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SOME PROPERTIES OF STRONG CHAIN TRANSITIVE MAPS

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Abstract. Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and for an arbitrary $x \in X$,

$$\mathcal{SC}_d(x, f) := \{y \,|\, x \text{ can be strong } d\text{-chain to } y\}.$$

We give an example to show that $\mathcal{SC}_d(x, f)$ is dependent on the metric d on X but it is a closed and f-invariant set. We prove that if $\mathcal{SC}_d(x, f) \supseteq \Omega(f)$ or f has the asymptotic-average shadowing property, then $\mathcal{SC}_d(x, f) = X$. Also, we show that if f has the shadowing property, then $\limsup_{n\in\mathbb{N}}\{f^n\} = \mathcal{SC}_d(f)$ where $\mathcal{SC}_d(f) = \{(x, y) \,|\, y \in \mathcal{SC}_d(x, f)\}$. For each $n \in \mathbb{N}$, we give an example in which $\mathcal{SCR}_d(f^n) \neq \mathcal{SCR}_d(f)$. In spite of it, we prove that if $f^{-1} : (X, d) \to (X, d)$ is an equicontinuous map, then $\mathcal{SCR}_d(f^n) = \mathcal{SCR}_d(f)$ for all $n \in \mathbb{N}$.

1. Introduction

Strong chains and strong chain recurrent points have been introduced by Easton [2]. He obtained a relation between strong chain transitivity and Lipschitz ergodicity (namely, any Lipschitz function which is constant along orbits is globally constant). The strong chain recurrent set depends on the metric. To eliminate the dependence on the metric in $\mathcal{SCR}_d$, Fathi and Pageault [4] have introduced two different sets the *Mañe* set $\mathcal{M}(f) = \bigcup_{d'} \mathcal{SCR}_{d'}(f)$ and the generalized recurrent set $GR(f) = \bigcap_{d'} \mathcal{SCR}_d(f)$, where the union and the intersection are both over all metrics d' compatible with the topology of X. Wiseman [10] has shown that there is a metric for which the strong chain recurrent set equals $GR(f)$. Fakhari, Ghane and Sarizadeh [3] have exhibited some general properties of the strong chain recurrent set, and studied strong chain transitivity for a map having a shadowing property. Some properties of strong chain recurrent set have been discussed by Yokoi [11] where he has given a necessary and sufficient condition for the coincidence of $\mathcal{SCR}(f)$ and $\mathcal{CR}(f)$. Motivated by the results above, we continue the study of strong chain recurrence.

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In this paper, we assume that $f:(X,d)\to(X,d)$ is a continuous map on compact metric space (X,d). The paper is organized as follows. In Section 2, we consider the set

$$\mathcal{SC}_d(x,f)=\{y \mid x \text{ can be strong } d\text{-chain to } y\}.$$

This section consists of three subsections. In Subsection 2.1, we give some properties of $\mathcal{SC}_d(x,f)$. Proposition 2.3 shows that $\mathcal{SC}_d(x,f)$ is a closed f-invariant set. We give Example 2.2 to show that $\mathcal{SC}_d(x,f)$ is dependent on the metric. Also, in Proposition 2.4, we show that if $\mathcal{SC}_d(x,f)\supseteq\Omega(f)$, then $\mathcal{SC}_d(x,f)=X$. Since $\mathcal{SC}_d(x,f)$ is a closed set, the map $\mathcal{SC}_d(\cdot,f):X\to 2^X$ which maps x to $\mathcal{SC}_d(x,f)$ is well-defined. In Subsection 2.2, we give some properties of $\mathcal{SC}_d(\cdot,f):X\to 2^X$. Indeed in Proposition 2.6 we show that

$$\bigcup_{n=1}^{\infty}\mathcal{SC}_d^n(x,f)=\mathcal{SC}_d(x,f) \quad\text{and}\quad \bigcap_{\epsilon>0}\overline{\mathcal{SC}_d(B_\epsilon(x),f)}=\mathcal{SC}_d(x,f),$$

where

$$\mathcal{SC}_d^n(x,f)=\bigcup_{y\in\mathcal{SC}_d^{n-1}(x,f)}\mathcal{SC}_d(y,f) \quad\text{and}\quad \mathcal{SC}_d(B_\epsilon(x),f)=\bigcup_{y\in B_\epsilon(x)}\mathcal{SC}_d(y,f).$$

This means that $\mathcal{SC}_d(\cdot,f):X\to 2^X$ is a transitive and cluster map. In Subsection 2.3, we consider the relation

$$\mathcal{SC}_d(f)=\{(x,y):y\in\mathcal{SC}_d(x,f)\}.$$

In Proposition 2.7, we show that $\mathcal{SC}_d(f)\subseteq X\times X$ is a transitive relation and it is a closed set in $X\times X$, also we show that if $\Omega(f)\times\Omega(f)\subseteq\mathcal{SC}_d(f)$, then $\mathcal{SC}_d(f)=X\times X$. Supposing $\mathcal{C}(f)=\{(x,y)\mid x \text{ can be chain to } y\}$, we can see that $\mathcal{C}(f)$ is a transitive relation and a closed set in $X\times X$. Example 2.10 shows that reverse of the following inclusions does not hold in general

$$\limsup\{f^n\}\subseteq\mathcal{SC}_d(f)\subseteq\mathcal{C}(f).$$

If f has the shadowing property, then the reverse of the above inclusions holds (see Proposition 2.11). Since Shadowing property is a generic property on compact manifolds, we can say that for a generic homeomorphism f on a compact manifold X, $\limsup\{f^n\}=\mathcal{SC}_d(f)=\mathcal{C}(f)$ (see Corollary 2.12). Recall that f has the shadowing property if for every $\epsilon>0$; there is $\delta>0$ such that for any δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ there is $y\in X$ such that $d(f^n(y),x_n)<\epsilon$ for all $n\in\mathbb{N}$.

In Section 3, we state some properties of strong chain transitive maps. In Proposition 3.1, we show that for an f-invariant set $Y\subset X$ with $\overline{Y}=X$, the system (X,f) is strong chain transitive if and only if subsystem $(Y,f|Y)$ is strong chain transitive. It is known that if $f:X\to X$ is chain transitive map, then for every $\delta>0$, there is $k(\delta)\in\mathbb{N}$ such that for every $x\in X$, $k(\delta)$ is the greatest common denominator of the lengths of δ-chain from x to x, see [9, Lemma 6]. In Proposition 3.2, we show that this property holds for strong chain transitive maps. In Subsection 3.1, we recall the notion of asymptotic-average shadowing property and in Proposition 3.3, we show that every system with asymptotic-average shadowing property is strong chain transitive map. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\mathrm{Homeo}_0(\mathbb{T}^2)$ denote the set of homeomorphisms homotopic to the identity. We shall say that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is non-resonant if the rotation set of f is a unique vector (α, β) and the values $1, \alpha, \beta$ are irrationally independent (i.e., α, β and $\frac{\alpha}{\beta}$ are not rational), for further information see [8]. We show that if $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is a non-resonant torus homeomorphism, then f is strong chain transitive map, see Corollary 3.5.

