

## ON A TYPE OF GENERALIZED SYMMETRIC MANIFOLDS

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ABSTRACT. The object of the present paper is to study generalized pseudo-projectively symmetric manifolds and Einstein generalized pseudo-projectively symmetric manifolds. Finally, the existence of generalized pseudo-projectively symmetric manifolds have been proved by two non-trivial examples.

### 1. Introduction

A Riemannian manifold  $(M^n, g)$  of dimension  $n$  is said to be locally symmetric if its curvature tensor  $R$  satisfies  $\nabla R = 0$ , where  $\nabla$  is the Levi-Civita connection of the Riemannian metric [4]. This condition of locally symmetric is equivalent to the fact that at every point  $p \in M^n$ , the local geodesic symmetry is an isometry [22]. The class of Riemannian symmetric manifolds is very natural generalization of the class of constant curvature manifolds. Many authors have been investigated the notion of locally symmetric manifolds in several way to a different extent such as conformally symmetric manifolds by Chaki and Gupta [6], for recent results on conformally symmetric manifolds we refer to [1, 16, 17], semi-symmetric manifolds by Szabo [25], pseudo symmetric manifolds introduced by Chaki [5], weakly symmetric manifolds by Tamassy and Binh [26] and also by De and Bandyopadhyay [8], Symmetric space by Desai and Amur [14], Decomposable weakly symmetric manifold introduced by Binh [3] and also by Hui et al. [19] etc.

A non-flat Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) is said to be a pseudo symmetric manifold [5] if its curvature tensor  $R$  satisfies the condition

$$(1) \quad \begin{aligned} (\nabla_X R)(Y, Z, U, V) &= 2A(X)R(Y, Z, U, V) + A(Y)R(X, Z, U, V) \\ &+ A(Z)R(Y, X, U, V) + A(U)R(Y, Z, X, V) \\ &+ A(V)R(Y, Z, U, X), \end{aligned}$$

for all vector fields  $X, Y, Z, U$  on  $M^n$ , where  $\nabla$  is the operator of covariant differentiation with respect to the metric tensor  $g$  and  $A$  is a non-zero associated

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1-form of the manifold defined by

$$(2) \quad g(X, \rho) = A(X),$$

where  $\rho$  is a vector field. If  $A = 0$ , then the manifold reduces to a symmetric manifold. An  $n$ -dimensional pseudo symmetric manifold is denoted by  $(PS)_n$ . This is to be noted that the notion of pseudo symmetric manifold studied by Chaki [5] is different from that of Deszcz and Grycak [15]. Pseudo symmetric manifolds have been studied by several authors [5, 9–11, 13, 28].

In 2008, De and Gazi [9] introduced the notion of almost pseudo symmetric manifolds on a non-flat Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) whose curvature tensor  $R$  of type (0,4) satisfies the condition:

$$(3) \quad \begin{aligned} (\nabla_X R)(Y, Z, U, V) &= [A(X) + B(X)]R(Y, Z, U, V) + A(Y)R(X, Z, U, V) \\ &+ A(Z)R(Y, X, U, V) + A(U)R(Y, Z, X, V) \\ &+ A(V)R(Y, Z, U, X), \end{aligned}$$

where  $A$  and  $B$  are non-zero 1-forms such that

$$(4) \quad A(X) = g(X, \rho) \quad \text{and} \quad B(X) = g(X, \sigma)$$

for all  $X$  where the vector field  $\rho$  and  $\sigma$  are called the basic vector fields of the manifold corresponding to the associated 1-forms  $A$  and  $B$  respectively. An  $n$ -dimensional almost pseudo symmetric manifold has been denoted by  $A(PS)_n$ . If  $A = B$  in (3), then the manifold reduces to a pseudo symmetric manifold  $(PS)_n$ . It is pointed out that the almost pseudo symmetric manifold is a particular case of pseudo symmetric manifold but almost pseudo symmetric manifold is not a particular case of weakly symmetric manifold define by Tamassy and Binh [26]. De and Gazi extend the  $A(PS)_n$  in the branch of almost pseudo conformally symmetric manifold [10] and almost pseudo Ricci symmetric manifold [11] and showed the physical significant in general relativity with some examples.

A Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) is said to be an Einstein manifold if its Ricci tensor  $S$  is of the form

$$(5) \quad S(X, Y) = \frac{r}{n}g(X, Y).$$

It is also called Einstein metric condition ([2], p. 432). Einstein manifolds play an important role in Riemannian Geometry as well as in the general theory of relativity. It form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([3], pp. 432–433).

