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LORENTZIAN MANIFOLDS: A CHARACTERIZATION WITH SEMICONFORMAL CURVATURE TENSOR

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Abstract. In this paper we characterize semiconformally flat spacetimes and a spacetime with harmonic semiconformal curvature tensor. At first in a semiconformally flat perfect fluid spacetime we obtain a state equation and prove that in particular for dimension $n = 4$, the spacetime represents a model for incoherent radiation. Next we prove that perfect fluid spacetime with harmonic semiconformal curvature tensor is of Petrov type I, D or O and the spacetime is a GRW spacetime. As a consequence we obtain several corollaries.

1. Introduction

The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector, that is, a vector v satisfying $g_{ij}v^iv^j = 0$, where g_{ij} is the metric tensor. The signature of the metric g of a Riemannian manifold is $(+,+,+,\dots,+,+,+)$ and of a semi-Riemannian manifold is $(-,-,-,\dots,+,+,+)$. Lorentzian manifold is a special case of semi-Riemannian manifold. The signature of the metric of a Lorentzian manifold is $(-,+,+,\dots,+,+,+)$. In a Lorentzian manifold three types of vectors exist such as timelike, spacelike and null vector. In general, a Lorentzian manifold (M, g) may not have a globally timelike vector field. If (M, g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as spacetime.

In a recent paper Kim [20] introduced a curvature-like tensor named semiconformal curvature tensor such that its $(1, 3)$ components remain invariant under conharmonic transformation and obtains several results on harmonic semiconformal curvature tensor and pseudo semiconformally symmetric manifolds. In this paper we characterize an n-dimensional Lorentzian manifold admitting semiconformal curvature tensor. Semiconformal curvature tensor P^h_{ijk} of type

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$(1,3)$ is defined as [21]:

$$P^h_{ijk} = -(n-2)C^h_{ijk} + [a + (n-2)b]H^h_{ijk}, \tag{1.1}$$

where a, b are constants not simultaneously zero, C^h_{ijk} and H^h_{ijk} denote conformal curvature tensor and conharmonic curvature tensor, respectively.

On the other hand, generalized Robertson-Walker (GRW) spacetimes were introduced in 1995 by Alias, Romero and Sánchez ([1], [2]).

A Lorentzian manifold M of dimension $n \geq 3$ is named generalized Robertson-Walker (GRW) spacetime if it is the warped product $M = I \times_{q^2} M^*$ with base $(I, -dt^2)$, warping function q and the fibre (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold ([1], [2], [7], [32], [33]).

If M^* is a 3-dimensional Riemannian manifold of constant curvature, the spacetime is called a Robertson-Walker (RW) spacetime. Therefore, GRW spacetimes are a wide generalization of RW spacetimes on which standard cosmology is modelled. They include the Einstein-de Sitter spacetime, the static Einstein spacetime, the Friedman cosmological models, the de Sitter spacetime and hence applications as inhomogeneous spacetimes admitting an isotropic radiation ([7], [32]).

Lorentzian manifolds with the Ricci tensor of the form

$$R_{ij} = \alpha g_{ij} + \beta u_i u_j, \tag{1.2}$$

where α, β are scalar fields and u_i is a unit timelike vector (that is, $u^i u_i = -1$), are called perfect fluid spacetimes and are of interest in general relativity. In the language of pure differential geometry they are named quasi Einstein. Pseudo-Riemannian quasi Einstein spaces arose in the study of exact solutions of Einstein's equations and in the investigation of quasi-umbilical hypersurfaces of Pseudo-Euclidean spaces ([11], [12]). Form (1.2) of the Ricci tensor is implied by Einstein's equation if the energy momentum tensor of the spacetime is perfect fluid with velocity vector field u. A spacetime is called perfect fluid if the energy momentum tensor is of the form

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij},$$

where μ is the energy density, p is the isotropic pressure, u_i is a unit timelike vector ($u^i u_i = -1$). The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscocity [18].

In addition, p and μ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\mu, T_0)$, where T_0 is the absolute temperature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\mu)$. In this case, the perfect fluid is called isentropic [18]. Moreover, if $p = \mu$, then the perfect fluid is termed as stiff matter (see [36], page 66). Einstein's field equation is given by $R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}$, κ is the gravitational constant. Einstein's equation implies that matter determines the geometry of spacetime and conversely, the motion of matter is determined by the metric of the space which is non-flat.

