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ON PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY (ε, δ)-TRANS SASAKIAN MANIFOLD

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Abstract. The purpose of the paper is to study the notion of pseudo-slant submanifolds and the existence of some structures on a pseudo-slant submanifolds of nearly (ε, δ)-trans-Sasakian manifold. Totally umbilical proper-slant submanifolds are worked out. We discuss the integrability of distributions on pseudo-slant submanifolds of nearly (ε, δ)-trans-Sasakian manifold.

1. Introduction

The concept of (ε)-Sasakian manifolds has been defined by Bejancu and Duggal in [1]. Later, Xufeng and Xiaoli in [12] introduced and studied that these manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. Kumar et al. in [6] also studied the curvature conditions of these manifolds and Tripathi et al. in [11] investigated (ε)-almost paracontact manifolds. De and Sarkar in [4] also introduced (ε)-Kenmotsu manifolds and studied conformally flat, Weyl semi-symmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. On the other hand Chen in [3] introduced the idea of slant submanifolds in natural generalization of both holomorphic and totally real immersions. A. Lotta in [7] also studied the idea of slant submanifolds of a Riemannian manifold into an almost contact metric manifold. L. Cabrerizo et al. in [2] defined slant submanifolds of Sasakian manifolds. Semi-slant submanifolds of an almost Hermitian manifold have been studied by N. Papagiuc in [8]. V. A. Khan et al. in [5] also defined contact version of pseudo-slant submanifold in a Sasakian manifold. In [9, 10], the author and et al., studied slant and pseudo slant submanifolds in quasi Sasakian manifolds and CR-submanifolds of a nearly (ε, δ)-trans-Sasakian manifold.

The purpose of the paper is to study the notion of pseudo-slant submanifolds and the existence of some structures on a pseudo-slant submanifolds of a

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nearly (ε, δ)-trans-Sasakian manifold. Also the purpose of the paper is to study the notion of a pseudo-slant submanifold of a nearly quasi-Sasakian manifold. In Section 2 we recall some results and formula for later use. In Section 3 we define a pseudo-slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold. In Section 4 we prove the classification theorem for totally umbilical pseudo-slant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold. In Section 5 it is concern with the integrability of the distributions on the pseudo-slant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold and obtain some characterizations.

2. Preliminaries

Let $\bar{M}$ be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$-tensor ϕ, a vector ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0. \tag{2.1}$$

An almost contact metric manifold $\bar{M}$ is called an (ε)-*almost contact metric* manifold if

$$\eta(X) = \varepsilon g(X, \xi), \quad g(\xi, \xi) = \varepsilon,$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \forall X, Y \epsilon TM, \tag{2.2}$$

where $\varepsilon = g(\xi, \xi) = \pm 1$.

An (ε)-almost contact metric manifold is called an (ε, δ)-*trans-Sasakian* manifold if

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \varepsilon\eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \delta\eta(Y)\phi X\},$$

$$\bar{\nabla}_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$$

holds for some smooth functions α and β on $\bar{M}$ and $\varepsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0, \alpha = 1$, an (ε, δ)-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha = 0$, $\beta = 1$, it reduces to a (δ)-Kenmotsu manifold.

Further, an (ε)-almost contact metric manifold is called *nearly* (ε, δ)*-trans-Sasakian* manifold if

$$\begin{aligned}(\bar{\nabla}_X \phi)Y &+ (\bar{\nabla}_Y \phi)X \\ &= \alpha\{2g(X, Y)\xi - \varepsilon\eta(Y)X - \varepsilon\eta(X)Y\} - \beta\delta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned} \tag{2.3}$$

The covariant derivative of the tensor filed ϕ is defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi\bar{\nabla}_X Y. \tag{2.4}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.6}$$

for any X, $Y \epsilon TM$ and $N \epsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.7}$$

For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\phi X = TX + NX \quad (TX \in TM \quad \text{and} \quad NX \in T^{\perp}M), \tag{2.8}$$

$$\phi V = tV + nV \quad (tV \in TM \quad \text{and} \quad nV \in T^{\perp}M). \tag{2.9}$$

The submanifold M is *invariant* if N is identically zero. On the other hand, M is *anti-invariant* if T is identically zero. From (2.1) and (2.8), we have

$$g(X, TY) = -g(TX, Y) \tag{2.10}$$

for any $X, Y \in TM$. If we put $Q = T^2$, we have

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{2.11}$$

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.12}$$

$$(\nabla_X V)Y = \nabla^{\perp}_X VY - V\nabla_X Y \tag{2.13}$$

for any $X, Y \in TM$. In view of (2.5), (2.8), and (2.4) it follows that

$$\bar{\nabla}_X \xi = -\varepsilon\alpha TX + \beta\delta X - \eta(X)\xi, \tag{2.14}$$

$$h(X, \xi) = -\varepsilon\alpha NX. \tag{2.15}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.16}$$

where n is the dimension of M and $e_1, e_2, \ldots, e_n$ is a local orthonormal frame of M. A submanifold M of an contact metric manifold $\bar{M}$ is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.17}$$

where H is the mean curvature vector. A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and M is said to be *minimal* if $H = 0$.