In 2002, Prasad [23] defined the pseudo-projective curvature tensor  $\tilde{P}$  on a Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) of type (0,4) as follows

$$(6) \quad \begin{aligned} \tilde{P}(X, Y, Z, V) &= aR(X, Y, Z, V) + b[S(Y, Z)g(X, V) - S(X, Z)g(Y, V)] \\ &- \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)g(X, V) - g(X, Z)g(Y, V)], \end{aligned}$$

where  $R$  is the Riemannian curvature tensor of the manifold,  $S$  is the Ricci tensor of type (0,2) defined as  $S(X, Y) = g(QX, Y)$ ,  $Q$  is Ricci operator,  $r$  denote the scalar curvature of the manifold and  $a, b$  are constants such that  $a, b \neq 0$ .

If  $a = 1$  and  $b = -\frac{1}{n-1}$ , then (6) takes the form

$$\begin{aligned} \tilde{P}(X, Y, Z, V) &= R(X, Y, Z, V) - \frac{1}{n-1}[S(Y, Z)g(X, V) - S(X, Z)g(Y, V)] \\ (7) \qquad \qquad &= \mathcal{W}(X, Y, Z, V), \end{aligned}$$

which is known as Weyl projective curvature tensor [21, 29].

The concircular curvature tensor  $C$  of  $(M^n, g)$  ( $n > 2$ ) of type (0,4) is defined by [21, 29]

$$\begin{aligned} C(X, Y, Z, V) &= R(X, Y, Z, V) \\ (8) \qquad \qquad &- \frac{r}{n(n-1)}[g(Y, Z)g(X, V) - g(X, Z)g(Y, V)]. \end{aligned}$$

In spacial case, pseudo-projective curvature tensor  $\tilde{P}$  can be written as a linear combination of Weyl projective curvature tensor  $\mathcal{W}$  and concircular curvature tensor  $C$  by using (7) and (8) in (6) as follows

$$(9) \quad \tilde{P}(X, Y, Z, V) = -(n-1)b\mathcal{W}(X, Y, Z, V) + [a + (n-1)b]C(X, Y, Z, V).$$

The properties of pseudo-projective curvature tensor with different types of symmetric manifolds have been studied by many authors in [18, 20, 24].

The object of the present paper is to study a non pseudo-projectively flat Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) whose pseudo-projective curvature tensor  $\tilde{P}$  satisfies the condition

$$\begin{aligned} (\nabla_X \tilde{P})(Y, Z, U, V) &= [A(X) + B(X)]\tilde{P}(Y, Z, U, V) + A(Y)\tilde{P}(X, Z, U, V) \\ (10) \qquad \qquad &+ A(Z)\tilde{P}(Y, X, U, V) + A(U)\tilde{P}(Y, Z, X, V) \\ &+ A(V)\tilde{P}(Y, Z, U, X), \end{aligned}$$

where  $A$  and  $B$  have the meaning already stated in (4). Such a manifold will be called an generalized pseudo-projective symmetric manifold and an  $n$ -dimensional manifold ( $n > 2$ ) of this kind will be denoted by  $G(\tilde{P}S)_n$ . If the 1-form  $A$  and  $B$  vanish at each point of the manifold then the manifold reduces to pseudo projective symmetric manifold.

Several authors have studied almost symmetric structure with different curvature tensor such as almost pseudo concircularly symmetric manifolds by De and Mallick [12], almost pseudo conformally symmetric manifolds by De and Gazi [10], almost pseudo symmetric Sasakian manifolds by Vishnuvardhana and Venkatesha [27], almost pseudo symmetric manifolds by De [7]. Motivated by the above studies, we are interested to characterize the same structure with a pseudo-projective curvature tensor which is the generalization of well known projective curvature tensor.

The present paper is organized as follows: Section 1 is the review of some preliminary definitions and their related works. In Section 2, we study a  $G(\tilde{P}S)_n$  and it is shown that in such a manifold  $\frac{r}{n}$  is an eigenvalue of the Ricci tensor  $S$  corresponding to the eigenvector  $\rho$ , provided that  $a + (n - 1)b \neq 0$ . We also obtain a sufficient condition for a  $G(\tilde{P}S)_n$  to be an  $A(P\tilde{S})_n$ . In Section 3, it is shown first that the scalar curvature of Einstein  $G(\tilde{P}S)_n$  vanishes if an Einstein  $G(\tilde{P}S)_n$  is an  $A(P\tilde{S})_n$  and also obtained its adversely results. Some geometrical properties of vector fields  $\rho$  in an Einstein  $G(\tilde{P}S)_n$  have been studied under certain conditions. Finally non-trivial examples of  $G(\tilde{P}S)_n$  have been constructed in the last section.