The conformal curvature tensor is the traceless part of the Riemann tensor [30]:

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}[\delta^h_k R_{ij} - \delta^h_j R_{ik} + g_{ij}R^h_k - g_{ik}R^h_j] + \frac{R}{(n-1)(n-2)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}], \tag{1.3}$$

where R^h_{ijk} is the $(1,3)$ curvature tensor, $R_{ij} = R^h_{ijh}$ is the Ricci tensor of type $(0,2)$, R^h_k is the $(1,1)$ Ricci tensor and R denotes the scalar curvature. Since in general relativity only the Ricci tensor is coupled to matter by the Einstein's equations, the conformal curvature tensor describes the pure gravity degree of freedom.

Perfect fluid spacetimes in four dimensions with divergence free conformal curvature tensor (that is, $\nabla_h C^h_{ijk} = 0$) were firstly investigated by Shepley and Taub [35] and successively by Sharma [34] and Coley [8].

Recently in [27] Mantica, Molinari and De extended some results to n-dimensional perfect fluids and proved the following:

Theorem 1.1 ([27]). *Let (M, g) be a perfect fluid spacetime (that is, $R_{ij} = \alpha g_{ij} + \beta u_i u_j$). If $\nabla_k u_j = \nabla_j u_k$ and $\nabla_h C^h_{ijk} = 0$, then*

- (i) *u_i is a concircular vector field and it is rescalable to a timelike vector X_j such that $\nabla_k X_j = \rho g_{jk}$.*
- (ii) *(M, g) is a GRW spacetime with Einstein fibre.*
- (iii) *the velocity vector u_h satisfies the condition $u_h C^h_{ijk} = 0$.*

Also De et al. ([9], [10]) studied conformally flat almost pseudo-Ricci symmetric spacetimes and spacetimes with semisymmetric energy momentum tensor respectively. Also in [24] Mallick, Suh and De studied spacetime with pseudo-projective curvature tensor. Moreover in [26] Mantica and Molinari studied weakly Z symmetric manifolds. Also several authors studied spacetimes in different way such as ([10], [15], [16], [23], [29], [37]) and many others. In [6] Chaki and Ray studied spacetimes with covariant constant energy momentum tensor. Motivated by the above studies in the present paper Lorentzian manifolds is characterized with semiconformal curvature tensor.

The paper is organised in the following way:

After preliminaries in Section 3, we prove that in a semiconformally flat perfect fluid spacetime the state equation is $p = \frac{1}{n-1}\mu$ and for $n = 4$ dimension we have a model for incoherent radiation $p = \frac{\mu}{3}$. Section 4 is devoted to study perfect fluid spacetimes with harmonic semiconformal curvature tensor and prove that a perfect fluid spacetime with harmonic semiconformal curvature tensor is a GRW spacetime with Einstein fibre. As a consequence we obtain that a perfect fluid Yang pure space is a GRW spacetime with Einstein fibre and for $n = 4$ dimension a perfect fluid spacetime with harmonic semi-conformal curvature tensor is a RW spacetime.

2. Preliminaries

The conharmonic curvature tensor H is defined by [19]

$$H^h_{ijk} = R^h_{ijk} - \frac{1}{(n-2)}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + g_{ij}R^h_k - g_{ik}R^h_j). \tag{2.1}$$

A manifold whose conharmonic curvature tensor vanishes at each point of the manifold is called conharmonically flat. Thus this tensor represents the deviation of the manifold from conharmonic flatness.

Suppose the manifold is conharmonically flat. Then from (2.1) we get

$$R^h_{ijk} = \frac{1}{(n-2)}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + g_{ij}R^h_k - g_{ik}R^h_j).$$

Contracting h and k yields

$$\frac{R}{(n-2)}g_{ij} = 0, \tag{2.2}$$

which implies that the scalar curvature vanishes.