In Section 4, we recall the notion of strong chain recurrent point $\mathcal{SCR}_d(f)$ of (X, f). The strong chain recurrent set depends on the metric, see [11, Example 3.1] but similar to Corollary 2.12, we can say that for generic homeomorphism f on a generalized homogeneous compact metric space, the strong chain recurrent set does not depend on the metric. In Example 4.3 we show that $\mathcal{SCR}(f^n) \neq \mathcal{SCR}(f)$ for all $n \in \mathbb{N}$, but if $f^{-1} : (X, d) \to (X, d)$ is equicontinuous on a compact metric space (X, d), then $\mathcal{SCR}_d(f^n) = \mathcal{SCR}_d(f)$ for all $n \in \mathbb{N}$, see Proposition 4.4.

2. Structure of $\mathcal{SC}_d(x, f)$

A finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X is a strong (ϵ, d)-chain from x to y with length n if $x_0 = x$, $x_n = y$ and $\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \epsilon$.

For $\epsilon > 0$, let us define

$$\mathcal{SC}_d^{\epsilon}(x, f) = \{y \mid x \text{ can be strong } (\epsilon, d)\text{-chain to } y\}, \tag{2.1}$$

and

$$\mathcal{SC}_d^{\epsilon^+}(x, f) = \bigcap_{\delta > 0} \mathcal{SC}_d^{\epsilon+\delta}(x, f). \tag{2.2}$$

In the following lemma, we give some properties of the sets $\mathcal{SC}_d^{\epsilon}(x, f)$ and $\mathcal{SC}_d^{\epsilon^+}(x, f)$.

Lemma 2.1. *Let (X, d) be a compact metric space. The sets $\mathcal{SC}_d^{\epsilon}(x, f)$ and $\mathcal{SC}_d^{\epsilon^+}(x, f)$ have the following properties*:

(1) *$\mathcal{SC}_d^{\epsilon}(x, f)$ is an open set,*
(2) *$\mathcal{SC}_d^{\epsilon^+}(x, f)$ is a closed set,*
(3) *$\{z \mid d(z, \mathcal{SC}_d^{\epsilon^+}(x, f)) < \eta\} \subseteq \mathcal{SC}_d^{\epsilon+\eta}(x, f)$,*
(4) *$f(\mathcal{SC}_d^{\epsilon^+}(x, f)) \subseteq \mathcal{SC}_d^{\epsilon^+}(x, f)$,*
(5) *$\omega_f(\mathcal{SC}_d^{\epsilon^+}(x, f)) \subseteq \mathcal{SC}_d^{\epsilon^+}(x, f)$,*
(6) *$\omega_f(x) \subseteq \mathcal{SC}_d^{\epsilon^+}(x, f)$,*
(7) *if $y \in \mathcal{SC}_d^{\epsilon_1}(x, f)$, $z \in \mathcal{SC}_d^{\epsilon_2}(y, f)$, then $z \in \mathcal{SC}_d^{\epsilon_1+\epsilon_2}(x, f)$.*

Proof. (1) Let $y \in \mathcal{SC}_d^{\epsilon}(x, f)$, there is $\{x_n\}_{n=0}^{k}$ with $x_0 = x$, $x_k = y$ and $\sum_{n=0}^{k-1} d(f(x_n), x_{n+1}) < \epsilon$. Let $0 < \epsilon_1 < \epsilon - \sum_{n=0}^{k-1} d(f(x_n), x_{n+1})$. Then replacing y by every point $z \in B_{\epsilon_1}(y)$, we obtain a strong (ϵ, d)-chain from x to z.

(2) Let $z \in \overline{\mathcal{SC}_d^{\epsilon}(x,f)}$ and $\eta > 0$ be given. Choose $y \in \mathcal{SC}_d^{\epsilon^+}(x,f)$ with $d(z,y) < \frac{\eta}{2}$. There is a strong $(\epsilon + \frac{\eta}{2}, d)$-chain from x to y. We can replace y by z and obtain a strong (ϵ, η, d)-chain from x to z.

(3) Let $y \in \mathcal{SC}_d^{\epsilon^+}(x,f)$ with $d(z,y) < \eta$. Consider $\lambda = \eta - d(z,y)$, since $y \in \mathcal{SC}_d^{\epsilon^+}(x,f)$, there is a strong $(\epsilon + \lambda, d)$-chain from x to y, replace y by z and obtain a strong $(\epsilon + \eta, d)$-chain from x to z.

(4) It is clear.

(5) Since $\omega_f(\mathcal{SC}_d^{\epsilon^+}(x,f)) = \bigcap_n \overline{\bigcup_{m>n} f^m(\mathcal{SC}_d^{\epsilon^+}(x,f))}$ by item (4) and item (2) we have

$$\omega_f(\mathcal{SC}_d^{\epsilon^+}(x,f)) \subseteq \mathcal{SC}_d^{\epsilon^+}(x,f).$$

(6) Since $\omega_f(x) = \bigcap_n \overline{\bigcup_{m>n} f^m(x)}$ and $f(x) \in \mathcal{SC}_d^{\epsilon^+}(x,f)$, by item (4) and item (2) we have $\omega_f(x) \subseteq \mathcal{SC}_d^{\epsilon^+}(x,f)$.

(7) It is clear. □

2.1. Some properties of $\mathcal{SC}_d(x,f)$

Let us define

$$\mathcal{SC}_d(x,f) = \bigcap_{\epsilon>0} \mathcal{SC}_d^{\epsilon}(x,f). \tag{2.3}$$

It is easy to see that $\mathcal{SC}_d(x,f) = \bigcap_{\epsilon>0} \mathcal{SC}_d^{\epsilon^+}(x,f)$. This implies that $\mathcal{SC}_d(x,f)$ is a closed set and by Lemma 2.1, $\omega(x,f) \subseteq \mathcal{SC}_d(x,f)$ which shows $\mathcal{SC}_d(x,f) \neq \emptyset$ for all $x \in X$.