## 2. Generalized pseudo-projectively symmetric manifolds

Let us define

$$\star W(Y, Z) = \sum_{i=1}^n W(e_i, Y, Z, e_i) \text{ and } \star C(Y, Z) = \sum_{i=1}^n C(e_i, Y, Z, e_i),$$

where  $\{e_i, i = 1, 2, \dots, n\}$  is a local orthonormal basis for the vector fields on  $M^n$ . Then by putting  $X = Y = e_i$  in (7) and (8) and taking summation over  $i$  ( $1 \leq i \leq n$ ), we get

$$(11) \quad \star W(Y, Z) = 0,$$

and

$$(12) \quad \star C(Y, Z) = S(Y, Z) - \frac{r}{n}g(Y, Z).$$

By virtue of  $r = \sum_{i=1}^n S(e_i, e_i)$ , we obtain from (12) that

$$(13) \quad \sum_{i=1}^n \star C(e_i, e_i) = 0.$$

Putting  $X = Y = e_i$  in (9) and using (11) and (12), we get

$$(14) \quad \sum_{i=1}^n \tilde{P}(e_i, Y, Z, e_i) = [a + (n - 1)b] \star C(Y, Z).$$

In local co-ordinate system, equation (10) takes the following form

$$(15) \quad \nabla_i \tilde{P}_{klmj} = [A_i + B_i] \tilde{P}_{klmj} + A_k \tilde{P}_{ilmj} + A_l \tilde{P}_{kimj} + A_m \tilde{P}_{klij} + A_j \tilde{P}_{klmi},$$

and the equation (14) is reduced to the form

$$(16) \quad \tilde{P}_{lm} = [a + (n - 1)b] \star C_{lm} = [a + (n - 1)b] \left( S_{lm} - \frac{r}{n} g_{lm} \right).$$

Transvecting (15) with  $g^{kj}$ , we get

$$(17) \quad \nabla_i \tilde{P}_{lm} = [A_i + B_i] \tilde{P}_{lm} + A_k \tilde{P}_{ilm}^k + A_l \tilde{P}_{im} + A_m \tilde{P}_{li} + A_j \tilde{P}_{lmi}^j.$$

Again transvecting (17) with  $g^{lm}$  and using (13) and (16), we obtain

$$(18) \quad A_k \left( \tilde{P}_{ilm}^k + \tilde{P}_{lmi}^k \right) g^{lm} = 0.$$

By using (16) in (18), we get

$$(19) \quad 2[a + (n-1)b] A_k \left( S_i^k - \frac{r}{n} \delta_i^k \right) = 0,$$

which implies that

$$(20) \quad 2[a + (n-1)b] \overset{\star}{C}(X, \rho) = 0.$$

Therefore, we get

$$(21) \quad \overset{\star}{C}(X, \rho) = 0, \quad \text{provided } [a + (n-1)b] \neq 0.$$

From (12) and (21), it follows that

$$(22) \quad S(X, \rho) = \frac{r}{n} g(X, \rho).$$

Thus, we can state the following theorem:

**Theorem 2.1.** *The Ricci tensor  $S$  in a  $G(\tilde{P}S)_n$  has eigenvalue  $\frac{r}{n}$  corresponding to the eigenvector  $\rho$  defined by  $g(X, \rho) = A(X)$  for all  $X$  provided that  $a + (n-1)b \neq 0$ .*

Suppose that the scalar curvature  $r$  of a  $G(\tilde{P}S)_n$  is zero. Then from (22) it follows that  $S(X, \rho) = 0$ . Substitution of  $X$  by  $Y$  yields

$$(23) \quad S(Y, \rho) = 0.$$

Thus from (4), (7), (8) and (23), equation (9) yield

$$(24) \quad \begin{aligned} \tilde{P}(X, Y, \rho, V) &= a R(X, Y, \rho, V) \\ &- \frac{r}{n} \left( \frac{a}{n-1} + b \right) \{A(Y)g(X, V) - A(X)g(Y, V)\}. \end{aligned}$$

Hence, we can state the following:

**Theorem 2.2.** *The pseudo-projective curvature tensor  $\tilde{P}$  of a  $G(\tilde{P}S)_n$  with zero scalar curvature is given by (24) provided that  $a + (n-1)b \neq 0$ .*

Adversely, let us consider  $\tilde{P}$  be the form (24). Then from (9) it follows that

$$(25) \quad \frac{ar}{n(n-1)} \{A(Y)g(X, V) - A(X)g(Y, V)\} = 0.$$

Since  $A$  is nowhere vanishing 1-form, so from (25), we obtain

$$r = 0, \quad \text{provided } a \neq 0.$$

Thus, we can state the following theorem:

**Theorem 2.3.** *If in a  $G(\tilde{P}S)_n$  the pseudo-projective curvature tensor  $\tilde{P}$  is satisfied the condition (24) and  $a \neq 0$ , then it is of zero scalar curvature tensor.*