Also from (2.1) the divergence of H, (that is, $\nabla_h H^h_{ijk}$) is given by

$$\nabla_h H^h_{ijk} = \frac{n-3}{n-2}(\nabla_k R_{ij} - \nabla_j R_{ik}) - \frac{1}{2(n-2)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R). \tag{2.3}$$

Hence if $\nabla_h H^h_{ijk} = 0$, that is, if the conharmonic curvature tensor is harmonic, then we infer that

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-3)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R). \tag{2.4}$$

Again the divergence of the conformal curvature tensor is given by ([25], [28])

$$\nabla_h C^h_{ijk} = \frac{n-3}{n-2}[\nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R)]. \tag{2.5}$$

3. Semiconformally flat Lorentzian manifolds

The defining relation (1.1) of semiconformal curvature tensor can be rewritten as

$$P^h_{ijk} = aH^h_{ijk} - \frac{bR}{(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}]. \tag{3.1}$$

Suppose the manifold is semiconformally flat, that is, the semiconformal curvature tensor vanishes at each point of the manifold. Then from (3.1) it follows that

$$aH^h_{ijk} - \frac{bR}{(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] = 0. \tag{3.2}$$

Contracting h and k and using (2.2) we infer that

$$-a\frac{R}{n-2}g_{ij} - bRg_{ij} = 0,$$

which yields $R = 0$, provided $a + (n-2)b \neq 0$. Hence from (3.1) we get $H^h_{ijk} = 0$, provided $a \neq 0$. Conversely, if the conharmonic curvature tensor H^h_{ijk} vanishes, then by (2.2) we obtain $R = 0$ and hence by (3.1) we conclude that $P^h_{ijk} = 0$. Thus we can state the following:

Proposition 3.1. *In a Lorentzian manifold of dimension $n > 4$ semiconformally flatness implies conharmonically flatness under the constraints $a + (n-2)b \neq 0$ and $a \neq 0$.*

Now we consider semiconformally flat perfect fluid spacetimes obeying Einstein's field equations. Then we get

$$R_{ij} - \frac{R}{2}g_{ij} = \kappa T_{ij}, \tag{3.3}$$

where

$$T_{ij} = (\mu + p)u_i u_j + pg_{ij}. \tag{3.4}$$

Since $R = 0$ in a semiconformally flat manifold, therefore from (3.3) and (3.4) we infer

$$R_{ij} = \kappa[(\mu + p)u_i u_j + pg_{ij}].$$

Transvecting the above equation with g^{ij} and using $R = 0$, we get $p(n-1) = \mu$, since $\kappa \neq 0$ and $u^i u_i = -1$ which means $p = \frac{1}{n-1}\mu$.

Hence we obtain the following:

Theorem 3.1. *In a semiconformally flat perfect fluid spacetime the state equation is $p = \frac{1}{n-1}\mu$.*

Corollary 3.1. *In $n = 4$ dimension we have a model for incoherent radiation*: $p = \frac{\mu}{3}$ [36].

Recall that a vector field V in a Lorentzian manifold M is called curvature collineation (CC) if the curvature tensor remains invariant under V. This means

$$\pounds_V R^h_{ijk} = 0, \tag{3.5}$$

where $\pounds_V$ denotes the Lie derivation. Obviously, (3.5) implies

$$\pounds_V R_{ij} = 0. \tag{3.6}$$

The vector field V is called Ricci colineation (RC) if (3.6) holds.

If a vector field V leaves the matter tensor invariant, that is, $\pounds_V T_{ij} = 0$, then we say that M has symmetry called matter colineation (MC). Well known examples are Killing and homothetic symmetries.

A conformal vector field V on a Lorentzian manifold M is defined by

$$\pounds_V g_{ij} = 2\sigma g_{ij}$$

for a smooth function σ on M.

If $\sigma = constant$, then the vector field is called homothetic vector field.

Since in a semiconformally flat spacetime $R = 0$, therefore from (3.3) we obtain

$$\pounds_\xi R_{ij} = \kappa \pounds_\xi T_{ij}.$$

Thus Ricci collineation ensures matter collineation and conversely.

In [17] Hall et al. proved that any vector field V in a spacetime M which simultaneously satisfies CC and MC is a homothetic vector field. Therefore from the above discussion we arrive to the following conclusion:

Theorem 3.2. *If a semiconformally flat spacetime obeying Einstein's equation admits a RC vector field, then the vector field reduces to a homothetic vector field.*

It is known [13] that if a four dimensional spacetime with $R = 0$ and $R_{ij} \neq 0$ admits a RC vector field V, then there exists a covariant conservation law generator of the form

$$(\sqrt{g} T^i_j V^j)_{,i} = 0, \;\; g = \mid g_{ij} \mid, \tag{3.7}$$

where T is the energy-momentum tensor.