Since $\mathcal{SC}_d(x,f)$ is a closed set, for $\epsilon > 0$ there is a finite set $\{x_i\}_{i=0}^n \subseteq \mathcal{SC}_d(x,f)$ such that $\{B(x_i, \frac{\epsilon}{2})\}$ is a finite open cover for $\mathcal{SC}(x,f)$. For every x_i, there is a strong $(\frac{\epsilon}{2}, d)$-chain from x to x_i with length l_i, thus if $l = \max\{l_i\}_{i=0}^n$, then for every $y \in \mathcal{SC}_d(x,f)$, we can find a strong (ϵ, d)-chain from x to y with length at most l. Hence for every $\epsilon > 0$ and $x \in X$, there is $n(x,\epsilon) > 0$ such that for every $y \in \mathcal{SC}_d(x,f)$ there is a strong (ϵ, d)-chain from x to y with length $\leq n(x,\epsilon)$.

In the next example, we show that $\mathcal{SC}_d(x,f)$ is dependent on the metric.

Example 2.2. Let

$$X = \bigcup_{p=0}^{\infty} I_p, \tag{2.4}$$

where $I_0 = \{(x,0) \,|\, x \in [0,1]\}$ and $I_p = \{(\frac{q}{2^p}, \frac{1}{2^{p-1}}) \,|\, q = 0,1,\ldots,2^p\}$ for $p \in \mathbb{N}$. Define a map f on X with the fixed points set $Fix(f) = I_0$, $f(\frac{q}{2^p}, \frac{1}{2^{p-1}}) = (\frac{q-1}{2^p}, 0)$ for $q = 1,2,\ldots,2^p$, $f(0, \frac{1}{2^{p-1}}) = (0,0)$. Consider the following metrics on X:

$$d_2((x_1,x_2),(y_1,y_2)) = d_1((x_1,x_2^2),(y_1,y_2^2)) \text{ and } d_1(x,y) = ||x-y||.$$

Let $x_0, x_1, \ldots, x_m$ be any strong d_1-chain from $x_0 = (1,0)$ to $x_m = (0,0)$. Let $\pi : X \to [0,1]$ be defined by $\pi(x,y) = x$. For $\{x_n\}_{n=0}^m \subseteq X$, we have

$$
\begin{aligned}
1 &= |\pi(f(x_0)) - \pi(x_m)| \\
&\leq \sum_{n=0}^{m-1} |\pi(f(x_n)) - \pi(x_{n+1})| + \sum_{n=1}^{m} |\pi(f(x_n)) - \pi(x_n)|.
\end{aligned} \tag{2.5}
$$

Using the inequalities $|\pi(f(x_n))-\pi(x_n)| \leq \frac{1}{2}||f(x_n)-x_{n+1}||$ and $|\pi(f(x_n))-\pi(x_{n+1})| \leq ||f(x_n)-x_{n+1}||$ for $0 \leq n \leq m-1$, we have $\sum_{n=0}^{m-1} ||f(x_n)-x_{n+1}|| > \frac{2}{3}$. Thus $\{x_n\}_{n=0}^m$ can not be strong d_1-chain transitive from $(1,0)$ to $(0,0)$, i.e., $(0,0) \notin \mathcal{SC}_{d_1}((1,0), f)$.

In the following we show that $(0,0) \in \mathcal{SC}_{d_2}((1,0), f)$.

Let $\epsilon > 0$ be given. Choose $p \in \mathbb{N}$ with $\frac{1}{2^p} < \epsilon$. Let $x_0 = (1,0)$, $x_{2^p+1} = (0,0)$ and for $n = 1, \ldots, 2^p$, take $x_n = (\frac{2^p-n+1}{2^p}, \frac{1}{2^{p-1}})$.

Claim.

$$
\{x_n\}_{n=0}^{2^p+1} = \{(1,0), x_1, x_2, \ldots, x_{2^p}, x_{2^p+1} = (0,0)\} \tag{2.6}
$$

is a strong (ϵ, d_2)-chain from $(1,0)$ to $(0,0)$.

Proof of Claim. By definition of $\{x_n\}_{n=0}^{2^p+1}$, for $x_n = (\frac{q}{2^p}, \frac{1}{2^{p-1}})$ we have $x_{n+1} = (\frac{q-1}{2^p}, \frac{1}{2^{p-1}})$, thus $d_2(f(x_n), x_{n+1}) = d_2((\frac{q-1}{2^p}, 0), (\frac{q-1}{2^p}, \frac{1}{2^{p-1}})) = \frac{1}{2^{2p-2}}$ for $n = 1, 2, \ldots, 2^p$, this means that $\sum_{n=0}^{2^p} d_2(f(x_n), x_{n+1}) = \sum_{n=0}^{2^p} \frac{1}{4^{p-1}} < \epsilon$, i.e., $(1,0)$ can be strong d_2-chain to $(0,0)$. It implies that $\{x_i\}_{i=0}^{2^p+1}$ is a strong (ϵ, d_2)-chain transitive. □

Note that by Example 2.2, it may be happen that $f(y) \in \mathcal{SC}_d(x,f)$, but $y \notin \mathcal{SC}_d(x,f)$. In the following we show that if $z \in \mathcal{SC}_d(x,f)$, there is $y \in \mathcal{SC}_d(x,f)$ such that $z = f(y)$.

Proposition 2.3. *Let $f : (X,d) \to (X,d)$ be a continuous map on the compact metric space X. Then $f(\mathcal{SC}_d(x,f)) = \mathcal{SC}_d(x,f)$.*

Proof. The inclusion $f(\mathcal{SC}_d(x,f)) \subseteq \mathcal{SC}_d(x,f)$ is clear. We show that $\mathcal{SC}_d(x,f) \subseteq f(\mathcal{SC}_d(x,f))$. Let $y \in \mathcal{SC}_d(x,f)$ and $\epsilon > 0$ be given. For every $n \in \mathbb{N}$, there is a strong $(\frac{1}{n}, d)$-chain $\{x_i^n\}_{i=0}^{m_n}$ from x to y, i.e.,

$$
\sum_{i=0}^{m_n-2} d(f(x_i^n), x_{i+1}^n) + d(f(x_{m_n-2}^n), x_{m_n-1}^n) + d(f(x_{m_n-1}), y) < \frac{1}{n}.
$$

The compactness of X implies that $\{x_{m_n-1}^n\}_{n=1}^\infty$ has a convergence subsequence. We can assume that $x_{m_n-1}^n \to z$ as $n \to \infty$. Since $d(f(x_{m_n-1}), y) < \frac{1}{n}$, hence from the continuity of f, we have $y = f(z)$. For $\epsilon > 0$ choose $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \frac{\epsilon}{2}$ and $d(x_{m_{n_0}-1}^{n_0}, z) < \frac{\epsilon}{2}$. Take

$$
\{z_i\}_{i=0}^{m_{n_0}-1} = \{x_0^{n_0} = x, x_1^{n_0}, x_2^{n_0}, \ldots, x_{m_{n_0}-2}^{n_0}, z\}.
$$