Let us consider that the Ricci tensor  $S$  of a  $G(\tilde{P}S)_n$  vanishes. Then from (9), it follows that

$$\tilde{P}(X, Y, Z, V) = a R(X, Y, Z, V).$$

Thus by virtue of (3) and (10) for  $a \neq 0$ , we obtain a sufficient condition for a  $G(\tilde{P}S)_n$  to be an  $A(PS)_n$ . This can be lead the following:

**Theorem 2.4.** *If the Ricci tensor of a  $G(\tilde{P}S)_n$  vanishes and  $a \neq 0$ , then it is an  $A(PS)_n$ .*

### 3. Einstein $G(\tilde{P}S)_n$

Let us consider a  $G(\tilde{P}S)_n$  defined by (10) is an Einstein manifold. Then its Ricci tensor  $S$  satisfies the condition

$$(26) \quad S(Y, Z) = \frac{r}{n}g(Y, Z).$$

It follows that

$$(27) \quad dr(Y) = 0 \quad \text{and} \quad (\nabla_X S)(Y, Z) = 0.$$

Using (26) in (6), we obtain

$$(28) \quad \begin{aligned} \tilde{P}(X, Y, Z, V) &= a R(X, Y, Z, V) \\ &- \frac{ar}{n(n-1)} \{g(Y, Z)g(X, V) - g(X, Z)g(Y, V)\}. \end{aligned}$$

Taking covariant derivative of (28) with respect to  $E$  and using (27), we get

$$(29) \quad (\nabla_E \tilde{P})(X, Y, Z, V) = a (\nabla_E R)(X, Y, Z, V).$$

Differentiating (9) covariantly, we get

$$(30) \quad \begin{aligned} (\nabla_E \tilde{P})(X, Y, Z, V) &= -(n-1)b (\nabla_E \mathcal{W})(X, Y, Z, V) \\ &+ [a + (n-1)b] (\nabla_E C)(X, Y, Z, V). \end{aligned}$$

In view of (10), (28), (29) and (30), we get

$$(31) \quad \begin{aligned} &a (\nabla_E R)(X, Y, Z, V) \\ &= -(n-1)b \left[ \{A(E) + B(E)\} \mathcal{W}(X, Y, Z, V) \right. \\ &\quad + A(X) \mathcal{W}(E, Y, Z, V) + A(Y) \mathcal{W}(X, E, Z, V) \\ &\quad \left. + A(Z) \mathcal{W}(X, Y, E, V) + A(V) \mathcal{W}(X, Y, Z, E) \right] \\ &\quad + [a + (n-1)b] \left[ \{A(E) + B(E)\} C(X, Y, Z, V) + A(X) C(E, Y, Z, V) \right. \\ &\quad \left. + A(Y) C(X, E, Z, V) + A(Z) C(X, Y, E, V) + A(V) C(X, Y, Z, E) \right]. \end{aligned}$$

Since a  $G(\tilde{P}S)_n$  is an Einstein manifold, so its Weyl projective curvature tensor (7) reduce to the following form

$$(32) \quad \begin{aligned} \mathcal{W}(X, Y, Z, V) &= R(X, Y, Z, V) \\ &- \frac{r}{n(n-1)} [g(Y, Z)g(X, V) - g(X, Z)g(Y, V)]. \end{aligned}$$

Its shows that the Weyl projective curvature tensor  $\mathcal{W}$  equal to the concircular curvature tensor  $C$ . Thus for  $a \neq 0$ , equation (31) follows from (32), that

$$(33) \quad \begin{aligned} &(\nabla_E R)(X, Y, Z, V) \\ &= [A(E) + B(E)]\mathcal{W}(X, Y, Z, V) + A(X)\mathcal{W}(E, Y, Z, V) \\ &\quad + A(Y)\mathcal{W}(X, E, Z, V) + A(Z)\mathcal{W}(X, Y, E, V) \\ &\quad + A(V)\mathcal{W}(X, Y, Z, E). \end{aligned}$$

Using (32) in (33), we obtain

$$(34) \quad \begin{aligned} &(\nabla_E R)(X, Y, Z, V) \\ &= [A(E) + B(E)] \left[ R(X, Y, Z, V) - \frac{r}{n(n-1)} \{g(Y, Z)g(X, V) - g(X, Z)g(Y, V)\} \right] \\ &\quad + A(X) \left[ R(E, Y, Z, V) - \frac{r}{n(n-1)} \{g(Y, Z)g(E, V) - g(E, Z)g(Y, V)\} \right] \\ &\quad + A(Y) \left[ R(X, E, Z, V) - \frac{r}{n(n-1)} \{g(E, Z)g(X, V) - g(X, Z)g(E, V)\} \right] \\ &\quad + A(Z) \left[ R(X, Y, E, V) - \frac{r}{n(n-1)} \{g(Y, E)g(X, V) - g(X, E)g(Y, V)\} \right] \\ &\quad + A(V) \left[ R(X, Y, Z, E) - \frac{r}{n(n-1)} \{g(Y, Z)g(X, E) - g(X, Z)g(Y, E)\} \right]. \end{aligned}$$