3. Pseudo-slant submanifolds of nearly (ε, δ)-trans-Sasakian manifold

Definition 3.1. Let M be a submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. For each non-zero vector X tangent to M at x, the angle $\theta(x) \in [0, \pi/2]$ between ϕX and TX is called the slant angle or the Wirtinger angle of M. If the slant angle is constant for each $X \in \Gamma(TM)$ and $x \in M$, then the submanifold is also called the *slant submanifold.* If $\theta = 0$, the submanifold is invariant submanifold. If $\theta = \pi/2$, then it is called an *anti-invariant submanifold.* If $\theta(x) \in [0, \pi/2]$, then it is called a *proper-slant submanifold.*

Now, we will give the definition of pseudo-slant submanifold which are a generalization of the slant submanifolds.

Definition 3.2. We say that M is a *pseudo-slant submanifold* of nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ if there exist two orthogonal distributions $D^{\perp}$ and D_θ on M such that

(a) TM admits the orthogonal direct decomposition $TM = D^{\perp} \oplus D_\theta$, $\xi = \Gamma(D)$,
(b) The distribution D_θ is anti-invariant, that is, $f(D_\theta) \subseteq T^{\perp}M$,
(c) The distribution $D^{\perp}$ is a slant with slant angle $\theta \neq 0$, that is, the angle between $f(D^{\perp})$ and $D^{\perp}$ is a constant.

From the definition, it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if $\theta = \pi/2$, submanifold becomes an anti-invariant. On the other hand we suppose that M is a pseudo-slant submanifold of nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ and we denote the dimensions of distributions $D^{\perp}$ and D_θ by d_1 and d_2 respectively, then we have the following cases:

(a) If $d_2 = 0$, then M is an anti-invariant submanifold,
(b) If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold,
(c) If $d_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ,
(d) If $d_1.d_2 \neq 0$ and $\theta \in [0, \pi/2]$, then M is a proper pseudo-slant submanifold.

Let M be a proper pseudo-slant submanifold of a contact metric manifold $\bar{M}$ and the projections on $D^{\perp}$ and D_θ by P_1 and P_2 respectively. Then for any vector field $X \in \Gamma(TM)$, we can write as

$$X = P_1X + P_2X + \eta(X)\xi. \tag{3.1}$$

Now applying ϕ on both sides of equation (3.1), we obtain

$$\phi X = \phi P_1X + \phi P_2X,$$

that is,

$$PX + NX = NP_1X + PP_2X + NP_2X. \tag{3.2}$$

We can easily to see

$$PX = PP_2X, \quad NX = NP_1X + NP_2X \quad \text{and} \tag{3.3}$$

$$\phi P_1X = NP_1X, \quad TP_1X = 0, \quad \phi P_2X = TP_2X + NP_2X, \tag{3.4}$$

$$TP_2X \in \Gamma(D_\theta). \tag{3.5}$$

If we denote the orthogonal complementary of ϕTM in $D^\perp M$ by μ, then the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = N(D^\perp) \oplus N(D_\theta) \oplus \mu, \tag{3.6}$$

where μ is an invariant subbundle of $T^\perp M$ as $N(D^\perp)$ and $N(D_\theta)$ are orthogonal distribution on M. Indeed, $g(Z, X) = 0$ for each $Z \in \Gamma(D^\perp)$ and $X \in \Gamma(D_\theta)$. Thus, by equation (2.1) and (3.5), we can write

$$g(NZ, NX) = g(\phi Z, \phi X) = g(Z, X) = 0, \tag{3.7}$$

that is, the distributions $N(D^\perp)$ and $N(D_\theta)$ are mutually perpendicular. In fact, the decomposition (3.6) is an orthogonal direct decomposition.

Theorem 3.1. *Let M be a slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda\{I - \varepsilon\eta \otimes \xi\}. \tag{3.8}$$

Furthermore, in such a case if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Let M be a slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with slant angle $\theta, \xi \in TM$. Then

$$\cos\theta = \frac{g(TX, \phi X)}{||TX||||\phi X||} = \frac{||TX||}{||\phi X||} = \text{constant} \tag{3.9}$$

for $X \in TM$. In this case, from (3.9) we have

$$\cos\theta||\phi X|| = ||TX||. \tag{3.10}$$

If we take the square (3.10) we get

$$g(TX, TX) = g(\phi X, \phi X)\cos^2\theta. \tag{3.11}$$

Moreover,

$$\cos\theta = \frac{g(TX, \phi X)}{||TX||||\phi X||} = \frac{g(\phi X, TX)}{||TX||||\phi X||} = -\frac{g(X, \phi TX)}{||TX||||\phi X||} = -\frac{g(X, T^2X)}{||TX||||\phi X||} \tag{3.12}$$

for all vector field X. Then from (2.1), (3.9) and (3.12) we find

$$\cos^2\theta\{g(X - \varepsilon\eta(X)\xi, X)\} = -g(T^2X, X).$$

In this case, we have

$$T^2X = -\cos^2\theta\{X - \varepsilon\eta(X)\xi, X\}$$

for any vector field $X \in TM$. That is, for all $X \in TM$

$$T^2 = -\lambda\{I - \varepsilon\eta \otimes \xi\}, \quad \lambda = \cos^2\theta. \tag{3.13}$$