One can check that

$$\sum_{i=0}^{m_{n_0}-2} d(f(z_i), z_{i+1}) \leq \sum_{i=0}^{m_{n_0}-1} d(f(x_i^{n_0}), x_{i+1}^{n_0}) + d(x_{m_{n_0}-1}^{n_0}, z) < \frac{\epsilon}{2} + \frac{1}{n_0} < \epsilon.$$

This implies that $y = f(z) \in f(\mathcal{SC}_d(x, f))$. □

By Lemma 2.1, $\omega(x, f) \subseteq \mathcal{SC}_d(x, f)$. This implies that $\mathcal{SC}_d(x, f) \neq \emptyset$, but it may be happen that $\mathcal{SC}_d(x, f) \neq X$. In the following we show that if $\mathcal{SC}_d(x, f)$ contains the set of non-wandering points, then $\mathcal{SC}_d(x, f) = X$. Recall that a point $x \in X$ is a non-wandering point of f, if for every open set U of x, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points of f is denoted by $\Omega(f)$.

Proposition 2.4. *Let (X, d) be a compact metric space and $\Omega(f) \subseteq \mathcal{SC}_d(x, f)$. Then $\mathcal{SC}_d(x, f) = X$.*

Proof. Let $y \in X$ and $\epsilon > 0$ be given. We show that x can be strong (ϵ, d)-chain to y.

If $d(y, \Omega(f)) < \frac{\epsilon}{4}$, then there is $z \in \Omega(f) \subseteq \mathcal{SC}_d(x, f)$ such that $d(y, z) = d(y, \Omega(f)) < \frac{\epsilon}{4}$ because $\Omega(f)$ is a closed set. This implies that $y \in \mathcal{SC}_d^{\epsilon}(x, f)$. Let

$$y \in K = \{z : d(z, \Omega(f)) \geq \frac{\epsilon}{4})\}. \tag{2.7}$$

It is clear that for every $z \in K$, there is an open set $U(z)$ such that $f^n(U(z)) \cap U(z) = \emptyset$ for all $n \in \mathbb{N}$. Since K is a compact set, there is a finite set $\{z_1, z_2, \ldots, z_m\} \subset K$ such that $K \subseteq \bigcup_{i=1}^{m} U(z_i)$. Take $p \in f^{-m}(y)$, by the pigeonhole principle there is $0 \leq n \leq m$ such that $f^n(p) \notin K$, this implies that $d(f^n(p), \Omega(f)) < \frac{\epsilon}{4}$. Hence we can say that $y \in \mathcal{SC}_d^{\epsilon}(x, f)$. □

The system $f : X \to X$ is called conjugate with the system $g : Y \to Y$, under $h : X \to Y$ whenever $h : X \to Y$ is a homeomorphism and $h \circ f = g \circ h$. In Example 2.2, the system $f : (X, d_1) \to (X, d_1)$ is conjugate with system $f : (X, d_2) \to (X, d_2)$ under the homeomorphism $id_X : (X, d_1) \to (X, d_2)$ and $\mathcal{SC}_{d_1}(x, f) \neq \mathcal{SC}_{d_2}(x, f)$.

Suppose h is a Lipschitz map with Lipschitz constant $k > 0$. For every $\epsilon > 0$, put $\delta = \frac{\epsilon}{2k}$. For every $n \in \mathbb{N}$, if $\sum_{i=1}^{n} d(x_i, y_i) < \delta$, then $\sum_{i=1}^{n} \rho(h(x_i), h(y_i)) < k \sum_{i=0}^{n} d(x_i, y_i) < \epsilon$. This implies that if $f : (X, d) \to (X, d)$ and $g : (Y, \rho) \to (Y, \rho)$ are conjugate under a Lipschitz homeomorphism h, then $h(\mathcal{SC}_d(x, f)) = \mathcal{SC}_\rho(h(x), g)$. Also, if $id : (X, d_1) \to (X, d_2)$ is a Lipschitz homeomorphism, then $\mathcal{SC}_{d_1}(x, f) = \mathcal{SC}_{d_2}(x, f)$.

2.2. Some properties of $\mathcal{SC}_d(\cdot, f) : X \to 2^X$

Let $\Gamma : X \to 2^X$ be a set-valued map and $A \subseteq X$. We define $\Gamma(A)$ as:

$$\Gamma(A) = \cup_{x \in A} \Gamma(x).$$

We let the composition $\Gamma^2 = \Gamma \circ \Gamma$ be given by

$$\Gamma^2(x) = \Gamma(\Gamma(x)) = \cup_{y \in \Gamma(x)} \Gamma(y)$$

so we can naturally define the iteration $\Gamma^n : X \to 2^X$ inductively by $\Gamma^1(x) = \Gamma(x)$ and $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$.

Since for all $x \in X$, $\mathcal{SC}_d(x, f)$ is a closed set, the map $\mathcal{SC}_d(\cdot, f) : X \to 2^X$ defined by $x \mapsto \mathcal{SC}_d(x, f)$ is well-defined.

Lemma 2.5. *If $x_n \to x$, $y_n \to y$ and $y_n \in \mathcal{SC}_d(x_n, f)$, then $y \in \mathcal{SC}_d(x, f)$.*

Proof. By the uniform continuity of f, for $\epsilon > 0$, there is $0 < \delta < \frac{\epsilon}{3}$ such that

$$d(a, b) < \delta \Rightarrow d(f(a), f(b)) < \frac{\epsilon}{3}.$$

Choose $n \in \mathbb{N}$ such that $d(x_n, x) < \delta$ and $d(y_n, y) < \delta$. $y_n \in \mathcal{SC}_d(x_n, f)$ implies that there is a finite sequence $\{z_0 = x_n, z_1, \ldots, z_m = y_n\}$ such that

$$\sum_{i=0}^{m-1} d(f(z_i), z_{i+1}) < \frac{\epsilon}{3}.$$

Also, $d(x_n, x) < \delta$ implies that $d(f(x), z_1) \leq \frac{\epsilon}{3} + d(f(z_0), z_1)$. Hence by $d(y_n, y) < \delta < \frac{\epsilon}{3}$, we have

$$d(f(x), z_0) + \sum_{i=1}^{m-1} d(f(z_i), z_{i+1}) + d(y_n, y) < \epsilon.$$

This means that $y \in \mathcal{SC}_d(x, f)$. □

We say that

- $\Gamma : X \to 2^X$ is transitive provided $S\Gamma = \Gamma$ where $S\Gamma(x) := \overline{\bigcup_{n=1}^{\infty} \Gamma^n(x)}$,
- $\Gamma : X \to 2^X$ is a cluster map if $D\Gamma = \Gamma$ where $D\Gamma(x) := \cap_{\epsilon>0} \overline{\Gamma(B_\epsilon(x))}$.