Hence if in a semi-conformally flat spacetime admits matter collineation (MC) vector field V, then there exists a conservation law generator of the above form (3.7).

4. Perfect fluid spacetimes with harmonic semiconformal curvature tensor

In [20] Kim proved that if the semiconformal curvature tensor is harmonic in a Riemannian manifold, then the scalar curvature R is constant provided $a + (n-2)b \neq 0$. The same result holds for a Lorentzian manifold. In this section we suppose that the semiconformal curvature tensor P^h_{ijk} is harmonic. Then from (3.1) we get $a \nabla_h H^h_{ijk} = 0$, since R =constant.

Therefore from (2.3) it follows that

$$\nabla_k R_{ij} = \nabla_j R_{ik},$$

which implies that the Ricci tensor is a Codazzi tensor [4].

In [31] Ray proved that in a perfect fluid spacetime with Codazzi type of Ricci tensor is of Petrov type I, D or O [3].

From the above discussions we have the following:

Theorem 4.1. *A perfect fluid spacetime with harmonic semi-conformal curvature tensor is of Petrov type I, D or O.*

Next we prove that an n-dimensional perfect fluid spacetime with harmonic semiconformal curvature tensor is a GRW spacetime. As a consequence, a Yang Pure Space [14] is a GRW spacetime.

Since harmonic semiconformal curvature tensor implies that the Ricci tensor is a Codazzi tensor, therefore from (2.5) we infer that

$$\nabla_h C^h_{ijk} = 0, \text{ since } \nabla_k R = 0.$$

Therefore harmonic semiconformal curvature tensor implies harmonic conformal curvature tensor.

Lemma 4.1. *Let (M, g) be a perfect fluid spacetime, that is, $R_{ij} = \alpha g_{ij} + \beta u_i u_j$, where u_k is a unit timelike vector and $\beta \neq 0$. If $\nabla_h C^h_{ijk} = 0$ and $\nabla_k R = 0$, then u_j is irrotational (i.e., $\nabla_k u_j = \nabla_j u_k$).*

Proof. The harmonic conformal curvature tensor is given by (2.5). Therefore the conditions $\nabla_h C^h_{ijk} = 0$ and $\nabla_k R = 0$ implies $\nabla_k R_{ij} = \nabla_j R_{ik}$. From $R_{ij} = \alpha g_{ij} + \beta u_i u_j$ by transvecting with g^{ij} we get $R = n\alpha - \beta$, so that $n\nabla_k \alpha = \nabla_k \beta$.

Now taking covariant derivative of the Ricci tensor and using $n\nabla_k \alpha = \nabla_k \beta$, we obtain

$$\nabla_j(\beta u_i u_k) - \nabla_k(\beta u_i u_j) = \frac{1}{n}(\nabla_j \beta g_{ik} - \nabla_k \beta g_{ij}). \tag{4.1}$$

Transvecting the above equation with u^i and using $u^i \nabla_k u_i = 0$, we get

$$\beta(\nabla_j u_k - \nabla_k u_j) = \frac{n-1}{n}(u_j \nabla_k \beta - u_k \nabla_j \beta). \tag{4.2}$$

We show that the right hand side of (4.2) is zero. Transvecting (4.2) with g^{ij} we get

$$u^j \nabla_j(\beta u_k) + \beta u_k \nabla_j u^j = \frac{1}{n}\nabla_k \beta.$$

Further transvecting with u^k implies

$$-\beta \nabla_j u^j = \frac{n-1}{n} u^j \nabla_j \beta.$$

The previous equation becomes

$$\beta u^j \nabla_j u_k = -\frac{1}{n}(\nabla_k \beta + u_k u^j \nabla_j \beta). \tag{4.3}$$

On the other hand, transvecting (4.1) with $u^j u^k$ infers

$$-\beta u^k \nabla_k u_i = \frac{n-1}{n}(\nabla_i \beta + u_i u^k \nabla_k \beta). \tag{4.4}$$