Conversely, we now assume that the (3.8) holds. Then from (2.2) and (2.8) we obtain

$$\cos\theta = \frac{g(TX, \phi X)}{||TX||||\phi X||} = \frac{g(T^2X, X)}{||TX||||\phi X||} = -\frac{-\lambda g(X - \varepsilon\eta(X)\xi, X)}{||TX||||\phi X||} = -\lambda\frac{||\phi X||}{||TX||}.$$

Also by using (3.9), we conclude that

$$\lambda = \cos^2\theta, \tag{3.14}$$

where θ is constant because λ is a constant, and so M is slant. □

Corollary 3.1. *Let M be a slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$ we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \varepsilon\eta(X)\eta(Y)), \tag{3.15}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \varepsilon\eta(X)\eta(Y)). \tag{3.16}$$

Proof. For any $X, Y \in TM$, we have

$$g(TX, TY) = -g(X, T^2Y).$$

Then by virtue of (3.8), we obtain (3.15). The proof of (3.16) follows from (2.2) and (2.8). □

4. Totally umbilical pseudo-slant submanifolds

Theorem 4.1. *Let M be a totally umbilical pseudo-slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Then at least one of the following statements is true.*

(i) $\dim(D^{\perp}) = 1$,
(ii) $H \in \Gamma(\mu)$,
(iii) *M is a proper pseudo-slant submanifold.*

Proof. Let $Z \in \Gamma(D^{\perp})$ and using (2.3), we obtain

$$(\bar{\nabla}_Z\phi)Z = \alpha\{g(Z, Z)\xi - \varepsilon\eta(Z)Z\} - \beta\delta\eta(Z)\phi Z,$$

$$\bar{\nabla}_Z NZ - \phi(\bar{\nabla}_Z Z + h(Z, Z)) = \alpha\{g(Z, Z)\xi - \varepsilon\eta(Z)Z\} - \beta\delta\eta(Z)\phi Z.$$

From the last equation, we have

$$\begin{aligned} & - A_{NZ}Z + \nabla^{\perp}_Z NZ - N\nabla_Z Z - th(Z, Z) - nh(Z, Z) \\ & = \alpha\{g(Z, Z)(T\xi + N\xi) - \varepsilon\eta(Z)Z\} - \beta\delta\eta(Z)(TZ + NZ). \end{aligned} \tag{4.1}$$

From (2.12) and from the tangential components of (4.1), we obtain

$$-A_{NZ}Z - th(Z, Z) = \alpha g(Z, Z)T\xi - \alpha\varepsilon\eta(Z)Z - \beta\delta\eta(Z)TZ. \tag{4.2}$$

Taking the product by $W \in \Gamma(D^{\perp})$, we obtain

$$g(A_{NZ}Z + th(Z, Z) + \alpha g(Z, Z)T\xi - \alpha\varepsilon\eta(Z)Z - \beta\delta\eta(Z)TZ, W) = 0.$$

It implies that

$$g(h(Z,W),NZ) + g(th(Z,Z),W) + \alpha g(Z,Z)g(T\xi,W) -\alpha\varepsilon\eta(Z)g(Z,W) - \beta\delta\eta(Z)g(TZ,W) = 0. \tag{4.3}$$

Since M is a totally umbilical submanifold, we obtain

$$g(Z,W)g(H,NZ) + g(Z,Z)g(tH,W) + \alpha g(Z,Z)g(T\xi,W) - \alpha\varepsilon g(Z,\xi)g(Z,W) + \beta\delta g(Z,\xi)g(Z,W) = 0, \tag{4.4}$$

that is

$$g(tH,Z)W + g(tH,W)Z + \alpha g(T\xi,W)Z - \alpha\varepsilon\eta(Z)W + \beta\delta\eta(Z)W = 0. \tag{4.5}$$

Here tH is either zero or Z and W are linearly dependent vector fields. If $tH = 0$, then $\dim\Gamma(D^\perp) = 1$. Otherwise $H \in \Gamma(\mu)$. Since $D_\theta = 0$, M is a pseudo-slant submanifold. Since $\theta = 0$ and $d_1.d_2 = 0$, M is a proper pseudo-slant submanifold. □

Theorem 4.2. *Let M be a totally umbilical proper pseudo-slant submanifold of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$. Then M is neither a totally geodesic submanifold nor an anti-invariant if $H, \nabla^\perp_X H \in \Gamma(\mu)$.*

Proof. Since the ambient space is a nearly quasi-Sasakian manifold, by using (2.3) we have for any $X \in \Gamma(TM)$,

$$\begin{aligned}(\bar{\nabla}_X\phi)X &= \alpha\{g(X,X)\xi - \varepsilon\eta(X)X\} - \beta\delta\eta(X)\phi X,\\ \bar{\nabla}_X\phi X - \phi\bar{\nabla}_X X &= \alpha\{g(X,X)\xi - \varepsilon\eta(X)X\} - \beta\delta\eta(X)\phi X.\end{aligned} \tag{4.6}$$