It is clear that if $\Gamma^2 = \Gamma$, then Γ is transitive. Also from the definition of $D\Gamma(x)$, we immediately obtain that $\Gamma(x) \subseteq D\Gamma(x)$.

Proposition 2.6. *The map $SC : X \to 2^X$ is transitive and it is a cluster map.*

Proof. It is clear that if $z \in \mathcal{SC}_d(y, f)$ and $y \in \mathcal{SC}_d(x, f)$, then $z \in \mathcal{SC}_d(x, f)$. So $SC^2 = SC$. Let $y \in \cap_{\epsilon>0} \overline{SC(B_\epsilon(x))}$. This means that there are $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$, $y_n \to y$ and $y_n \in \mathcal{SC}_d(x_n, f)$. Lemma 2.5 implies that $y \in \mathcal{SC}_d(x, f)$. Hence SC is a cluster map. □

2.3. Some properties of $\mathcal{SC}_d(f)$

Let us define

$$\mathcal{SC}_d(f) = \{(x, y) \mid y \in \mathcal{SC}_d(x, f)\}. \tag{2.8}$$

Some properties of the relation $\mathcal{SC}_d(f)$ are illustrated in the next Proposition:

Proposition 2.7. *The relation $\mathcal{SC}_d(f)$ has the following properties*:

(1) *$\mathcal{SC}_d(f)$ is a transitive relation,*

(2) $(x,y) \in \mathcal{SC}_d(f)$ *if and only if* $(f(x),y) \in \mathcal{SC}_d(f)$,
(3) $\mathcal{SC}_d(f)$ *is a closed set,*
(4) *If* (X,f) *is conjugate with* (Y,g) *under a Lipschitz homeomorphism* $h:(X,d) \to (Y,\rho)$, *then* $h(\mathcal{SC}_d(f)) = \mathcal{SC}_\rho(g)$,
(5) $\limsup_{n\in\mathbb{N}}\{f^n\} \subseteq \mathcal{SC}_d(f)$,
(6) *If* $\Omega(f)\times\Omega(f) \subseteq \mathcal{SC}_d(f)$, *then* $\mathcal{SC}_d(f) = X \times X$.

Proof. (1) The transitivity of the relation $\mathcal{SC}_d(f)$ follows directly from Lemma 2.1.

(2) Let $(f(x),y) \in \mathcal{SC}_d(f)$. Since for every $x \in X$, $(x,f(x)) \in \mathcal{SC}_d(f)$, by the transitivity of $\mathcal{SC}_d(f)$ we have $(x,y) \in \mathcal{SC}_d(f)$. Conversely, let $(x,y) \in \mathcal{SC}_d(f)$ and $\epsilon > 0$ be given. Choose $0 < \delta < \frac{\epsilon}{4}$ such that

$$d(a,b) < \delta \Rightarrow d(f(a),f(b)) < \frac{\epsilon}{4}. \tag{2.9}$$

Since $(x,y) \in \mathcal{SC}_d(f)$, there is a strong (δ,d)-chain, $\{x_n\}_{n=0}^m$, from x to y. One can check that $\{f(x),x_2,x_3,\ldots,x_k\}$ is a strong (ϵ,d)-chain from $f(x)$ to y.

(3) Let $(x,y) \in \overline{\mathcal{SC}_d(f)}$ and $\epsilon > 0$ be given. Choose $\delta > 0$ satisfies in (2.9) and $(z,w) \in \mathcal{SC}_d(f)$ such that $d(x,z) < \delta$ and $d(y,w) < \delta$. The inequalities $d(f(x),f(z)) < \frac{\epsilon}{4}$, $d(w,y) < \frac{\epsilon}{4}$ and $(z,w) \in \mathcal{SC}_d(f)$ imply that $(x,y) \in \mathcal{SC}_d(f)$.

(4) It is clear.

(5) Note that $\limsup_{n\in\mathbb{N}}\{f^n\} = \bigcap_{n=1}^\infty \overline{\bigcup_{k=n}^\infty\{f^k\}}$. Hence if

$$(x,y) \in \bigcap_{n=1}^\infty \overline{\bigcup_{k=n}^\infty\{f^k\}},$$

then there are $\{x_n\}$ and $\{y_n\}$ in X and $\{k_n\} \subseteq \mathbb{N}$ such that $y_n = f^{k_n}(x_n)$, $x_n \to x$ and $y_n \to y$. For $\epsilon > 0$, there is $0 < \delta < \epsilon$ such that

$$d(a,b) < \delta \Rightarrow d(f(a),f(b)) < \epsilon.$$

Choose $n \in \mathbb{N}$ such that $d(x_n,x) < \delta$ and $d(y_n,y) < \delta$. It is easy to see that the finite sequence

$$\{x,,f(x_n),f^2(x_n),\ldots,f^{k_n-1}(x_n),y\}$$

is a strong (ϵ,d)-chain from x to y. Hence, $(x,y) \in \mathcal{SC}_d(f)$.

(6) Since $\Omega(f)\times\Omega(f) \subseteq \mathcal{SC}_d(f)$, for every $x \in X$, we have $\Omega(f) \subseteq \mathcal{SC}_d(x,f)$. Hence, Proposition 2.4 implies that $\mathcal{SC}_d(x,f) = X$. □

A finite sequence $\{x_n\}_{n=0}^k$ is called ϵ-chain from x to y, if $x_0 = x$, $x_k = y$ and $d(f(x_n),x_{n+1}) < \epsilon$ for $n = 0,1,\ldots,k-1$. Let $\mathcal{C}_\epsilon(x,f)$ denote the set of points $y \in X$ such that x can be ϵ-chain to y, also let $\mathcal{C}_{\epsilon^+}(x,f) = \bigcap_\delta \mathcal{C}_{\epsilon+\delta}(x,f)$. Similarly to the proof of Lemma 2.1, one can see that $\mathcal{C}_\epsilon(x,f)$ is an open and $\mathcal{C}_{\epsilon^+}(x,f)$ is a closed set. Let us define $\mathcal{C}(x,f) = \bigcap_\epsilon \mathcal{C}_\epsilon(x,f)$. Hence $\mathcal{C}(x,f)$ is a closed set. Similarly to the proof of Proposition 2.6, we can say that:

Proposition 2.8. *The map* $\mathcal{C}(\cdot,f) : X \to 2^X$ *defined by* $x \mapsto \mathcal{C}(x,f)$ *is a transitive and cluster map.*