If $\beta \neq 0$, the equations (4.3) and (4.4) imply

$$u^k \nabla_i u_k = 0$$

and

$$\nabla_i \beta + u_i u^k \nabla_k \beta = 0.$$

Multiplying this equation by u_j and taking the antisymmetric part we obtain

$$u_k \nabla_j \beta - u_j \nabla_k \beta = 0.$$

Therefore from equation (4.2) we conclude that

$$\nabla_j u_k = \nabla_k u_j,$$

which implies that u_k is irrotational. Hence the integral curves of the vector field u_k are geodesic. □

Now using the above Lemma and Theorem 1.1 of [27] we are in a position to state the following:

Theorem 4.2. *Let (M, g) be a perfect fluid spacetime with harmonic semi-conformal curvature tensor. Then (M, g) is a GRW spacetime with Einstein fibre.*

Since an n-dimensional spacetime with $\nabla_h C^h_{ijk} = 0$ and $\nabla_k R = 0$ is a Yang pure space [14], Theorem 4.2 can be restated as:

Corollary 4.1. *Any $n(n \geq 3)$-dimensional perfect fluid Yang pure space with $\beta \neq 0$ is a GRW spacetime with Einstein fibre.*

Remark 1. For the dimension $n = 4$, the condition $u_h C^h_{ijk} = 0$ is equivalent to

$$u_i C_{jklm} + u_j C_{kilm} + u_h C_{ijlm} = 0$$

([22], p. 128) and the contraction with u^i gives $C_{jklm} = 0$. But it is known [5] that a GRW spacetime M is conformally flat if and only if M is a RW spacetime. Thus we conclude that for $n = 4$ dimension a perfect fluid spacetime with harmonic semiconformal curvature tensor is a RW spacetime.

References

[1] L. Alías, A. Romero, and M. Sánchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **27** (1995), no. 1, 71–84. https://doi.org/10.1007/BF02105675

[2] ———, *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, In Geometry and topology of submanifolds, VII (Leuven, 1994/Brussels, 1994), 67–70, World Sci. Publ., River Edge, NJ, 1995.

[3] A. Barnes, *On shear free normal flows of a perfect fluid*, General Relativity and Gravitation **4** (1973), no. 2, 105–129. https://doi.org/10.1007/bf00762798

[4] J. P. Bourguigon, *Codazzi tensor fields and curvature operators*, Global Diff. geometry and global analysis, (Proceedings, Berlin 1979), 249–250, Lecture Notes in Maths. 838, Spriger Verlag, Berlin, New York, 2006.

[5] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo, *Some remarks on locally conformally flat static space-times*, J. Math. Phys. **46** (2005), no. 2, 022501, 11 pp. https://doi.org/10.1063/1.1832755

[6] M. C. Chaki and S. Ray, *Space-times with covariant-constant energy-momentum tensor*, Internat. J. Theoret. Phys. **35** (1996), no. 5, 1027–1032. https://doi.org/10.1007/BF02302387

[7] B.-Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **46** (2014), no. 12, Art. 1833, 5 pp. https://doi.org/10.1007/s10714-014-1833-9

[8] A. A. Coley, *Fluid spacetimes admitting a conformal Killing vector parallel to the velocity vector*, Classical Quantum Gravity **8** (1991), no. 5, 955–968.

[9] A. De, C. Özgür, and U. C. De, *On conformally flat almost pseudo-Ricci symmetric spacetimes*, Internat. J. Theoret. Phys. **51** (2012), no. 9, 2878–2887. https://doi.org/10.1007/s10773-012-1164-0

[10] U. C. De and L. Velimirović, *Spacetimes with semisymmetric energy-momentum tensor*, Internat. J. Theoret. Phys. **54** (2015), no. 6, 1779–1783. https://doi.org/10.1007/s10773-014-2381-5

[11] R. Deszcz, M. Glogowska, M. Hotloś, and Z. Sentürk, *On certain quasi-Einstein semisymmetric hypersurfaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **41** (1998), 151–164.

[12] R. Deszcz, M. Hotloś, and Z. Ṡentürk, *Quasi-Einstein hypersurfaces in semi-Riemannian space forms*, Colloq. Math. **89** (2001), no. 1, 81–97. https://doi.org/10.4064/cm89-1-6

[13] K. L. Duggal and R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds*, Mathematics and its Applications,**487**, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/978-1-4615-5315-1

[14] B. S. Guilfoyle and B. C. Nolan, *Yang's gravitational theory*, Gen. Relativity Gravitation **30** (1998), no. 3, 473–495. https://doi.org/10.1023/A:1018815027071

[15] S. Güler and S. Altay Demirbağ, *On Ricci symmetric generalized quasi Einstein spacetimes*, Miskolc Math. Notes **16** (2015), no. 2, 853–868. https://doi.org/10.18514/MMN.2015.1447

[16] ———, *A study of generalized quasi-Einstein spacetimes with applications in general relativity*, Int. J. Theor. Phys. **55** (2016), 548–562.