Using (2.5), (2.7), (2.8) and (2.12) in (4.6), we get

$$\begin{aligned}&\nabla_X TX + g(X,TX)H - A_{NX}X + \nabla^\perp_X NX\\ &= \phi\nabla_X X + g(X,X)\phi H + \alpha\{g(X,X)\xi - \varepsilon\eta(X)X\}\\ &\quad - \beta\delta\eta(X)TX - \beta\delta\eta(X)NX.\end{aligned} \tag{4.7}$$

Applying product ϕH to the above equation we get

$$g(\nabla^\perp_X NX,\phi H) = g(N\nabla_X X,\phi H) + g(X,X)\|H\|^2 + \alpha g(X,X)g(N\xi,\phi H) - \beta\delta\eta(X)g(NX,\phi H). \tag{4.8}$$

Taking into account (2.6), we get

$$g(\bar{\nabla}^\perp_X NX,\phi H) = g(X,X)\|H\|^2 + \alpha g(X,X)g(N\xi,\phi H) - \beta\delta\eta(X)\eta(X)g(N\xi,\phi H). \tag{4.9}$$

Now, for any $X \in \Gamma(TM)$, we obtain

$$(\bar{\nabla}_X\phi)H = \bar{\nabla}_X\phi H - \phi\bar{\nabla}_X H. \tag{4.10}$$

In view of (2.6), (2.8), (2.9), (2.17) and (4.10) we obtain

$$-A_{\phi H}X + \nabla^\perp_X\phi H = (\bar{\nabla}_X\phi)H - TA_H X - NA_H X + t\nabla^\perp_X H + n\nabla^\perp_X H. \tag{4.11}$$

Applying product NX to the above equation we get

$$g(\nabla^{\perp}_X \phi H, NX) = g((\bar{\nabla}_X \phi)H, NX) - g(NA_H X, NX),$$

$$g(\bar{\nabla}_X \phi H, NX) = g((\nabla_X n)H + h(tH, X) + NA_H X, NX) - g(NA_H X, NX).$$

By using (2.7), (2.17) and (3.16), we have

$$g(\bar{\nabla}_X \phi H, NX) = -\sin^2\theta\{g(X,X)\|H\|^2 - g(h(X,\xi), H)\eta(X)\}.$$

From (2.15), we obtain

$$\begin{aligned} g(\bar{\nabla}_X \phi H, NX) &= -\sin^2\theta\{g(X,X)\|H\|^2 - g(-\varepsilon\alpha NX, H)\eta(X)\}, \\ g(\bar{\nabla}_X NX, \phi H) &= -\sin^2\theta\{g(X,X)\|H\|^2 + \varepsilon\alpha g(NX, H)\eta(X)\}. \end{aligned} \tag{4.12}$$

Thus, (4.9) and (4.12) imply

$$\begin{aligned} &g(X,X)\|H\|^2 + \alpha g(X,X)g(N\xi, \phi H) - \beta\delta\eta(X)g(NX, \phi H) \\ &= \sin^2\theta\{g(X,X)\|H\|^2\}, \end{aligned}$$

$$\cos^2\theta g(X,X)\|H\|^2 + \alpha g(X,X)g(N\xi,\phi H) - \beta\delta\eta(X)g(NX,\phi H) = 0. \tag{4.13}$$

Since M is a proper pseudo-slant submanifold of nearly (ε, δ)-trans-Sasakian manifold, we can not obtain $H = 0$. This tells us that M is not totally geodesic in $\bar{M}$. □

Theorem 4.3. *Let M be a totally umbilical proper pseudo-slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Then at least one of the following statements is true.*

(i) $H \in \mu$,
(ii) $g(\nabla_{TX}\xi, X) = 0$,
(iii) $\eta((\nabla_X T)X) = 0$,
(iv) *M is an anti-invariant submanifold,*
(v) *If M is a proper slant submanifold, then* $\dim(M) > 3$ *for any* $X \in \Gamma(TM)$.

Proof. From equation (2.3) and M is a nearly (ε, δ)-trans-Sasakian manifold, we have

$$\bar{\nabla}_X \phi X - \phi\bar{\nabla}_X X = \alpha\{g(X,X)\xi - \varepsilon\eta(X)X\} - \beta\delta\eta(X)\phi X.$$

By using (2.5), (2.6), (2.8) and (2.9), we have

$$\begin{aligned} &\nabla_X TX + h(X, TX) - A_{NX}X + \nabla^{\perp}_X NX - T\nabla_X X \\ &- N\nabla_X X - th(X,X) - nh(X,X) \\ &= \alpha\{g(X,X)\xi - \varepsilon\eta(X)X\} - \beta\delta\eta(X)\phi X. \end{aligned} \tag{4.14}$$

Taking the tangential components of (4.14), we obtain

$$\nabla_X TX - T\nabla_X X - th(X,X) - A_{NX}X = -\alpha\varepsilon\eta(X)X - \beta\delta\eta(X)TX. \tag{4.15}$$

Since M is a totally umbilical pseudo-slant submanifold, by using (2.7) and (2.17), we can write

$$g(A_{NX}X, X) = 0. \tag{4.16}$$

If $H \in \Gamma(\mu)$, then from (4.15), we obtain

$$\nabla_X TX - T\nabla_X X = -\alpha\varepsilon\eta(X)X - \beta\delta\eta(X)TX.$$

