generically, $\overline{\mathrm{Per}(f)} = \Omega(f)$, and C. Bonatti and S. Crovisier [8] showed that C^1-generically, $\overline{\mathrm{Per}(f)} = \mathcal{CR}(f)$. Furthermore, M. Lee [15] showed that if a diffeomorphism f has the periodic shadowing property on the set $\mathcal{CR}(f)$, then the closure of the periodic set is the chain recurrent set.

M. Lee [16] showed that for an f-invariant dense subset Λ of a compact metric space X, a continuous map $f : X \to X$ has the eventual shadowing property in Λ if and only if f has the eventual shadowing property introduced by C. Good and J. Meddaugh [12].

M. L. Blank introduced the notion of the average shadowing property as a generalization of the shadowing property for studying properties of orbits of perturbed hyperbolic dynamical systems (see [6,7]). It is easy to see that f has the shadowing if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every finite δ-pseudo orbit is ε-shadowed by the finite orbit of some point (see [4, Remark 3.1] or [2]). A. V. Osipov et al. [18] studied the relations between periodic shadowing property, Lipschitz periodic shadowing property and Ω-stability.

P. Oprocha and X. Wu introduced the notion of average tracing of finite pseudo orbits and showed that a discrete semidynamical system (X, f) has the finite average shadowing property if and only if $(\mathrm{supp}(X, f), f)$ has the finite average shadowing property [17, Theorem 3.1]. Here $\mathrm{supp}(X, f)$ is the *measure center* of f.

In this paper we show that a continuous map f has the finite average (periodic) shadowing property if and only if $f|_A : A \to A$ (i.e., its restriction to an f-invariant dense subspace A of X) has the finite average(periodic) shadowing property, where A is an f-invariant dense subspace of X. Also, we give some examples to illustrate our results.

2. Preliminaries

The set of natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We recall some basic notions of dynamical systems which are used in the sequel.

A point z is said to be a *periodic point* if $f^k(z) = z$ for some $k \in \mathbb{N}$. The period of z is the smallest nonnegative integer of $\{k \in \mathbb{N} \mid f^k(z) = z\}$. Denote by $\mathrm{Per}(f)$ the set of all periodic points of f. A point x is said to be a *nonwandering point* if for any neighborhood U of x there is a positive integer $m \in \mathbb{N}$ such that $U \cap f^m(U) \neq \emptyset$. The set $\Omega(f)$ of all nonwandering points is called the *nonwandering set.* A sequence $\{x_i\}_{i\in\mathbb{N}_0}$ of points in X is said to be a *δ-pseudo orbit* if $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \in \mathbb{N}_0$. A point x is said to be *chain recurrent* if for any $\delta > 0$ there is a periodic δ-pseudo orbit which passes through x. The set $\mathcal{CR}(f)$ of all chain recurrent points is called the *chain recurrent set.* A subset $A \subset X$ is said to be f*-invariant* for a continuous map f if $f(A) = A$.

We say that a continuous map $f : X \to X$ has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ there is $z \in X$ such that $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$.

We say that a continuous map $f : X \to X$ has the *periodic shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that if $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ is a periodic δ-pseudo orbit with period N ($x_{i+N} = x_i, \forall i \in \mathbb{N}_0$), then there is $z \in \mathrm{Per}(f)$ such that $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$.

Let $\delta > 0$. We say that a sequence $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ in X is a δ*-average pseudo orbit* of f if there exists a positive integer N such that for all $n \geq N$ and $k \in \mathbb{N}_0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

A δ-average pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ is *periodic* if it is formed by a periodic sequence, i.e., $x_i = x_{i+s}$ for some positive integer s and all $i \in \mathbb{N}_0$.