Now, let us consider that an Einstein  $G(\tilde{P}S)_n$  is an  $A(PS)_n$ . Then by virtue of (34), we get

$$(35) \quad \begin{aligned} &\frac{r}{n(n-1)} \left[ \{A(E) + B(E)\} \{g(Y, Z)g(X, V) - g(X, Z)g(Y, V)\} \right. \\ &\quad + A(X) \{g(Y, Z)g(E, V) - g(E, Z)g(Y, V)\} \\ &\quad + A(Y) \{g(E, Z)g(X, V) - g(X, Z)g(E, V)\} \\ &\quad + A(Z) \{g(Y, E)g(X, V) - g(X, E)g(Y, V)\} \\ &\quad \left. + A(V) \{g(Y, Z)g(X, E) - g(X, Z)g(Y, E)\} \right] = 0. \end{aligned}$$

Putting  $X = Y = e_i$  in (35) and taking summation over  $i$  ( $1 \leq i \leq n$ ), we get

$$(36) \quad \begin{aligned} &r[\{(n+1)A(E) + (n-1)B(E)\}g(Y, Z) + (n-2)A(Y)g(E, Z) \\ &\quad + (n-2)A(Z)g(Y, E)] = 0. \end{aligned}$$

Again summing over  $i$  ( $1 \leq i \leq n$ ) for  $Y = Z = e_i$  in (36), we obtain

$$(37) \quad r[(n+4)A(E) + n B(E)] = 0.$$

Also by substituting  $Y = E = e_i$  in (36) and summing over  $i$  ( $1 \leq i \leq n$ ), we get

$$r[(n+1)A(Z) + B(Z)] = 0,$$

replacing  $Z$  by  $E$  in above equation, we have

$$(38) \quad r[(n+1)A(E) + B(E)] = 0.$$

Similarly, putting  $E = Z = e_i$  in (36), we get

$$r[(n+1)A(Y) + B(Y)] = 0,$$

replacing  $Y$  by  $E$  in above equation, we have

$$(39) \quad r[(n+1)A(E) + B(E)] = 0.$$

Adding (37), (38) and (39), we get

$$r = 0 \text{ if } 3A(E) + B(E) \neq 0.$$

Thus, in view of the above results, the following theorem is obtained:

**Theorem 3.1.** *If an Einstein  $G(\tilde{P}S)_n$  is an  $A(PS)_n$  and  $a \neq 0$  and  $3A(E) + B(E) \neq 0$ , then its scalar curvature vanishes.*

Adversely, suppose that the scalar curvature of an Einstein  $G(\tilde{P}S)_n$  vanishes. Then (34) follows that an Einstein  $G(\tilde{P}S)_n$  is an  $A(PS)_n$  provided  $a \neq 0$ . Thus we can state the following:

**Theorem 3.2.** *If the scalar curvature of an Einstein  $G(\tilde{P}S)_n$  vanishes and  $a \neq 0$ , then such a manifold is an  $A(PS)_n$ .*

Let us consider that the vector field  $\rho$  associated with 1-form  $A$  defined in (4) is parallel in an Einstein  $G(\tilde{P}S)_n$ . Then we get

$$\nabla_X \rho = 0 \text{ for all } X.$$

Also, we have

$$R(X, Y)\rho = \nabla_X \nabla_Y \rho - \nabla_Y \nabla_X \rho - \nabla_{[X, Y]}\rho = 0,$$

which gives

$$(40) \quad R(X, Y, \rho, V) = 0.$$

Contracting (40), we have

$$(41) \quad S(Y, \rho) = 0.$$

By virtue of (41), we have from (22)

$$r g(Y, \rho) = 0.$$

If  $\|\rho\|^2 \neq 0$ , then above equation follows that  $r = 0$ .



Conversely, if  $r = 0$ , then from (22) it is clear that the manifold is an  $A(PS)_n$  for  $a \neq 0$ .