Taking the product of (4.16) by ξ, we obtain

$$g(\nabla_X TX, \xi) - \eta(T\nabla_X X) = -\alpha\varepsilon\eta(X)\eta(X) - \beta\delta\eta(X)\eta(TX),$$

$$g(\nabla_X TX, \xi) = 0. \tag{4.17}$$

Interchanging X by TX in (4.17), we derive

$$g(\nabla_{TX} T^2 X, \xi) = 0 \quad \Rightarrow \quad g(\nabla_{TX}\xi, T^2 X) = 0.$$

By using (3.1), we have

$$g(\nabla_{TX}\xi, -\cos^2\theta(X - \eta(X)\xi)) = 0 \quad \Rightarrow \quad \cos^2\theta g(\nabla_{TX}\xi, (X - \eta(X)\xi)) = 0.$$

Since, M is a proper pseudo-slant submanifold, we have

$$g(\nabla_{TX}\xi, (X - \eta(X)\xi) = 0,$$

from which

$$g(\nabla_{TX}\xi, X) = \eta(X) g(\nabla_{TX}\xi, \xi). \tag{4.18}$$

Now, we have $g(\xi, \xi) = 1$. Taking the covariant derivative of above equation with respect to TX for any $X \in \Gamma(TM)$, we obtain $g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0$, which implies $g(\nabla_{TX}\xi, \xi) = 0$ and then (4.18) gives

$$g(\nabla_{TX}\xi, X) = 0. \tag{4.19}$$

This proves (ii) of theorem.

Now, interchanging X by TX in the equation (4.19), we derive

$$g(\nabla_{T^2X}\xi, TX) = g(\nabla_{\cos^2\theta(-X+\eta(X)\xi)}\xi, TX) = 0,$$

$$\cos^2\theta g(\nabla_{(-X+\eta(X)\xi)}\xi, TX) = 0,$$

$$-\cos^2\theta g(\nabla_X\xi, TX) + \cos^2\theta\eta(X) g(\nabla_\xi\xi, TX) = 0.$$

Since $\nabla_\xi\xi = 0$, we obtain

$$\cos^2\theta g(\nabla_X\xi, TX) = 0. \tag{4.20}$$

From (4.20), if $\cos\theta = 0, \theta = \pi/2$, then M is an anti-invariant submanifold. On the other hand, $g(\nabla_X\xi, TX) = 0$, that is, $\nabla_X\xi = 0$. This implies that ξ is a the Killing vector field on M. If the vector field ξ is not Killing, then we can take at least two linearly independent vectors X and PX to span D_θ, that is, the $\dim(M) > 3$. □

5. Integrability of distributions

Theorem 5.1. *Let M be a pseudo-slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Then we have*

$$\begin{aligned} A_{\phi Y}X - A_{\phi X}Y = & -\alpha\varepsilon g(X,Y)\xi - \beta\delta g(\phi X, Y)\xi - \nabla_X(TY) \\ & - h(X,TY) + A_{NY}X - \nabla^\perp_X(NY) + T(\nabla_X Y) \\ & + N(\nabla_X Y) + N(h(X,Y)) \end{aligned} \tag{5.1}$$

for all $X, Y \in D^\perp$.

Proof. In view of (2.7), we get

$$g(A_{\phi Y}X, Z) = g(h(X,Z), \phi Y) = -g(\phi h(X,Z), Y). \tag{5.2}$$

From (2.5) and (5.2), since $\phi\nabla_Z X \in T^\perp M$ we get

$$\begin{aligned} g(A_{\phi Y}X, Z) &= -g(\phi\bar{\nabla}_Z X, Y) + g(\phi\nabla_Z X, Y) \\ &= -g(\phi\bar{\nabla}_Z X, Y) \\ &= -g(\bar{\nabla}_Z \phi X, Y) + g((\bar{\nabla}_Z\phi)X, Y). \end{aligned} \tag{5.3}$$

Now, for $X \in D_1$, $\phi X \in T^\perp M$. Hence, from (2.6) we have

$$\bar{\nabla}_Z \phi X = -A_{\phi X}Z + \nabla^\perp_Z \phi X. \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$g(A_{\phi Y}X, Z) = g((\bar{\nabla}_Z\phi)X, Y) + g(A_{\phi X}Z, Y). \tag{5.5}$$

Since $h(X,Y) = h(Y,X)$, if follows from (2.7) that

$$g(A_{\phi X}Z, Y) = g(A_{\phi X}Y, Z).$$

Hence, from (5.5) we obtain, with the help of (2.3),

$$\begin{aligned} & g(A_{\phi Y}X, Z) - g(A_{\phi X}Y, Z) = 2\alpha\eta(Y)g(X,Z) - \varepsilon\alpha\eta(Z)g(X,Y) \\ & \quad - \varepsilon\alpha\eta(X)g(Z,Y) - \beta\delta\eta(Z)g(\phi X, Y) + \eta(X)g(\phi Z, Y) - g((\bar{\nabla}_X\phi)Z, Y) \\ & = 2\alpha\eta(Y)g(X,Z) - \varepsilon\alpha\eta(Z)g(X,Y) - \varepsilon\alpha\eta(X)g(Z,Y) - \beta\delta\eta(Z)g(\phi X, Y) \\ & \quad + \eta(X)g(\phi Z, Y) + g(\nabla_X(TY) + h(X,TY) - A_{NY}X + \nabla^\perp_X NY \\ & \quad - T(\nabla_X Y) - N(\nabla_X Y) - Th(X,Y) - Nh(X,Y), Z). \end{aligned} \tag{5.6}$$