[17] G. S. Hall and A. D. Rendall, *Uniqueness of the metric from the Weyl and energy-momentum tensors*, J. Math. Phys. **28** (1987), no. 8, 1837–1839. https://doi.org/10.1063/1.527443

[18] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, London, 1973.

[19] Y. Ishii, *On conharmonic transformations*, Tensor (N.S.) **7** (1957), 73–80.

[20] J. Kim, *On pseudo semiconformally symmetric manifolds*, Bull. Korean Math. Soc. **54** (2017), no. 1, 177–186. https://doi.org/10.4134/BKMS.b151007

[21] ———, *A type of conformal curvature tensor*, to appear in Far East J. Math. Soc.

[22] D. Lovelock and H. Rund, *Tensor, Differential Forms, and Variational Principles*, Wiley-Interscience, New York, 1975.

[23] S. Mallick and U. C. De, *Spacetimes admitting W_2-curvature tensor*, Int. J. Geom. Methods Mod. Phys. **11** (2014), no. 4, 1450030, 8 pp. https://doi.org/10.1142/S0219887814500303

[24] S. Mallick, Y. J. Suh, and U. C. De, *A spacetime with pseudo-projective curvature tensor*, J. Math. Phys. **57** (2016), no. 6, 062501, 10 pp. https://doi.org/10.1063/1.4952699

[25] C. A. Mantica and L. G. Molinari, *A second-order identity for the Riemann tensor and applications*, Colloq. Math. **122** (2011), no. 1, 69–82. https://doi.org/10.4064/cm122-1-7

[26] ———, *Weakly Z-symmetric manifolds*, Acta Math. Hungar. **135** (2012), no. 1-2, 80–96. https://doi.org/10.1007/s10474-011-0166-3

[27] C. A. Mantica, L. G. Molinari, and U. C. De, *A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time*, J. Math. Phys. **57** (2016), no. 2, 022508, 6 pp. https://doi.org/10.1063/1.4941942

[28] C. A. Mantica and Y. J. Suh, *Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors*, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 1, 1250004, 21 pp. https://doi.org/10.1142/S0219887812500041

[29] ———, *Pseudo-Z symmetric space-times*, J. Math. Phys. **55** (2014), no. 4, 042502, 12 pp. https://doi.org/10.1063/1.4871442

[30] M. M. Postnikov, *Geometry VI*, translated from the 1998 Russian edition by S. A. Vakhrameev, Encyclopaedia of Mathematical Sciences, 91, Springer-Verlag, Berlin, 2001. https://doi.org/10.1007/978-3-662-04433-9

[31] S. Ray, *On a type of general relativistic perfect fluid space-time with divergence-free projective curvature tensor*, Tensor (N.S.) **61** (1999), no. 2, 176–180.

[32] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relativity Gravitation **30** (1998), no. 6, 915–932. https://doi.org/10.1023/A:1026664209847

[33] ———, *On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields*, J. Geom. Phys. **31** (1999), no. 1, 1–15. https://doi.org/10.1016/S0393-0440(98)00061-8

[34] R. Sharma, *Proper conformal symmetries of space-times with divergence-free Weyl conformal tensor*, J. Math. Phys. **34** (1993), no. 8, 3582–3587. https://doi.org/10.1063/1.530046

[35] L. C. Shepley and A. H. Taub, *Space-times containing perfect fluids and having a vanishing conformal divergence*, Comm. Math. Phys. **5** (1967), no. 4, 237–256. https://doi.org/10.1007/BF01646477

[36] H. Stephani, *General Relativity*, translated from the German by Martin Pollock and John Stewart, Cambridge University Press, Cambridge, 1982.

[37] F. Zengin, *M-projectively flat spacetimes*, Math. Rep. (Bucur.) **14(64)** (2012), no. 4, 363–370.

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