A point x can be chain to y if $y \in \mathcal{C}(x, f)$ and it is clear that $\mathcal{C}(x, f) = \bigcap_\epsilon \mathcal{C}_{\epsilon^+}(x, f)$, hence $\mathcal{C}(x, f)$ is a closed set. By a similar argument as the proof of Proposition 2.7, we can say that:

Proposition 2.9. *Let* $\mathcal{C}(f) = \{(x, y) \mid y \in \mathcal{C}(x, f)\}$. *Then*

(1) $\mathcal{C}(f)$ *is a transitive relation on* X *and a closed subset of* $X \times X$.
(2) $(x, y) \in \mathcal{C}(f)$ *if and only if* $(f(x), y) \in \mathcal{C}(f)$.
(3) *If* $\Omega(f) \times \Omega(f) \subseteq \mathcal{C}(f)$, *then* $\mathcal{C}(f) = X \times X$.
(4) *If* (X, f) *is conjugate with* (Y, g) *under a homeomorphism* $h : X \to Y$, *then* $h(\mathcal{C}(f)) = \mathcal{C}(g)$.
(5) $\mathcal{C}(f)$ *does not depend on the metric on* X.

Proposition 2.7 and Proposition 2.9 imply that

$$\limsup_{n\in\mathbb{N}}\{f^n\} \subseteq \mathcal{SC}_d(f) \subseteq \mathcal{C}(f). \tag{2.10}$$

Example 2.10. Let $f : X \to X$ be as in Example 2.2 and $x = (1, 0)$, $y = (0, 0)$. Then

- $(x, y) \in \mathcal{C}(f)$ while $(x, y) \notin \mathcal{SC}_{d_1}(f)$.
- Since $\limsup_{n\in\mathbb{N}}\{f^n\}$ does not depend on the metric on X, $(x, y) \notin \mathcal{SC}_{d_1}(f)$ implies that $(x, y) \notin \limsup_{n\in\mathbb{N}}\{f^n\}$ while $(x, y) \in \mathcal{SC}_{d_2}(f)$.

Note that Example 2.10 shows that for a continuous map $f : X \to X$, it may happen that $\mathcal{C}(f) - \mathcal{SC}_d(f) \neq \emptyset$ and $\mathcal{SC}_d(f) - \limsup_{n\in\mathbb{N}}\{f^n\} \neq \emptyset$.

Proposition 2.11. *If* $f : (X, d) \to (X, d)$ *has the shadowing property, then* $\limsup_{n\in\mathbb{N}}\{f^n\} = \mathcal{SC}_d(f) = \mathcal{C}(f)$.

Proof. Let $(x, y) \in \mathcal{C}(f)$ and $\epsilon > 0$ be given. There is a $\delta > 0$ that satisfies in the definition of shadowing property of f. Let $\{x_n\}_{n=0}^k$ be a δ-chain with $x_0 = x$ and $x_k = y$. Take $x_n = f^{n-k}(y)$ for $n > k$, by shadowing property of f, there is $p \in X$ with $d(f^n(p), x_n) < \epsilon$ This implies that $d(x, p) < \epsilon$ and $d(f^k(p), y) < \epsilon$, i.e., $(x, y) \in \limsup_{n\in\mathbb{N}} f^n$. □

Corless and Pilyugin in [1] proved C^0-genericity of the shadowing on a compact smooth manifold (without boundary) and later Mazur [7] proved the same result for generalized homogeneous compact metric spaces with no isolated point. It implies that:

Corollary 2.12. *Let* X *be a generalized homogeneous compact metric space. Then for any generic homeomorphism* f *on* X, $\limsup_{n\in\mathbb{N}}\{f^n\} = \mathcal{SC}_d(f) = \mathcal{C}(f)$.

3. Strong chain transitive map

We say that

- f is a chain transitive map on $A \subseteq X$, if $A \times A \subseteq \mathcal{C}(f)$ and
- f is a strong chain transitive map on $A \subseteq X$, if $A \times A \subseteq \mathcal{SC}_d(f)$.

So, a homeomorphism $f : X \to X$ is chain transitive on A if and only if f^{-1} is a chain transitive map on A. Also if f is a bi-Lipschitz homeomorphism, then $f : X \to X$ is strong chain transitive on A if and only if f^{-1} is a strong chain transitive map on A. We say that f is a chain transitive map (resp. strong chain transitive) if $\mathcal{C}(f) = X \times X$ (resp. $\mathcal{SC}_d(f) = X \times X$). It is clear that if f is a chain transitive map on a dense subset of X, then it is chain transitive. Also, if f is a strong chain transitive map on a dense subset of X, then it is a strong chain transitive map.

Proposition 3.1. *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map and Y be a dense subset of X with $f(Y) = Y$. Then f is a strong d-chain transitive map if and only if $f_{|Y}$ is a strong d-chain transitive map.*

Proof. Suppose f is a strong d-chain transitive map, let $a, b \in Y$ and let $\epsilon > 0$. There is a strong $(\frac{\epsilon}{3}, d)$-chain $(\{z_n\}_{n=0}^k)$ from a to b in X. Since X is compact and Y is dense in X, for every $n = 1, \dots, k$, there is $t_n \in N(z_n, \frac{\epsilon}{3^{n+2}}) \cap Y$ such that $f(t_n) \in N(f(z_n), \frac{\epsilon}{3^{n+2}})$. Thus

$$d(f(t_n), t_{n+1}) \leq d(f(t_n), f(z_n)) + d(f(z_n), z_{n+1}) + d(z_{n+1}, t_{n+1}) < \frac{\epsilon}{3^{n+1}}. \tag{3.1}$$

Therefore $\sum_{n=0}^{k-1} d(f(t_n), t_{n+1}) < \epsilon$. This means that $f|_Y$ is a strong d-chain transitive map. □

If (X, d) and (Y, ρ) are compact metric spaces and (X, f) is conjugate with (Y, g) under a bi-Lipschitz surjective map $h : (X, d) \to (Y, \rho)$, then f is a strong d-chain transitive map if and only if g is strong ρ-chain transitive.

If $f : X \to X$ is a chain transitive map, then for $\delta > 0$, there is $k_\delta \in \mathbb{N}$ such that for every $x \in X$, k_δ is the greatest common denominator of the lengths of δ-chain from x to x, see [9, Lemma 6]. In the following we show that this property holds for strong chain transitive maps.