Since $X, Y, Z \in D^\perp$ an orthonormal distribution to the distribution $\langle\xi\rangle$, it follows that $\eta(X) = \eta(Y) = 0$. Therefore, the above equation reduces to

$$\begin{aligned} A_{\phi Y}X - A_{\phi X}Y = & -\alpha\varepsilon g(X,Y)\xi - \beta\delta g(\phi X, Y)\xi - \nabla_X(TY) \\ & - h(X,TY) + A_{NY}X - \nabla^\perp_X(NY) + T(\nabla_X Y) \\ & + N(\nabla_X Y) + N(h(X,Y)). \end{aligned}$$

□

Theorem 5.2. *In a pseudo-slant submanifold of a nearly (ε, δ)-trans-Sasakian manifold, we have*

$$
\begin{aligned}
(5.7) \qquad (\nabla_X T)Y = {} & A_{NY}X + A_{NX}Y + th(X,Y) - (\nabla_Y T)X + T(h(Y,X)) \\
& + 2\alpha g(X,Y)\xi - \alpha\varepsilon\eta(Y)X - \alpha\varepsilon\eta(X)Y - \beta\delta\eta(X)TY \\
& - \beta\delta\eta(Y)TX.
\end{aligned}
$$

Proof. Let $X, Y \in TM$, we have

$$(\bar\nabla_X \phi)Y = \bar\nabla_X \phi Y - \phi(\bar\nabla_X Y).$$

From (2.7) and (2.8), we obtain

$$\bar\nabla_X TY + \bar\nabla_X NY = (\bar\nabla_X \phi)Y + \phi(\nabla_X Y + h(X,Y)).$$

Also from (2.8) and (2.9), we obtain

$$\bar\nabla_X TY + \bar\nabla_X NY = (\bar\nabla_X \phi)Y + T(\nabla_X Y) + N(\nabla_X Y) + th(X,Y) + nh(X,Y).$$

Using (2.5) and (2.6) from above, we obtain

$$
\begin{aligned}
(5.8) \qquad & \nabla_X TY + h(X,TY) - A_{NY}X + \nabla^\perp_X NY \\
& = 2\alpha g(X,Y)\xi - \alpha\varepsilon\eta(Y)X - \alpha\varepsilon\eta(X)Y - \beta\delta\eta(Y)(TX + NX) \\
& \quad - \beta\delta\eta(X)(TY + NY) - \nabla_Y TX - h(Y,TX) + A_{NX}Y \\
& \quad - \nabla^\perp_Y NX + T\nabla_Y X + N\nabla_Y X + T(h(Y,X)) + N(h(Y,X)) \\
& \quad + T\nabla_X Y + N\nabla_X Y + th(X,Y) + nh(X,Y).
\end{aligned}
$$

Comparing tangential and normal parts of the above equation, we obtain

$$
\begin{aligned}
(5.9) \qquad \nabla_X TY - A_{NY}X = {} & 2\alpha g(X,Y)\xi - \alpha\varepsilon\eta(Y)X - \alpha\varepsilon\eta(X)Y \\
& - \beta\delta\eta(Y)TX - \beta\delta\eta(X)TY - \nabla_Y TX + A_{NX}Y \\
& + T(\nabla_Y X) + T(h(Y,X) + T\nabla_X Y + th(Y,X).
\end{aligned}
$$

That is,

$$
\begin{aligned}
(5.10) \qquad (\nabla_X T)Y = {} & A_{NY}X + A_{NX}Y + th(X,Y) - (\nabla_Y T)X + T(h(Y,X)) \\
& + 2\alpha g(X,Y)\xi - \alpha\varepsilon\eta(Y)X - \alpha\varepsilon\eta(X)Y \\
& - \beta\delta\eta(X)TY - \beta\delta\eta(Y)TX.
\end{aligned}
$$

□

Theorem 5.3. *Let M be a pseudo-slant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold $\bar M$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$
\begin{aligned}
(5.11) \qquad & A_{NW}Z + A_{NZ}W + 2T\nabla_Z W + 2th(W,Z) \\
& = -2\alpha g(Z,W)\xi + \alpha\varepsilon\eta(W)Z + \alpha\varepsilon\eta(Z)W + \beta\delta\eta(W)TZ + \beta\delta\eta(Z)TW
\end{aligned}
$$

for any $Z, W \in \Gamma(D^\perp)$.