Definition 2.1 ([17])**.** A continuous map $f : X \to X$ has the *average shadowing of periodic* (*or finite*) *pseudo orbit property* (abbrev. FinASP) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every periodic δ-average pseudo orbit $\{x_i\}_{i\in\mathbb{N}_0}$ is ε-shadowed on average by a point z in X, i.e.,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), z_i) < \epsilon, \ \forall n \geq N.$$

Proposition 2.2 ([17, Proposition 2.1])**.** *Let $\delta > 0$. If a finite sequence $\{w_i\}_{i=0}^{s-1}$ satisfies*

$$\frac{1}{s-1}\sum_{i=0}^{s-2} d(f(w_i), w_{i+1}) < \tfrac{\delta}{2}$$

and $\frac{\mathrm{diam}(X)}{s} < \frac{\delta}{2}$, then the periodic sequence given by $x_{js+i} = w_i$ for all $j \in \mathbb{N}_0$ and $0 \leq i < s$ is a δ-average pseudo orbit.

We say that a finite δ-pseudo orbit $\{x_i\}_{i=0}^{k}$ of f is a δ*-chain* from x_0 to x_k with length k. A non-empty subset A of X is said to be *chain transitive* whenever for any $x, y \in A$ and any $\delta > 0$ there exists a δ-chain from x to y. A continuous map $f : X \to X$ is said to be *chain transitive* if X is a chain transitive set.

3. Main results

In this section we show that if $f|_A : A \to A$ has the periodic shadowing property on a dense subspace of X, then $f : X \to X$ has the periodic shadowing property. Also, we show that f has the finite average shadowing property if and only if $f|_A$ has the finite average shadowing property.

Theorem 3.1. *Let A be an f-invariant dense subspace of a compact metric space (X,d). If a continuous map $f|_A$ has the periodic shadowing property, then a continuous map $f : X \to X$ has the periodic shadowing property.*

Proof. Suppose that $f|_A : A \to A$ has the periodic shadowing property. Let $\varepsilon > 0$ and $\delta = \delta(\frac{\varepsilon}{2}) > 0$ be given by the periodic shadowing property of $f|_A$. Let $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ be a periodic $\frac{\delta}{3}$-pseudo orbit with period N (i.e., $x_{i+N} = x_i$ for all $i \in \mathbb{N}_0$) in X and $\frac{\delta}{3} < \frac{\varepsilon}{2}$. Since f is continuous and X is compact, there is $r > 0$ with $r < \frac{\delta}{3}$ such that $d(f(x), f(y)) < \frac{\delta}{3}$ for any $x, y \in X$ with $d(x, y) < r$, $0 \le i < N$. Since A is dense in X, we have $B(x_i, r) \cap A \ne \emptyset$ for each $0 \le i < N$. Let $y_i \in B(x_i, r) \cap A$. Since f is uniformly continuous, we have $d(f(x_i), f(y_i)) < \frac{\delta}{3}$, $0 \le i < N$. Now we define a periodic sequence $\eta = \{y'_i\}_{i\in\mathbb{N}_0}$ given by $y'_{jN+i} = y_i$ for all $j \in \mathbb{N}_0$ and $0 \le i < N$. Then we have

$$\begin{aligned} d(f(y'_i), y'_{i+1}) &\le d(f(y'_i), f(x_i)) + d(f(x_i), x_{i+1}) + d(x_{i+1}, y'_{i+1}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \quad i \in \mathbb{N}_0. \end{aligned}$$

Thus $\eta = \{y'_i\}_{i\in\mathbb{N}_0}$ is a periodic δ-pseudo orbit with period N in A. Since $f|_A$ has the periodic shadowing property, there is a periodic point $z \in A$ such that $d(f^i(z), y'_i) < \varepsilon/2$ for all $i \in \mathbb{N}_0$, Then we have

$$\begin{aligned} d(f^i(z), x_i) &< d(f^i(z), y'_i) + d(y'_i, x_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad i \in \mathbb{N}_0. \end{aligned}$$

Thus the periodic point z of A ε-shadowes ξ. Hence f has the periodic shadowing property. □

Theorem 3.2. *Let A be an f-invariant subspace of a compact metric space (X,d) and containing* $\mathrm{Per}(f)$. *If a continuous map $f : X \to X$ has the periodic shadowing property, then $f|_A$ has the periodic shadowing property.*