Hence we have the following theorem:

**Theorem 3.3.** *If the vector field  $\rho$  is a parallel vector field in an Einstein  $G(\tilde{P}S)_n$  with  $a \neq 0$  and  $\|\rho\|^2 \neq 0$ , then it is an  $A(PS)_n$ .*

#### 4. Example of 5-dimensional $G(\tilde{P}S)_5$

In this section we construct an example of an  $G(\tilde{P}S)_5$  on coordinate space  $\mathbb{R}^5$  (with coordinates  $(x^1, x^2, x^3, x^4, x^5)$ ) and calculate the components of the curvature tensor, the Ricci tensor, the pseudo-projective curvature tensor and its covariant derivatives. Then verify the relation (10).

Let us consider a Riemannian metric  $g$  defined on 5-dimensional manifold  $M^5 = \{(x^1, x^2, x^3, x^4, x^5) \in \mathbb{R}^5 : x^1 \neq -1\}$  given by [11]

$$(42) \quad ds^2 = (x^1 + 1)(x^3)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2.$$

Then the covariant and contravariant components of the metric are as follows:

$$(43) \quad \begin{aligned} g_{11} &= (x^1 + 1)(x^3)^2, & g_{12} &= g_{21} = 1, & g_{22} &= 0, \\ g_{33} &= g_{44} = g_{55} = 1, & g^{11} &= 0, & g^{22} &= -(x^1 + 1)(x^3)^2, \\ g^{12} &= g^{21} = g^{33} = g^{44} = g^{55} = 1. \end{aligned}$$

In the metric considered, the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$(44) \quad \begin{aligned} \Gamma_{11}^2 &= \frac{1}{2}(x^3)^2, & \Gamma_{11}^3 &= -(x^1 + 1)(x^3), & \Gamma_{13}^2 &= (x^1 + 1)(x^3). \\ R_{1331} &= x^1 + 1. \end{aligned}$$

By (43) and (44) the non-vanishing components of the Ricci tensor are

$$(45) \quad S_{11} = x^1 + 1.$$

The scalar curvature of the resulting manifold  $(M^5, g)$  is given by

$$r = g^{ij}S_{ij} = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} + g^{55}S_{55}.$$

By the use of (43) and (45) it can be easily seen that

$$(46) \quad r = 0.$$

Therefore  $(M^5, g)$  is of zero scalar curvature.

By virtue of (7) and (8), we obtained that the only non-vanishing components of the projective curvature tensor  $\mathcal{W}$  and the concircular curvature tensor  $C$  of  $(M^5, g)$  are given by  $\mathcal{W}_{1331} = x + 1$  and  $C_{1331} = x^1 + 1$  respectively.

Then from (9), it follows that

$$(47) \quad \tilde{P}_{1331} = a(x^1 + 1) \neq 0.$$

Hence  $(M^5, g)$  is not pseudo-projectively flat.

From (47), it can be easily shown that the only non-zero term of  $\nabla_l \tilde{P}_{ikjm}$  are

$$(48) \quad \nabla_l \tilde{P}_{1331} = a \neq 0,$$

and all other components of  $\nabla_l \tilde{P}_{ikjm}$  vanishes identically.

Thus the manifold  $M^5$  with considered metric  $g$  in (42) is a Riemannian manifold with vanishing scalar curvature which is neither pseudo-projectively symmetric nor pseudo-projectively flat.

In term of local coordinate system, let us consider the components of the 1-form  $A$  and  $B$  as

$$(49) \quad A_i = \begin{cases} \frac{1}{6(x^1 + 1)} & \text{for } i = 1 \\ 0 & \text{for otherwise,} \end{cases} \quad \text{and } B_i = \begin{cases} \frac{1}{2(x^1 + 1)} & \text{for } i = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

at any point of  $M^5$ .

In  $(M^5, g)$  the considered 1-form reduces (10) in to the following equations:

$$(50) \quad \nabla_1 \tilde{P}_{1331} = [3A_1 + B_1] \tilde{P}_{1331} + A_3 \tilde{P}_{1131} + A_3 \tilde{P}_{1311},$$

$$(51) \quad \nabla_3 \tilde{P}_{1131} = [2A_3 + B_3] \tilde{P}_{1131} + A_1 \tilde{P}_{3131} + A_1 \tilde{P}_{1331} + A_1 \tilde{P}_{1133},$$

$$(52) \quad \nabla_3 \tilde{P}_{1311} = [2A_3 + B_3] \tilde{P}_{1311} + A_1 \tilde{P}_{3311} + A_1 \tilde{P}_{1331} + A_1 \tilde{P}_{1313}.$$

Since, for the case other than (50), (51) and (52), the components of each term of (10) either vanishes identically or the relation (10) holds trivially.