Proposition 3.2. *Let $f : (X, d) \to (X, d)$ be a strong d-chain transitive map. Then for every $\delta > 0$, there is $K_\delta \in \mathbb{N}$ such that for every $x \in X$, k_δ is the greatest common denominator of the lengths of strong (δ, d)-chain from x to x.*

Proof. Fix $x \in X$, let $K_\delta(x) \in \mathbb{N}$ be the greatest common denominator of the lengths of all strong (δ, d)-chains from x to x. Let $y \in X$ be given and $\{y_n\}_{n=0}^m$ be a strong (δ, d)-chain from y to y. We will show that $K_\delta(x) \,|\, m$. Since f is a strong d-chain transitive map, for $0 < \eta < \min\{d(f(y), y_1), d(f(y_{m-1}), y)\}$, there is a strong (η, d)-chain $\{z_n\}_{n=1}^p$from x to y and a strong (η, d)-chain $\{w_n\}_{n=1}^q$ from y to x. Hence $\{z_1, z_2, \dots, z_{p-1}, w_1 w_2, \dots, w_q\}$ is a strong (δ, d)-chain from x to x of length $p + q$ and

$$\{z_1, z_2, \dots, z_{p-1}, y_1, y_2, \dots, y_{m-1}, w_1 w_2, \dots, w_q\}$$

is a strong (δ, d)-chain from x to x of length $p+q+m$. By the definition of $K_\delta(x)$, we have $K_\delta(x) \,|\, p + q + m$ and $K_\delta(x) \,|\, p + q$, this implies that $K_\delta(x) \,|\, m$. □

3.1. Two examples of strong chain transitive map

In this subsection, we show that every homeomorphism on a compact metric space with the asymptotic-average shadowing property and every non-resonant torus homeomorphism are strong chain transitive.

Definition ([6]). A sequence $\{x_i\}_{i=0}^{\infty}$ in X is called an asymptotic-average pseudo orbit of f if $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f(x_i),x_{i+1})=0$. A sequence $\{x_i\}_{i=0}^{\infty}$ is said to be asymptotically shadowed in average by the point z in X if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f^i(z),x_i)=0.$$

A map f is said to have the asymptotic-average shadowing property on X (Abbrev. AASP), if every asymptotic-average pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of f can be asymptotically shadowed in average by some points in X .

Proposition 3.3. *Let (X,d) be a compact metric space and f be a homeomorphism on X. If f has the asymptotic-average shadowing property, then f is a strong chain transitive map.*

Proof. Suppose x,y are two distinct points in X and $\delta>0$ is a positive number. It is sufficient to prove that there is a δ-chain from x to y. For every $n\in\{1,2,\ldots\}$ choose $k\in\{0,1,2,\ldots\}$ such that $2^k\le n\le 2^{k+1}$. If $0\le i=n-2^k<2^{k-1}$, take $x_n=f^i(x)$ and for $2^{k-1}\le i=n-2^k<2^k$ take $x_n=y_{-i}$ where $f(y_{-i})=y_{-i+1}$ where $i>0$ and $y_0=y$.

Claim. The sequence $\{x_n\}_{n=0}^{\infty}$ is a δ-asymptotic-average pseudo orbit of f.

Proof of Claim. If D is the diameter of X, that is $D=\max\{d(x,y):x,y\in X\}$, then for $2^k\le n\le 2^{k+1}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}d(f(x_i),x_{i+1})<\frac{2(k+1)D}{2^k}. \tag{3.2}$$

Hence,

$$\lim_{n\to\infty}\sum_{i=1}^{n}d(f(w_i),w_{i+1})=0.$$

This means that $\{x_i\}_{i=0}^{\infty}$ is an asymptotic-average pseudo orbit of f. Since f has the asymptotic-average shadowing property, there is a point z in X such that

$$\lim_{n\to\infty}\sum_{i=1}^{n}d(f^i(z),w_i)=0.$$

□

To continue the proof, we need to apply the following statements that have been proved in [6, Theorem 3.1].

(1) There exist infinitely many positive integers j such that $w_{n_j}\in\{x,f(x),\ldots,f^{2^j-1}(x)\}$ and $d(f^{n_j}(z),w_{n_j})<\frac{\delta}{3}$.

(2) There exist infinitely many positive integers l such that
$w_{n_l} \in \{y_{-2^l+1}, \dots, y_{-1}, y\}$ and $d(f^{n_l}(z), w_{n_l}) < \frac{\delta}{3}$.

Thus we can pick two positive integers j_0 and l_0 such that $n_{j_0} < n_{l_0}$,

$$w_{n_{j_0}} \in \{x, f(x), \dots, f^{2^{j_0}-1}(x)\} \text{ with } d(f^{n_{j_0}}(z), w_{n_{j_0}}) < \frac{\delta}{3},$$

and

$$w_{n_{l_0}} \in \{y_{-2^l+1}, \dots, y_{-1}, y\} \text{ with } d(f^{n_{l_0}}(z), w_{n_l}) < \frac{\delta}{3}.$$

It may be assumed

$$w_{n_{j_0}} = f^{j_1}(x) \text{ for some } j_1 > 0, \text{ and } w_{n_{l_0}} = y_{-l_1} \text{ for some } l_1 > 0.$$

The finite sequence

$$\{x, f(x), \dots, f^{j_1}(x) = w_{n_{j_0}}, f^{n_{j_0}+1}(z), f^{n_{j_0}+2}(z), \dots, f^{n_{l_0}-1}(z),$$
$$w_{n_{l_0}} = y_{-l_1}, \dots, f^{-1}(y), y\}$$

is a strong δ-chain from x to y. □

Recall that for a homeomorphism $f : X \to X$ and an f-invariant compact set K, we say that $f|_K$ is weakly transitive if given two open sets U and V of X intersecting K, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. Note that the difference with being transitive is that for transitivity one requires the open sets to be considered relative to K.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\mathrm{Homeo}_0(\mathbb{T}^2)$ denote the set of homeomorphisms homotopic to the identity. We shall say that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is non-resonant if the rotation set of f is a unique vector (α, β) and the values $1, \alpha, \beta$ are irrationally independent (i.e., α, β and $\frac{\alpha}{\beta}$ are not rational), for further information see [8].

Proposition 3.4 ([8]). *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism. Then $f|_{\Omega(f)}$ is weakly transitive.*

As a corollary, we can say that:

Corollary 3.5. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism. Then f is a strong chain transitive map.*

Proof. Let $x, y \in \Omega(f)$ and $\epsilon > 0$ be given. By continuity of f, for $\epsilon > 0$ there is $0 < \delta < \frac{\epsilon}{2}$ such that

$$d(a, b) < \delta \Rightarrow d(f(a), f(b)) < \frac{\epsilon}{2}.$$

Since $B(x, \delta)$ and $B(y, \delta)$ intersecting $\Omega(f)$ and f is weakly transitive on $\Omega(f)$, there is $n \in \mathbb{N}$ such that $f^n(B(x, \delta)) \cap B(y, \delta) \neq \emptyset$. Hence there is $p \in B(x, \delta)$ such that $f^n(p) \in B(x, \delta)$. This implies that $\{x, f(p), \dots, f^{n-1}(p), y\}$ is a strong ϵ-chain from x to y. Therefore f is a strong chain transitive map on $\Omega(f)$ and by Proposition 2.7, we can say that f is a strong chain transitive map. □

4. Strong chain recurrent points

A point $x \in X$ is called a strong d-chain recurrent point if x can be strong d-chain to itself. The set of strong d-chain recurrent points is called strong d-chain recurrent set of f and denoted by $\mathcal{SCR}_d(f)$.