Proof. Suppose that f has the periodic shadowing property. Let $\varepsilon > 0$ and choose $\delta > 0$ such that every periodic δ-pseudo orbit in X is ε-shadowed. Let $\xi = \{x_i\}_{i\in\mathbb{N}_0} \subset A$ be a periodic δ-pseudo orbit in A. Then it is clear that ξ is a periodic δ-pseudo orbit in X. Since f has the periodic shadowing property, there is a periodic point $z \in A \subset X$ such that $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$. This means that $f|_A$ has the periodic shadowing property. □

Remark 3.3. Let $X = \Sigma_2^+$ and the shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$ be defined as $\sigma(x_0x_1x_2\cdots) = (x_1x_2x_3\cdots)$ (see §4. Examples). Then $A = \{\sigma^n(x) \,|\, n \in \mathbb{N}_0\}$ is a σ-invariant dense subset of Σ_2^+, where x is not a periodic point for σ. But A does not contain any periodic point for σ.

In general, Theorem 3.1 is not true because there exists an f-invariant dense subspace A of X such that A does not containing any periodic point for f.

Theorem 3.4. *Let A be an f-invariant dense subspace of a compact metric space (X, d). Then a continuous map $f : X \to X$ has the FinASP if and only if $f|_A$ has the FinASP.*

Proof. ($\Longleftarrow$) Suppose that $f|_A : A \to A$ has the FinASP. Let $\varepsilon > 0$ and $\delta(< \frac{\varepsilon}{2}) > 0$ be given by the FinASP of $f|_A$. Let $N \in \mathbb{N}$ with $\frac{\mathrm{diam}(X)}{N} = \frac{\delta}{2}$ and $\xi = \{x_i\}_{i \in \mathbb{N}_0}$ be a periodic $\frac{\delta}{6}(< \frac{\varepsilon}{2})$-average pseudo orbit with period N (i.e., $x_{i+N} = x_i$ for all $i \in \mathbb{N}$) in X. As in the proof of Theorem 3.1, we can find $\eta' = \{y_0, y_1, \ldots, y_{N-1}\}$. Then η' is a finite $\frac{\delta}{2}$-pseudo orbit in A. Define a periodic sequence $\eta = \{y'_i\}_{i \in \mathbb{N}_0}$ with period N. By Proposition 2.2, $\eta = \{y'_i\}_{i \in \mathbb{N}_0}$ is a periodic δ-average pseudo orbit with period N ($y'_{jN+i} = y_i$). Since $f|_A$ has the FinASP, there is a point $z \in A$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), {y'}_i) < \frac{\varepsilon}{2}, \ n \geq N.$$

Then we have

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) &\leq \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), {y'}_i) + \frac{1}{n} \sum_{i=0}^{n-1} d(x_i, {y'}_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ n \geq N. \end{aligned}$$

Thus the point z ε-shadows ξ, and so, f has the FinASP.

($\Longrightarrow$) Suppose that f has the FinASP. Let $\varepsilon > 0$ with $\varepsilon > 2r_{N-1} > 0$ and choose $\delta > 0$ with $\delta < r_0$ and $B(x, \delta) \cap A \neq \emptyset$ for each $x \in X$ such that every periodic δ-pseudo orbit in X is $\frac{\varepsilon}{2}$-shadowed. Let $\xi = \{x_i\}_{i \in \mathbb{N}_0} \subset A$ be a periodic δ-average pseudo orbit with period N ($x_i = x_{i+N}$, $i \in \mathbb{N}_0$) in A. Then it is clear that ξ is a periodic δ-average pseudo orbit in X. Since f has the FinASP, there is a point $z \in X$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \frac{\varepsilon}{2}, \ n \geq N.$$