By (49), we get

$$\begin{aligned} \text{R.H.S. of (50)} &= (3A_1 + B_1) \tilde{P}_{1331} + A_3 \tilde{P}_{1131} + A_3 \tilde{P}_{1311} \\ &= \left[ \frac{3}{6(x^1 + 1)} + \frac{1}{2(x^1 + 1)} \right] \cdot a(x^1 + 1) \\ &= a \\ &= \nabla_1 \tilde{P}_{1331} \\ &= \text{L.H.S. of (50)}. \end{aligned}$$

By similar argument it can be shown that (51) and (52) are also true. Thus  $(M^5, g)$  is a  $G(\tilde{P}S)_5$  with vanishing scalar curvature.

Hence we can state the following:

**Theorem 4.1.** *If  $(M^5, g)$  is a Riemannian manifold equipped with the metric*

$$ds^2 = (x^1 + 1)(x^3)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2,$$

*then  $(M^5, g)$  is an  $G(\tilde{P}S)_5$  with vanishing scalar curvature which is neither pseudo-projectively symmetric nor pseudo-projectively flat.*

**5. Example of  $n$ -dimensional  $G(\tilde{P}S)_n$**

In this section we construct an example of an  $n$ -dimensional  $G(\tilde{P}S)_n$  on coordinate space  $\mathbb{R}^n$  (with coordinates  $(x^1, x^2, \dots, x^n)$ ). We define a Riemannian manifold  $V^n$  and calculate the components of the curvature tensor, the Ricci tensor, the pseudo-projective curvature tensor and its covariant derivatives. Then verify the relation (10).

Let us consider each Latin index runs over  $1, 2, \dots, n$  and each Greek index runs over  $2, 3, \dots, n - 1$ . We define a Riemannian metric on the  $\mathbb{R}^n$  ( $n \geq 4$ ) by the formula [11]

$$(53) \quad ds^2 = \phi(dx^1)^2 + \kappa_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n,$$

where  $[\kappa_{\alpha\beta}]$  is a non-singular symmetric matrix consisting of constants and  $\phi$  is a function of  $x^1, x^2, \dots, x^{n-1}$  and independent of  $x^n$ .

The covariant and contravariant components of the metric (53) are in the follows:

$$(54) \quad \begin{aligned} g_{11} &= \phi, & g_{\alpha\alpha} &= 1, & g_{1n} &= g_{n1} = 1, \\ g^{11} &= 0, & g^{\alpha\alpha} &= 1, & g^{1n} &= g^{n1} = 1, & g^{nn} &= -\phi. \end{aligned}$$

In the metric considered, the only non-vanishing components of the Christoffel symbols, curvature tensors and Ricci tensor are in the following

$$(55) \quad \begin{aligned} \Gamma_{11}^\beta &= -\frac{1}{2}\kappa^{\alpha\beta}\phi_{\cdot\alpha}, & \Gamma_{11}^n &= \frac{1}{2}\phi_{\cdot 1}, & \Gamma_{1\alpha}^n &= \frac{1}{2}\phi_{\cdot\alpha}, \\ R_{1\alpha\beta 1} &= \frac{1}{2}\phi_{\cdot\alpha\beta}, \\ S_{11} &= \frac{1}{2}\kappa^{\alpha\beta}\phi_{\cdot\alpha\beta}, \end{aligned}$$

where  $(\cdot)$  denotes the partial differentiation with respect to the coordinates and  $[\kappa^{\alpha\beta}]$  is the inverse matrix of the matrix  $[\kappa_{\alpha\beta}]$ .

Let us consider  $\kappa_{\alpha\beta}$  as the Kronecker symbol  $\delta_{\alpha\beta}$  and

$$(56) \quad \phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta (x^1)^{2/3},$$

where  $M_{\alpha\beta}$  are constants and satisfy the relations

$$(57) \quad M_{\alpha\beta} \begin{cases} = 0 & \text{for } \alpha \neq \beta, \\ \neq 0 & \text{for } \alpha = \beta, \end{cases} \quad \text{and} \quad \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0.$$

Thus from (56) and (57), it follows that

$$(58) \quad \begin{cases} \phi_{\cdot\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta (x^1)^{2/3}, \\ \delta_{\alpha\beta}\delta^{\alpha\beta} = n - 2, \\ \delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \end{cases}$$

Therefore due to (55), (56) and (57) the only non-zero components of the curvature tensor and the Ricci tensor are in the following

$$(59) \quad \begin{aligned} R_{1\alpha\alpha 1} &= \frac{1}{2}\phi_{\cdot\alpha\alpha} = (1 + M_{\alpha\alpha})(x^1)^{2/3}, \\ S_{11} &= \frac{1}{2}\phi_{\cdot\alpha\beta}\delta^{\alpha\beta} = (n - 2)(x^1)^{2/3}. \end{aligned}$$

In virtue of (54) and (59), it can be easily seen that the scalar curvature

$$r = g^{ij}S_{ij} = g^{11}S_{11} = 0.$$

Therefore,  $V^n$  will be a space whose scalar curvature is zero.