Proof. Let $Z, W \in \Gamma(D^{\perp})$ and using (2.3), we obtain

$$(\bar{\nabla}_Z\phi)W + (\bar{\nabla}_W\phi)Z = \alpha\{2g(Z,W)\xi - \varepsilon\eta(W)Z - \varepsilon\eta(Z)W\} - \beta\delta\{\eta(W)\phi Z + \eta(Z)\phi W\},$$

which is equivalent to

$$\begin{aligned} &\bar{\nabla}_Z\phi W - \phi\bar{\nabla}_Z W + \bar{\nabla}_W\phi Z - \phi\bar{\nabla}_W Z \\ &= \alpha\{2g(Z,W)\xi - \varepsilon\eta(W)Z - \varepsilon\eta(Z)W\} - \beta\delta\{\eta(W)\phi Z + \eta(Z)\phi W\}. \end{aligned}$$

By using (2.5), (2.6), (2.8) and (2.9), we obtain

$$\begin{aligned} &\alpha\{2g(Z,W)\xi - \varepsilon\eta(W)Z - \varepsilon\eta(Z)W\} - \beta\delta\{\eta(W)\phi Z + \eta(Z)\phi W\} \\ &= \bar{\nabla}_Z NW - T\nabla_Z W - N\nabla_Z W - th(W,Z) - nh(W,Z) \\ &\quad + \bar{\nabla}_W NZ - T\nabla_W Z - N\nabla_W Z - th(W,Z) - nh(W,Z). \end{aligned}$$

So we have

$$\begin{aligned} &\alpha\{2g(Z,W)\xi - \varepsilon\eta(W)Z - \varepsilon\eta(Z)W\} - \beta\delta\{\eta(W)\phi Z + \eta(Z)\phi W\} \\ &= -A_{NW}Z + \nabla^{\perp}_Z NW - T\nabla_Z W - N\nabla_Z W - 2th(W,Z) - A_{NZ}W \\ &\quad + \nabla^{\perp}_W NZ - T\nabla_W Z - N\nabla_W Z - 2nh(W,Z). \end{aligned}$$

Corresponding the tangential components of the last equation, we conclude

$$\begin{aligned} &-2\alpha g(Z,W)\xi + \alpha\varepsilon\eta(W)Z + \alpha\varepsilon\eta(Z)W + \beta\delta\eta(W)TZ + \beta\delta\eta(Z)TW \\ &= A_{NW}Z + T\nabla_Z W + 2th(W,Z) + A_{NZ}W + T\nabla_W Z. \end{aligned}$$

From the above equation, we can infer

$$\begin{aligned} &-2\alpha g(Z,W)\xi + \alpha\varepsilon\eta(W)Z + \alpha\varepsilon\eta(Z)W + \beta\delta\eta(W)TZ + \beta\delta\eta(Z)TW \\ &= A_{NW}Z + A_{NZ}W + 2T\nabla_Z W - T(\nabla_Z W - \nabla_W Z) + 2th(W,Z), \end{aligned}$$

$$\begin{aligned} T[Z,W] = &-A_{NW}Z - A_{NZ}W - 2T\nabla_W Z - 2th(W,Z) - 2\alpha g(Z,W)\xi \\ &+ \alpha\varepsilon\eta(W)Z + \alpha\varepsilon\eta(Z)W + \beta\delta\eta(W)TZ + \beta\delta\eta(Z)TW. \end{aligned}$$

Thus $[Z,W] \in \Gamma(D^{\perp})$ if and only if (5.11) is satisfied. □

Theorem 5.4. *Let M be a pseudo-slant submanifold of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if*

$$\begin{aligned} P_1\{&\nabla_X TY - T\nabla_X Y + (\nabla_Y T)X - A_{NX}Y - A_{NY}X - 2th(X,Y) \\ &+ \alpha\varepsilon\eta(Y)X + \alpha\varepsilon\eta(X)Y + \beta\delta\eta(X)TY + \beta\delta\eta(Y)TX\} = 0 \end{aligned} \tag{5.12}$$

for any $X, Y \in \Gamma(D_\theta)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and we denote the projections on $D^{\perp}$ and D_θ by P_1 and P_2 respectively. Then for any vector fields $X, Y \in \Gamma(D_\theta)$, by using equation (2.4), we obtain

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \varepsilon\eta(Y)X - \varepsilon\eta(X)Y\}$$

$$- \beta\delta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$

or

$$\begin{aligned}&\bar{\nabla}_X \phi Y - \phi\bar{\nabla}_X Y + \bar{\nabla}_Y \phi X - \phi\bar{\nabla}_Y X \\ &= \alpha\{2g(X,Y)\xi - \varepsilon\eta(Y)X - \varepsilon\eta(X)Y\} - \beta\delta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned}$$

By using equations (2.5), (2.6), (2.8), and (2.9), we can write

$$\begin{aligned}&\bar{\nabla}_X TY + \bar{\nabla}_X NY - \phi(\nabla_X Y + h(X,Y)) + \bar{\nabla}_Y TX + \bar{\nabla}_Y NX \\ &- \phi(\nabla_Y X + h(X,Y)) \\ &= \alpha\{2g(X,Y)\xi - \varepsilon\eta(Y)X - \varepsilon\eta(X)Y\} - \beta\delta\{\eta(Y)\phi X + \eta(X)\phi Y\},\end{aligned}$$

$$\begin{aligned}&\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y \\ &- th(X,Y) - nh(X,Y) + \nabla_Y TX + h(Y,TX) - A_{NX}Y \\ &+ \nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - th(X,Y) - nh(X,Y) \\ &= \alpha\{2g(X,Y)\xi - \varepsilon\eta(Y)X - \varepsilon\eta(X)Y\} - \beta\delta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned} \tag{5.13}$$