Remark 4.1. The strong chain recurrent set does depend on the metric, see [11, Example 3.1].

If for $x \in X$, x can be chain to itself, it is called a chain recurrent point for f and the set of chain recurrent points is denoted by $\mathcal{CR}(f)$ and it depends only on the topology, not on the metric (see [5]). Note that the following inclusions always hold.

$$\Omega(f) \subseteq \mathcal{SCR}_{d_X}(f) \subseteq \mathcal{CR}(f).$$

It is easy to see that

- $\Omega(f) = \{x : (x,x) \in \limsup\{f^n\}\}$,
- $\mathcal{SCR}_d(f) = \{x : x \in \mathcal{SC}_d(x,f)\}$,
- $\mathcal{CR}(f) = \{x : x \in \mathcal{C}(x,f)\}$.

Example 4.2. Let $S^1 = \{(x,y) \,|\, x^2 + y^2 = 1\}$ and $f : S^1 \to S^1$ be a homeomorphism that fixes every point on the left semicircle S^1 and moves points on the right semicircle clockwise. It can be seen that for any metric d, $\mathcal{SCR}_d(f) = \{(x,y) \in S^1 \,|\, x < 0\}$ but $\mathcal{CR}(f) = S^1$.

It is known that if $f : (X,d) \to (X,d)$ has shadowing property, then $\mathcal{CR}(f) = \mathcal{SCR}(f) = \Omega(f)$, also the chain recurrent set does not depend on the metric. Hence if $f : (X,d) \to (X,d)$ has shadowing property, then the strong chain recurrent set does not depend on the metric. Also similar to Corollary 2.12, we can say that for generic homeomorphism f on a generalized homogeneous compact metric space, the strong chain recurrent set does not depend on the metric. It is easy to see that if there is $k \in \mathbb{N}$ such that $\frac{1}{k}d_1(x,y) \leq d_2(x,y) \leq kd_1(x,y)$, then

$$\mathcal{SCR}_{d_1}(f) = \mathcal{SCR}_{d_2}(f).$$

There is an example to show that $\mathcal{SCR}(f^2) \neq \mathcal{SCR}(f)$, see [11, Example 3.4]. In the following we extend it to every $n \in \mathbb{N}$, indeed we use some techniques of [11] to show that $\mathcal{SCR}(f^n) \neq \mathcal{SCR}(f)$ for all $n \in \mathbb{N}$.

Example 4.3. Let $n \in \mathbb{N}$ be given. For $p = 0,1,\ldots$ and $q = 0,1,\ldots,n^p$, let $a_{p,q} = (1 + \frac{1}{n^{n^p}})e^{i\frac{\pi q}{n^{p+1}}}$. Take $X = S^1 \cup_{p=0}^{\infty} \{a_{p,q} : q = 0,1,\ldots,n^p\}$. Let $g : X \to X$ be defined by $g(a_{p,q}) = \frac{a_{p,q+1}}{|a_{p,q+1}|}$ for $q = 0,1,\ldots,n^p - 1$ and $g(a_{p,n^p}) = e^{\frac{i\pi}{n^p}}$. Assume that g is constant on $[\frac{(2m)\pi}{n}, \frac{(2m+1)\pi}{n}]$ for $m = 0,1,\ldots$ and moves clockwise on $[\frac{(2m+1)\pi}{n}, \frac{(2m+2)\pi}{n}]$ for $m = 1,\ldots$. It is easy to see that $g_{|S^1}$ is a homeomorphism which fixes z or satisfies $arg g(z) > arg z$ for $z \in S^1$. The map h (on S^1) is defined by rotation about the origin through angle $\frac{2\pi}{n}$.

Consider the composite map $f = h \circ g$, similar to the proof of [11, Example 3.4], it can be seen that $\mathcal{SCR}(f) = S^1$ and

$$\mathcal{SCR}(f^n) = \{e^{i\theta} : \theta \in [\frac{(2m)\pi}{n}, \frac{(2m+1)\pi}{n}], m = 0, 1, \ldots\}.$$

Recall that a point $x \in X$ is an equicontinuity point of f if for every $\epsilon > 0$ there is a $\delta > 0$ such that given a point $y \in X$, $d(f^n(x), f^n(y)) < \epsilon$ holds for all $n \in \mathbb{N}$ whenever $d(x, y) < \delta$. If every $x \in X$ is a point of equicontinuity, then we say that f is equicontinuous.

Proposition 4.4. *For all $n \in \mathbb{N}$, $\mathcal{SCR}_d(f^n) = \mathcal{SCR}_d(f)$ provided that one of the following conditions holds.*

(1) *(X, d) is a compact metric space and $f^{-1} : (X, d) \to (X, d)$ is equicontinuous,*
(2) *(X, d) is a compact metric space and $f : (X, d) \to (X, d)$ is Lipschitz,*
(3) *(X, d) is a compact manifold and $f : X \to X$ is a diffeomorphism.*

Proof. (1) Let $f^{-1} : X \to X$ be equicontinuous. We claim every $x \in X$ is recurrent under f, i.e., $x \in \omega(x)$. By contradiction assume that there is $x \in X$ with $x \notin \omega(x)$ and $d(x, \omega(x, f)) = \epsilon > 0$. If $y \in \omega(x, f)$, then, by the equicontinuity of f^{-1}, there is $\delta > 0$ such that $d(f^{-i}(y), f^{-i}(z)) < \frac{\epsilon}{2}$, for all i provided that $d(y, z) < \delta$. But there is $m > 0$ such that $d(f^m(x), y) < \delta$ which is a contradiction. This implies that f is a recurrent homeomorphism and by [3, Proposition 3.4], $\mathcal{SCR}_d(f^n) = \mathcal{SCR}_d(f)$ for all $n \in \mathbb{N}$.

(2) See [11, Proposition 3.3].

(3) It is known that if $f : X \to X$ is a diffeomorphism on a compact smooth manifold X, then f is bi-Lipschitz with respect to the Riemannian distance functions, hence by item (2), $\mathcal{SCR}_d(f^n) = \mathcal{SCR}_d(f)$ for all $n \in \mathbb{N}$. □

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