From (7) and (8), it can be shown that the only non-vanishing components of the Weyl projective curvature tensor  $\mathcal{W}$  and the concircular curvature tensor  $C$  are

$$(60) \quad \mathcal{W}_{1\alpha\alpha 1} = \left[1 + M_{\alpha\alpha} - \frac{n - 2}{n - 1}\right] (x^1)^{2/3} \text{ and } C_{1\alpha\alpha 1} = (1 + M_{\alpha\alpha})(x^1)^{2/3},$$

which never vanish.

Thus from (9), it follows that

$$(61) \quad \tilde{P}_{1\alpha\alpha 1} = [a M_{\alpha\alpha} + (n - 2)b](x^1)^{2/3} \neq 0.$$

Hence  $(M^n, g)$  is not pseudo-projectively flat.

We shall now show that  $V^n$  is a  $G(\tilde{P}S)_n$ . From (61), it can be easily shown that the only non-zero terms of  $\nabla_l \tilde{P}_{ikjm}$  are

$$\nabla_1 \tilde{P}_{1\alpha\alpha 1} = \frac{2[a M_{\alpha\alpha} + (n - 1)b]}{3(x^1)^{1/3}} \neq 0.$$

Hence  $V^n$  equipped with the metric considered is neither pseudo-projectively symmetric nor pseudo-projectively flat.

In term of local co-ordinate system, we consider the components of the 1-form  $A$  and  $B$  as follows:

$$(62) \quad A_i = \begin{cases} \frac{1}{9x^1} & \text{for } i = 1, \\ 0 & \text{for } i \neq 1, \end{cases} \quad \text{and } B_i = \begin{cases} \frac{1}{3(x^1)} & \text{for } i = 1, \\ 0 & \text{for } i \neq 1. \end{cases}$$

In  $(V^n, g)$  the considered 1-forms reduces (10) in the following equations:

$$(63) \quad \nabla_1 \tilde{P}_{1\alpha\alpha 1} = [3A_1 + B_1]\tilde{P}_{1\alpha\alpha 1} + A_\alpha \tilde{P}_{11\alpha 1} + A_\alpha \tilde{P}_{1\alpha 11},$$

$$(64) \quad \nabla_\alpha \tilde{P}_{11\alpha 1} = [2A_\alpha + B_\alpha]\tilde{P}_{11\alpha 1} + A_1 \tilde{P}_{\alpha 1\alpha 1} + A_1 \tilde{P}_{1\alpha\alpha 1} + A_1 \tilde{P}_{11\alpha\alpha},$$

$$(65) \quad \nabla_\alpha \tilde{P}_{1\alpha 11} = [2A_\alpha + B_\alpha]\tilde{P}_{1\alpha 11} + A_1 \tilde{P}_{\alpha\alpha 11} + A_1 \tilde{P}_{1\alpha\alpha 1} + A_1 \tilde{P}_{1\alpha 1\alpha}.$$

Since, for the case other than (63), (64) and (65), the components of each term of (10) either vanishes identically or the relation (10) holds trivially.

By (62), we get

$$\begin{aligned}
 \text{R.H.S. of (63)} &= (3A_1 + B_1)\tilde{P}_{1\alpha\alpha 1} + A_\alpha\tilde{P}_{11\alpha 1} + A_\alpha\tilde{P}_{1\alpha 11} \\
 &= \left[ \frac{3}{9x^1} + \frac{1}{3x^1} \right] \{a M_{\alpha\alpha} + (n-2)b\}(x^1)^{2/3} \\
 &= \frac{2a M_{\alpha\alpha} + (n-2)b}{3(x^1)^{(1/3)}} \\
 &= \nabla_1\tilde{P}_{1331} \\
 &= \text{L.H.S. of (63)}.
 \end{aligned}$$

By similar argument it can be shown that (64) and (65) are also true. Thus  $(V^n, g)$  is a  $G(\tilde{P}S)_n$  with vanishing scalar curvature.

Hence we can state the following:

**Theorem 5.1.** *If  $(V^n, g)$  ( $n \geq 4$ ) is a Riemannian manifold equipped with the metric*

$$ds^2 = \phi(dx^1)^2 + \kappa_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n,$$

where  $\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta (x^1)^{2/3}$  and  $M_{\alpha\beta}$  are constant defined in (57), then  $(V^n, g)$  is a  $G(\tilde{P}S)_n$  with vanishing scalar curvature which is neither pseudo-projectively symmetric nor pseudo-projectively flat.

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