Tangential components of (5.13) reach

$$\begin{aligned}&\nabla_X TY - T\nabla_X Y + (\nabla_Y T)X - A_N XY - A_N YX - 2th(X,Y) \\ &= -\alpha\varepsilon\eta(Y)X - \alpha\varepsilon\eta(X)Y - \beta\delta\eta(X)TY - \beta\delta\eta(Y)TX,\end{aligned} \tag{5.14}$$

$$\begin{aligned}T[X,Y] = &\nabla_X TY - T\nabla_X Y + (\nabla_Y T)X - A_{NX}Y - A_{NY}X \\ &- 2th(X,Y) + \alpha\varepsilon\eta(Y)X + \alpha\varepsilon\eta(X)Y \\ &+ \beta\delta\eta(X)TY + \beta\delta\eta(Y)TX.\end{aligned} \tag{5.15}$$

Applying P_1 to (5.15), we get (5.12). □

Theorem 5.5. *Let M be a pseudo-slant submanifold of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp} \oplus \xi$ is integrable if and only if*

$$\begin{aligned}3\{A_{\phi Z}W - A_{\phi W}Z\} = &2\alpha(1-\varepsilon)\{\eta(W)Z - \eta(Z)W\} \\ &- \beta\delta\{\eta(TW)Z - \eta(TZ)W + \eta(W)TZ - \eta(Z)TW\}\end{aligned}$$

for any $Z, W \in \Gamma(D^{\perp} \oplus \xi)$.

Proof. For any $Z, W \in \Gamma(D^{\perp} \oplus \xi)$ and $U \in \Gamma(TM)$, by using (2.7), we can write

$$2g(A_{\phi Z}W, U) = g(h(U,W), \phi Z) + g(h(U,W), \phi Z).$$

By using (2.5), we have

$$\begin{aligned}2g(A_{\phi Z}W, U) &= g(\bar{\nabla}_W U, \phi Z) + g(\bar{\nabla}_U W, \phi Z) \\ &= -g(\phi\bar{\nabla}_W U, Z) - g(\phi\bar{\nabla}_U W, Z).\end{aligned}$$

So we have

$$2g(A_{\phi Z}W, U) = -g(\bar{\nabla}_W \phi U, Z) - g(\bar{\nabla}_U \phi W, Z)$$

$$+ g((\bar{\nabla}_W \phi)U + (\bar{\nabla}_U \phi)W, Z).$$

By using equation (2.3), we obtain

$$\begin{aligned}
2g(A_{\phi Z}W, U) = & -g(\bar{\nabla}_W \phi U, Z) - g(\bar{\nabla}_U \phi W, Z) + g(\alpha(2g(W,U)\xi \\
& - \varepsilon\eta(U)W - \varepsilon\eta(W)U\} - \beta\delta\{\eta(U)\phi W + \eta(W)\phi U), Z) \\
= & \, g(\bar{\nabla}_W Z, \phi U) - g(-A_{\phi W}U, Z) + g(\alpha(2g(W,U)\xi \\
& - \varepsilon\eta(U)W - \varepsilon\eta(W)U\} - \beta\delta\{\eta(U)\phi W + \eta(W)\phi U), Z) \\
= & -g(T\nabla_W Z + th(Z,W), U) + g(A_{\phi W}Z, U) + 2\alpha g(\eta(W)Z, U) \\
& - \alpha\varepsilon g(\eta(W)Z, U) - \alpha\varepsilon g(\eta(W)Z, U) - \beta\delta g(\eta(TW)Z, U) \\
& - \beta\delta g(\eta(W)TZ, U),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
2A_{\phi Z}W = & \, 2\alpha(1-\varepsilon)\eta(W)Z - \beta\delta\eta(TW)Z - \beta\delta\eta(W)TZ \\
& - T\nabla_W Z - th(Z,W) + A_{\phi W}Z.
\end{aligned} \tag{5.16}$$

Take $Z = W$ in (5.16), we infer

$$\begin{aligned}
2A_{\phi W}Z = & \, 2\alpha(1-\varepsilon)\eta(Z)W - \beta\delta\eta(TZ)W - \beta\delta\eta(T)TW \\
& - T\nabla_Z W - th(W,Z) + A_{\phi Z}W.
\end{aligned} \tag{5.17}$$

By using equation (5.16) and (5.17), we obtain

$$\begin{aligned}
3\{A_{\phi Z}W - A_{\phi W}Z\} = & \, 2\alpha(1-\varepsilon)\{\eta(W)Z - \eta(Z)W\} \\
& - \beta\delta\{\eta(TW)Z - \eta(TZ)W + \eta(W)TZ - \eta(Z)TW\}.
\end{aligned}$$

Thus the distribution $D^{\perp} \oplus \xi$ is integrable if and only if $T[Z, W] = 0$, which proves our assertion. □

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