Since A is dense in X and chosen δ, there is $z_1 \in B(z, \delta) \cap A$ such that $d(z, z_1) < \delta$. Since f is continuous, there is $r_{N-1} > 0$ with $r_{N-1} < \frac{\varepsilon}{2}$ and $f(B(f^{N-1}(z), r_{N-1})) \subseteq B(f^N(z), \frac{\varepsilon}{2})$. Also, there is $r_{N-2} > 0$ with $r_{N-2} < r_{N-1}$ and $f(B(f^{N-2}(z), r_{N-2})) \subseteq f(B(f^{N-1}(z), r_{N-1}))$. Continuing this process, we arrive at $r_1 > 0$ with $r_1 < r_2$ and $f(B(f(z), r_1)) \subseteq B(f^2(z), r_2)$. Finally, there is $r_0 > 0$ with $r_0 < r_1$ and $f(B(z, r_0)) \subseteq B(f(z), r_1)$. Thus $d(z, z_1) < \delta \Rightarrow d(f^i(z), f^i(z_1)) < \frac{\varepsilon}{2}$ for all $0 \leq i < N$. Then we have

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z_1), x_i) &\leq \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z_1), f^i(z)) + \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ n \geq N. \end{aligned}$$

Thus the point z_1 of A ε-shadowes ξ, and so, $f|_A$ has the FinASP. $\square$

Theorem 3.5. *Suppose that a continuous map $f : X \to X$ has the periodic shadowing property on a compact metric space (X, d). Then $\overline{\mathrm{Per}(f)} = \mathcal{CR}(f)$.*

Proof. The proof is similar to the proof of [15, Theorem 1.4]. $\square$

In case $f : X \to X$ is a homeomorphism (diffeomorphism) of a compact metric space (a closed C^∞ manifold), we obtain the following results.

Remark 3.6. Suppose that a map $f : X \to X$ is chain transitive.

(i) If a homeomorphism f has the periodic shadowing property, then it follows from Theorem 2.5 [10] that f has the shadowing property. Also, from [2, Theorem 3.1.2], we have $\Omega(f) = \mathcal{CR}(f)$.

(ii) If a diffeomorphism f has the periodic shadowing property on $\mathcal{CR}(f)$, then it follows from [15, Theorem 1.4] that we have $\overline{\mathrm{Per}(f)} = \mathcal{CR}(f)$.

From condition (i) in Remark 3.6, we have $\overline{\mathrm{Per}(f)} = \Omega(f) = \mathcal{CR}(f)$.

4. Examples

In this section M. Baloush et al. [3] classify subshifts of the full-2-shift Σ_2. Thus, we give some examples to illustrate the notions of some shadowing properties.

Full-2-shift Σ_2 is the collection of all infinite sequences of symbols 0 and 1. Therefore the elements of Σ_2 is in a form of $x = x_0x_1x_2\cdots$ where $x_i \in \{0, 1\}$ for every $i \in \mathbb{N}_0$. The space is a metric space which equipped with metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-i}, & \text{if } x \neq y, \end{cases}$$

where i is the smallest integer such that $x_i \neq y_i$ for every pair $x, y \in \Sigma_2$. Therefore Σ_2 is a topological space induced by the metric d and the basic open ball is a any subset of the full-2-shift Σ_2 of the form

$$X_w = \{x \in \Sigma_2 \,|\, x = x_0x_1x_2\cdots = w_0w_1w_2\cdots w_{n-1} = w\}$$

for any block w of length n. We now define a continuous shift map $\sigma : \Sigma_2 \to \Sigma_2$ on the full-2-shift by $\sigma(x_0x_1x_2\cdots) = x_1x_2x_3\cdots$. A shift space $Y \subset \Sigma_2$ is called a *shift of finite type* (SFT) if there exists a finite number of blocks from symbols 0 and 1 such that the blocks do not occur in any element of Y. The blocks are called *forbidden blocks* in Y. Since we only have four possible different blocks of length two, i.e., 00,01,10 and 11, we have 16 set of forbidden blocks, as follows:

$\mathcal{F}_1 = \emptyset$	$\mathcal{F}_2 = \{00\}$	$\mathcal{F}_3 = \{01\}$	$\mathcal{F}_4 = \{10\}$
$\mathcal{F}_5 = \{11\}$	$\mathcal{F}_6 = \{00, 01\}$	$\mathcal{F}_7 = \{00, 10\}$	$\mathcal{F}_8 = \{00, 11\}$
$\mathcal{F}_9 = \{01, 10\}$	$\mathcal{F}_{10} = \{01, 11\}$	$\mathcal{F}_{11} = \{10, 11\}$	$\mathcal{F}_{12} = \{00, 01, 10\}$
$\mathcal{F}_{13} = \{00, 01, 11\}$	$\mathcal{F}_{14} = \{00, 10, 11\}$	$\mathcal{F}_{15} = \{01, 10, 11\}$	$\mathcal{F}_{16} = \{00, 01, 10, 11\}$

For each $i \in \{1, 2, \ldots, 16\}, Y_i \subset \Sigma_2$ is the SFT with set of forbidden blocks $\mathcal{F}_i$. Then we can conclude that there are only six different dynamics of shift spaces, Y_2, Y_3, Y_6, Y_7, Y_8 and Y_9.

Example 4.1. $(Y_2, \sigma|_{Y_2}), (Y_8, \sigma|_{Y_8})$ and $(Y_9, \sigma|_{Y_9})$ have the PerSP.

Proof. First, we show that $(Y_2, \sigma|_{Y_2})$ has the PerSP. Fix $\varepsilon > 0$ take $N \geq 1$ such that $\frac{1}{2^N} < \varepsilon$. Put $\delta = \frac{1}{2^{N+1}}$. Suppose $\{x^{(n)} : n \in \mathbb{N}_0\} \subset Y_2$ is a periodic sequence with the property

$$d(\sigma(x^{(n)}), x^{(n+1)}) < \delta \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Since the periodic points of Y_2 are dense in Y_2 and $\mathrm{Per}(\sigma|_{Y_2}) \cap B(x^{(0)}, \delta) \neq \emptyset$, there is a periodic point $z \in Y_2$ with $z \in \mathrm{Per}(\sigma|_{Y_2}) \cap B(x^{(0)}, \delta)$. Put $z = (x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(k-1)}, \ldots)$. As in the Example 4.2 in [10], we can show that a periodic δ-pseudo orbit $\{x^{(n)}\}$ is ε-shadowed by periodic point z.

$Y_8 = \{\overline{01}, \overline{10}\}$ consists only two periodic points x^0 and x^1, where $x^0 = 010101 \cdots = \overline{01}$ and $x^1 = 101010 \cdots = \overline{10}$. Then there are periodic sequences $\xi_0 = \{x^0, x^1, x^0, x^1, \ldots\}$ and $\xi_1 = \{x^1, x^0, x^1, x^0, \ldots\}$. Take any $\varepsilon > 0$ and $\delta > 0$. It is clear that each ξ_i is a periodic δ-pseudo orbit for σ, $i = 0, 1$. Also, each periodic δ-pseudo orbit ξ_i is ε-shadowed periodic point x^i of Y_8 for $i = 0, 1$. Therefore $(Y_8, \sigma|_{Y_8})$ has the PerSP. Similarly, we can show that $(Y_9, \sigma|_{Y_9})$ has the PerSP. $\square$

Example 4.2 ([10, Example 4.2])**.** Let (X, d) be a compact metric space. We denote by $X^{\mathbb{N}_0}$ the product space of all sequences $x = (x_n)_{n \in \mathbb{N}_0}$ such that $x_n \in X$ for each $n \in \mathbb{N}_0$. Then we note that $X^{\mathbb{N}_0}$ is a compact metric space with metric $\tilde{d}$ which is given by

$$\tilde{d}(x, y) = \max_{i \in \mathbb{N}_0} \frac{d(x_i, y_i)}{2^i}, \quad x = (x_i),\ y = (y_i) \in X^{\mathbb{N}_0}.$$

Also, we note that the shift map $\sigma : X^{\mathbb{N}_0} \to X^{\mathbb{N}_0}$ given by $\sigma(x_n)_{n \in \mathbb{N}_0} = (x_{n+1})_{n \in \mathbb{N}_0}$ is a continuous map with the periodic shadowing property.

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